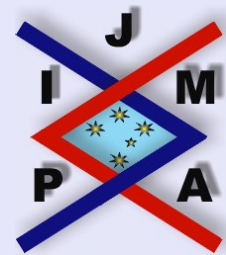


GROWTH OF SOLUTIONS OF CERTAIN NON-HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS WITH ENTIRE COEFFICIENTS

BENHARRAT BELAÏDI

Department of Mathematics
Laboratory of Pure and Applied Mathematics
University of Mostaganem
B. P 227 Mostaganem-(Algeria).
EMail: belaidi@univ-mosta.dz



volume 5, issue 2, article 40,
2004.

*Received 31 December, 2003;
accepted 06 March, 2004.*

Communicated by: H.M. Srivastava

Abstract

Contents

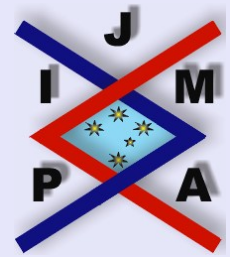


Home Page

Go Back

Close

Quit



Abstract

In this paper, we investigate the growth of solutions of the differential equation $f^{(k)} + A_{k-1}(z)f^{(k-1)} + \dots + A_1(z)f' + A_0(z)f = F$, where $A_0(z), \dots, A_{k-1}(z), F(z) \neq 0$ are entire functions, and we obtain general estimates of the hyper-exponent of convergence of distinct zeros and the hyper-order of solutions for the above equation.

2000 Mathematics Subject Classification: 34M10, 30D35.

Key words: Differential equations, Hyper-order, Hyper-exponent of convergence of distinct zeros, Wiman-Valiron theory.

Contents

1	Introduction and Statement of Results	3
2	Preliminary Lemmas	8
3	Proof of Theorem 1.1	11
4	Proof of Theorem 1.2	16
	References	

Growth Of Solutions Of Certain Non-Homogeneous Linear Differential Equations With Entire Coefficients

Benharrat Belaïdi

Title Page

Contents



Go Back

Close

Quit

Page 2 of 21

1. Introduction and Statement of Results

In this paper, we will use the standard notations of the Nevanlinna value distribution theory (see [8]). In addition, we use the notations $\sigma(f)$ and $\mu(f)$ to denote respectively the order and the lower order of growth of $f(z)$. Recalling the following definitions of hyper-order and hyper-exponent of convergence of distinct zeros.

Definition 1.1. ([3] – [6], [12]). *Let f be an entire function. Then the hyper-order $\sigma_2(f)$ of $f(z)$ is defined by*

$$(1.1) \quad \sigma_2(f) = \overline{\lim}_{r \rightarrow +\infty} \frac{\log \log T(r, f)}{\log r} = \overline{\lim}_{r \rightarrow +\infty} \frac{\log \log \log M(r, f)}{\log r},$$

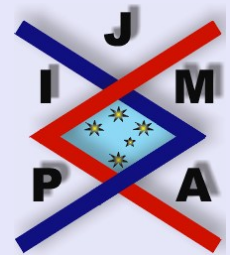
where $T(r, f)$ is the Nevanlinna characteristic function of f (see [8]), and $M(r, f) = \max_{|z|=r} |f(z)|$.

Definition 1.2. ([5]). *Let f be an entire function. Then the hyper-exponent of convergence of distinct zeros of $f(z)$ is defined by*

$$(1.2) \quad \bar{\lambda}_2(f) = \overline{\lim}_{r \rightarrow +\infty} \frac{\log \log \bar{N}\left(r, \frac{1}{f}\right)}{\log r},$$

where $\bar{N}\left(r, \frac{1}{f}\right)$ is the counting function of distinct zeros of $f(z)$ in $\{|z| < r\}$.

We define the linear measure of a set $E \subset [0, +\infty[$ by $m(E) = \int_0^{+\infty} \chi_E(t) dt$ and the logarithmic measure of a set $F \subset [1, +\infty[$ by $lm(F) = \int_1^{+\infty} \frac{\chi_F(t) dt}{t}$,



Growth Of Solutions Of Certain
Non-Homogeneous Linear
Differential Equations With
Entire Coefficients

Benharrat Belaïdi

Title Page

Contents



Go Back

Close

Quit

Page 3 of 21

where χ_H is the characteristic function of a set H . The upper and the lower densities of E are defined by

$$(1.3) \quad \overline{\text{dens}}E = \lim_{r \rightarrow +\infty} \frac{m(E \cap [0, r])}{r},$$

$$\underline{\text{dens}}E = \lim_{r \rightarrow +\infty} \frac{m(E \cap [0, r])}{r}.$$

The upper and the lower logarithmic densities of F are defined by

$$(1.4) \quad \overline{\log \text{dens}}(F) = \lim_{r \rightarrow +\infty} \frac{\text{lm}(F \cap [1, r])}{\log r},$$

$$\underline{\log \text{dens}}(F) = \lim_{r \rightarrow +\infty} \frac{\text{lm}(F \cap [1, r])}{\log r}.$$

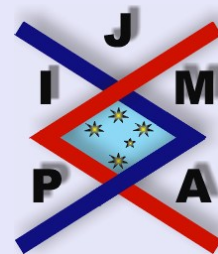
In the study of the solutions of complex differential equations, the growth of a solution is a very important property. Recently, Z. X. Chen and C. C. Yang have investigated the growth of solutions of the non-homogeneous linear differential equation of second order

$$(1.5) \quad f'' + A_1(z) f' + A_0(z) f = F,$$

and have obtained the following two results:

Theorem A. [5, p. 276]. Let E be a set of complex numbers satisfying $\overline{\text{dens}}\{|z| : z \in E\} > 0$, and let $A_0(z)$, $A_1(z)$ be entire functions, with $\sigma(A_1) \leq \sigma(A_0) = \sigma < +\infty$ such that for a real constant $C (> 0)$ and for any given $\varepsilon > 0$,

$$(1.6) \quad |A_1(z)| \leq \exp(o(1)|z|^{\sigma-\varepsilon})$$



Growth Of Solutions Of Certain
Non-Homogeneous Linear
Differential Equations With
Entire Coefficients

Benharrat Belaïdi

Title Page

Contents



Go Back

Close

Quit

Page 4 of 21

and

$$(1.7) \quad |A_0(z)| \geq \exp((1 + o(1))C|z|^{\sigma-\varepsilon})$$

as $z \rightarrow \infty$ for $z \in E$, and let $F \not\equiv 0$ be an entire function with $\sigma(F) < +\infty$. Then every entire solution $f(z)$ of the equation (1.5) satisfies $\overline{\lambda}_2(f) = \sigma_2(f) = \sigma$, with at most one exceptional solution f_0 satisfying $\sigma(f_0) < \sigma$.

Theorem B. [5, p. 276]. Let $A_1(z)$, $A_0(z) \not\equiv 0$ be entire functions such that $\sigma(A_0) < \sigma(A_1) < \frac{1}{2}$ (or A_1 is transcendental, $\sigma(A_1) = 0$, A_0 is a polynomial), and let $F \not\equiv 0$ be an entire function. Consider a solution f of the equation (1.5), we have

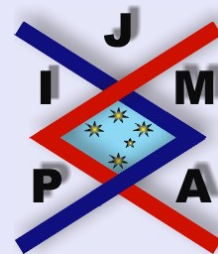
- (i) If $\sigma(F) < \sigma(A_1)$ (or F is a polynomial when A_1 is transcendental, $\sigma(A_1) = 0$, A_0 is a polynomial), then every entire solution $f(z)$ of (1.5) satisfies $\overline{\lambda}_2(f) = \sigma_2(f) = \sigma(A_1)$.
- (ii) If $\sigma(A_1) \leq \sigma(F) < +\infty$, then every entire solution $f(z)$ of (1.5) satisfies $\overline{\lambda}_2(f) = \sigma_2(f) = \sigma(A_1)$, with at most one exceptional solution f_0 satisfying $\sigma(f_0) < \sigma(A_1)$.

For $k \geq 2$, we consider the non-homogeneous linear differential equation

$$(1.8) \quad f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = F,$$

where $A_0(z), \dots, A_{k-1}(z)$ and $F(z) \not\equiv 0$ are entire functions. It is well-known that all solutions of equation (1.8) are entire functions.

Recently, the concepts of hyper-order [3] – [6] and iterated order [10] were used to further investigate the growth of infinite order solutions of complex



Growth Of Solutions Of Certain Non-Homogeneous Linear Differential Equations With Entire Coefficients

Benharrat Belaïdi

Title Page

Contents



Go Back

Close

Quit

Page 5 of 21

differential equations. The main purposes of this paper are to investigate the hyper-exponent of convergence of distinct zeros and the hyper-order of infinite order solutions for the above equation. We will prove the following two theorems:

Theorem 1.1. *Let E be a set of complex numbers satisfying $\overline{\text{dens}}\{|z| : z \in E\} > 0$, and let $A_0(z), \dots, A_{k-1}(z)$ be entire functions, with $\max\{\sigma(A_j) : j = 1, \dots, k\} \leq \sigma(A_0) = \sigma < +\infty$ such that for real constants $0 \leq \beta < \alpha$ and for any given $\varepsilon > 0$,*

$$(1.9) \quad |A_j(z)| \leq \exp(\beta|z|^{\sigma-\varepsilon}) \quad (j = 1, \dots, k-1)$$

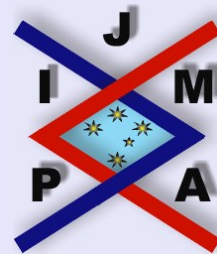
and

$$(1.10) \quad |A_0(z)| \geq \exp(\alpha|z|^{\sigma-\varepsilon})$$

as $z \rightarrow \infty$ for $z \in E$, and let $F \not\equiv 0$ be an entire function with $\sigma(F) < +\infty$. Then every entire solution $f(z)$ of the equation (1.8) satisfies $\overline{\lambda}_2(f) = \sigma_2(f) = \sigma$, with at most one exceptional solution f_0 satisfying $\sigma(f_0) < \sigma$.

Theorem 1.2. *Let $A_0(z), \dots, A_{k-1}(z)$ be entire functions with $A_0(z) \not\equiv 0$ such that $\max\{\sigma(A_j) : j = 0, 2, \dots, k-1\} < \sigma(A_1) < \frac{1}{2}$ (or A_1 is transcendental, $\sigma(A_1) = 0$, A_0, A_2, \dots, A_{k-1} are polynomials), and let $F \not\equiv 0$ be an entire function. Consider a solution f of the equation (1.8), we have*

- (i) *If $\sigma(F) < \sigma(A_1)$ (or F is a polynomial when A_1 is transcendental, $\sigma(A_1) = 0$, A_0, A_2, \dots, A_{k-1} are polynomials), then every entire solution $f(z)$ of (1.8) satisfies $\overline{\lambda}_2(f) = \sigma_2(f) = \sigma(A_1)$.*

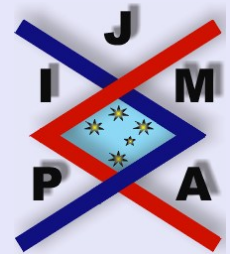


Growth Of Solutions Of Certain Non-Homogeneous Linear Differential Equations With Entire Coefficients

Benharrat Belaïdi

Title Page	
Contents	
◀◀	▶▶
◀	▶
Go Back	
Close	
Quit	
Page 6 of 21	

(ii) If $\sigma(A_1) \leq \sigma(F) < +\infty$, then every entire solution $f(z)$ of (1.8) satisfies $\lambda_2(f) = \sigma_2(f) = \sigma(A_1)$, with at most one exceptional solution f_0 satisfying $\sigma(f_0) < \sigma(A_1)$.



**Growth Of Solutions Of Certain
Non-Homogeneous Linear
Differential Equations With
Entire Coefficients**

Benharrat Belaïdi

Title Page

Contents



Go Back

Close

Quit

Page 7 of 21

2. Preliminary Lemmas

Our proofs depend mainly upon the following lemmas.

Lemma 2.1. ([3]). *Let E be a set of complex numbers satisfying $\overline{\text{dens}} \{ |z| : z \in E \} > 0$, and let $A_0(z), \dots, A_{k-1}(z)$ be entire functions, with $\max\{\sigma(A_j) : j = 1, \dots, k\} \leq \sigma(A_0) = \sigma < +\infty$ such that for some real constants $0 \leq \beta < \alpha$ and for any given $\varepsilon > 0$,*

$$(2.1) \quad |A_j(z)| \leq \exp(\beta |z|^{\sigma-\varepsilon}) \quad (j = 1, \dots, k-1)$$

and

$$(2.2) \quad |A_0(z)| \geq \exp(\alpha |z|^{\sigma-\varepsilon})$$

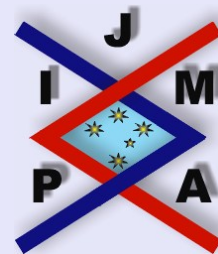
as $z \rightarrow \infty$ for $z \in E$. Then every entire solution $f \not\equiv 0$ of the equation

$$(2.3) \quad f^{(k)} + A_{k-1}(z) f^{(k-1)} + \dots + A_1(z) f' + A_0(z) f = 0$$

satisfies $\sigma(f) = +\infty$ and $\sigma_2(f) = \sigma(A_0)$.

Lemma 2.2. ([7]). *Let $f(z)$ be a nontrivial entire function, and let $\alpha > 1$ and $\varepsilon > 0$ be given constants. Then there exist a constant $c > 0$ and a set $E \subset [0, +\infty)$ having finite linear measure such that for all z satisfying $|z| = r \notin E$, we have*

$$(2.4) \quad \left| \frac{f^{(j)}(z)}{f(z)} \right| \leq c [T(\alpha r, f) r^\varepsilon \log T(\alpha r, f)]^j \quad (j \in \mathbb{N}).$$



Growth Of Solutions Of Certain
Non-Homogeneous Linear
Differential Equations With
Entire Coefficients

Benharrat Belaïdi

Title Page

Contents

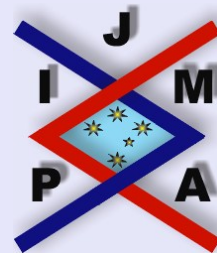


Go Back

Close

Quit

Page 8 of 21



Title Page

Contents



Go Back

Close

Quit

Page 9 of 21

Lemma 2.3. ([7]). Let $f(z)$ be a transcendental meromorphic function, and let $\alpha > 1$ be a given constant. Then there exists a set $E \subset (1, +\infty)$ of finite logarithmic measure and a constant $B > 0$ that depends only on α and (m, n) (m, n positive integers with $m < n$) such that for all z satisfying $|z| = r \notin [0, 1] \cup E$, we have

$$(2.5) \quad \left| \frac{f^{(n)}(z)}{f^{(m)}(z)} \right| \leq B \left[\frac{T(\alpha r, f)}{r} (\log^\alpha r) \log T(\alpha r, f) \right]^{n-m}.$$

Lemma 2.4. ([5]). Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of infinite order with the hyper-order $\sigma_2(f) = \sigma$, $\mu(r)$ be the maximum term, i.e $\mu(r) = \max\{|a_n| r^n; n = 0, 1, \dots\}$ and let $\nu_f(r)$ be the central index of f , i.e $\nu_f(r) = \max\{m, \mu(r) = |a_m| r^m\}$. Then

$$(2.6) \quad \lim_{r \rightarrow +\infty} \frac{\log \log \nu_f(r)}{\log r} = \sigma.$$

Lemma 2.5. (Wiman-Valiron, [9, 11]). Let $f(z)$ be a transcendental entire function and let z be a point with $|z| = r$ at which $|f(z)| = M(r, f)$. Then for all $|z|$ outside a set E of finite logarithmic measure, we have

$$(2.7) \quad \frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z} \right)^j (1 + o(1)) \quad (j \text{ is an integer, } r \notin E).$$

Lemma 2.6. ([1]). Let $f(z)$ be an entire function of order $\sigma(f) = \sigma < \frac{1}{2}$, and denote $A(r) = \inf_{|z|=r} \log |f(z)|$, $B(r) = \sup_{|z|=r} \log |f(z)|$. If $\sigma < \alpha < 1$, then

$$(2.8) \quad \underline{\log dens} \{r : A(r) > (\cos \pi \alpha) B(r)\} \geq 1 - \frac{\sigma}{\alpha}.$$

Lemma 2.7. ([2]). Let $f(z)$ be an entire function with $\mu(f) = \mu < \frac{1}{2}$ and $\mu < \sigma(f) = \sigma$. If $\mu \leq \delta < \min(\sigma, \frac{1}{2})$ and $\delta < \alpha < \frac{1}{2}$, then

$$(2.9) \quad \overline{\log dens} \{r : A(r) > (\cos \pi\alpha) B(r) > r^\delta\} > C(\sigma, \delta, \alpha),$$

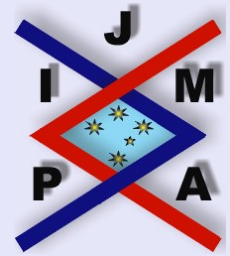
where $C(\sigma, \delta, \alpha)$ is a positive constant depending only on σ, δ and α .

Lemma 2.8. Suppose that $A_0(z), \dots, A_{k-1}(z)$ are entire functions such that $A_0(z) \not\equiv 0$ and

$$(2.10) \quad \max \{\sigma(A_j) : j = 0, 2, \dots, k-1\} < \sigma(A_1) < \frac{1}{2}.$$

Then every transcendental solution $f \not\equiv 0$ of (2.3) is of infinite order.

Proof. Using the same argument as in the proof of Theorem 4 in [6, p. 222], we conclude that $\sigma(f) = +\infty$. □



**Growth Of Solutions Of Certain
Non-Homogeneous Linear
Differential Equations With
Entire Coefficients**

Benharrat Belaïdi

Title Page

Contents



Go Back

Close

Quit

Page 10 of 21

3. Proof of Theorem 1.1

We affirm that (1.8) can only possess at most one exceptional solution f_0 such that $\sigma(f_0) < \sigma$. In fact, if f^* is a second solution with $\sigma(f^*) < \sigma$, then $\sigma(f_0 - f^*) < \sigma$. But $f_0 - f^*$ is a solution of the corresponding homogeneous equation (2.3) of (1.8). This contradicts Lemma 2.1. We assume that f is a solution of (1.8) with $\sigma(f) = +\infty$ and f_1, \dots, f_k are k entire solutions of the corresponding homogeneous equation (2.3). Then by Lemma 2.1, we have $\sigma_2(f_j) = \sigma(A_0) = \sigma$ ($j = 1, \dots, k$). By variation of parameters, f can be expressed in the form

$$(3.1) \quad f(z) = B_1(z) f_1(z) + \dots + B_k(z) f_k(z),$$

where $B_1(z), \dots, B_k(z)$ are determined by

$$B_1'(z) f_1(z) + \dots + B_k'(z) f_k(z) = 0$$

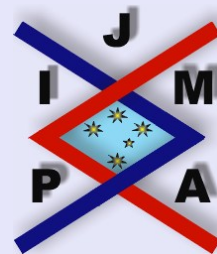
$$B_1'(z) f_1'(z) + \dots + B_k'(z) f_k'(z) = 0$$

.....

$$(3.2) \quad B_1'(z) f_1^{(k-1)}(z) + \dots + B_k'(z) f_k^{(k-1)}(z) = F.$$

Noting that the Wronskian $W(f_1, f_2, \dots, f_k)$ is a differential polynomial in f_1, f_2, \dots, f_k with constant coefficients, it easy to deduce that $\sigma_2(W) \leq \sigma_2(f_j) = \sigma(A_0) = \sigma$. Set

$$(3.3) \quad W_i = \begin{vmatrix} f_1, \dots, \overset{(i)}{0}, \dots, f_k \\ \vdots \\ \vdots \\ f_1^{(k-1)}, \dots, F, \dots, f_k^{(k-1)} \end{vmatrix} = F \cdot g_i \quad (i = 1, \dots, k),$$



Growth Of Solutions Of Certain
Non-Homogeneous Linear
Differential Equations With
Entire Coefficients

Benharrat Belaïdi

Title Page

Contents



Go Back

Close

Quit

Page 11 of 21

where g_i are differential polynomials in f_1, f_2, \dots, f_k with constant coefficients. So

$$(3.4) \quad \sigma_2(g_i) \leq \sigma_2(f_j) = \sigma(A_0), \quad B'_i = \frac{W_i}{W} = \frac{F \cdot g_i}{W} \quad (i = 1, \dots, k)$$

and

$$(3.5) \quad \sigma_2(B_i) = \sigma_2\left(B'_i\right) \leq \max(\sigma_2(F), \sigma(A_0)) = \sigma(A_0) \quad (i = 1, \dots, k),$$

because $\sigma_2(F) = 0$ ($\sigma(F) < +\infty$). Then from (3.1) and (3.5), we get

$$(3.6) \quad \sigma_2(f) \leq \max(\sigma_2(f_j), \sigma_2(B_i)) = \sigma(A_0).$$

Now from (1.8), it follows that

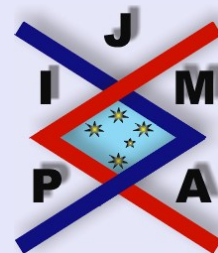
$$(3.7) \quad |A_0(z)| \leq \left| \frac{f^{(k)}}{f} \right| + |A_{k-1}(z)| \left| \frac{f^{(k-1)}}{f} \right| + \dots + |A_1(z)| \left| \frac{f'}{f} \right| + \left| \frac{F}{f} \right|.$$

Then by Lemma 2.2, there exists a set $E_1 \subset [0, +\infty)$ with a finite linear measure such that for all z satisfying $|z| = r \notin E_1$, we have

$$(3.8) \quad \left| \frac{f^{(j)}(z)}{f(z)} \right| \leq r [T(2r, f)]^{k+1} \quad (j = 1, \dots, k).$$

Also, by the hypothesis of Theorem 1.1, there exists a set E_2 with $\overline{\text{dens}}\{|z| : z \in E_2\} > 0$ such that for all z satisfying $z \in E_2$, we have

$$(3.9) \quad |A_j(z)| \leq \exp(\beta |z|^{\sigma-\varepsilon}) \quad (j = 1, \dots, k-1)$$



**Growth Of Solutions Of Certain
Non-Homogeneous Linear
Differential Equations With
Entire Coefficients**

Benharrat Belaïdi

Title Page

Contents



Go Back

Close

Quit

Page 12 of 21

and

$$(3.10) \quad |A_0(z)| \geq \exp(\alpha |z|^{\sigma-\varepsilon})$$

as $z \rightarrow \infty$. Since $\sigma(f) = +\infty$, then for a given arbitrary large $\rho > \sigma(F)$,

$$(3.11) \quad M(r, f) \geq \exp(r^\rho)$$

holds for sufficiently large r . On the other hand, for a given ε with $0 < \varepsilon < \rho - \sigma(F)$, we have

$$(3.12) \quad |F(z)| \leq \exp(r^{\sigma(F)+\varepsilon}),$$

$$\left| \frac{F(z)}{f(z)} \right| \leq \exp(r^{\sigma(F)+\varepsilon} - r^\rho) \rightarrow 0 \quad (r \rightarrow +\infty),$$

where $|f(z)| = M(r, f)$ and $|z| = r$. Hence from (3.7) – (3.10) and (3.12), it follows that for all z satisfying $z \in E_2$, $|z| = r \notin E_1$ and $|f(z)| = M(r, f)$

$$(3.13) \quad \exp(\alpha |z|^{\sigma-\varepsilon})$$

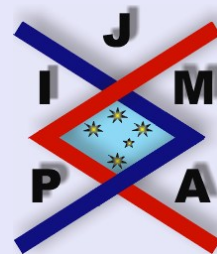
$$\leq |z| [T(2|z|, f)]^{k+1} [1 + (k-1) \exp(\beta |z|^{\sigma-\varepsilon})] + o(1)$$

as $z \rightarrow \infty$. Thus there exists a set $E \subset [0, +\infty)$ with a positive upper density such that

$$(3.14) \quad \exp(\alpha r^{\sigma-\varepsilon}) \leq dr \exp(\beta r^{\sigma-\varepsilon}) [T(2r, f)]^{k+1}$$

as $r \rightarrow +\infty$ in E , where $d(> 0)$ is some constant. Therefore

$$(3.15) \quad \sigma_2(f) = \overline{\lim}_{r \rightarrow +\infty} \frac{\log \log T(r, f)}{\log r} \geq \sigma - \varepsilon.$$



**Growth Of Solutions Of Certain
Non-Homogeneous Linear
Differential Equations With
Entire Coefficients**

Benharrat Belaïdi

Title Page

Contents



Go Back

Close

Quit

Page 13 of 21

Since ε is arbitrary, then by (3.15) we get $\sigma_2(f) \geq \sigma(A_0) = \sigma$. This and the fact that $\sigma_2(f) \leq \sigma$ yield $\sigma_2(f) = \sigma(A_0) = \sigma$.

By (1.8), it is easy to see that if f has a zero at z_0 of order $\alpha (> k)$, then F must have a zero at z_0 of order $\alpha - k$. Hence,

$$(3.16) \quad n\left(r, \frac{1}{f}\right) \leq k\bar{n}\left(r, \frac{1}{f}\right) + n\left(r, \frac{1}{F}\right)$$

and

$$(3.17) \quad N\left(r, \frac{1}{f}\right) \leq k\bar{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{F}\right).$$

Now (1.8) can be rewritten as

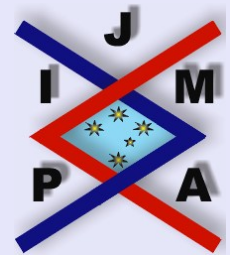
$$(3.18) \quad \frac{1}{f} = \frac{1}{F} \left(\frac{f^{(k)}}{f} + A_{k-1} \frac{f^{(k-1)}}{f} + \cdots + A_1 \frac{f'}{f} + A_0 \right).$$

By (3.18), we have

$$(3.19) \quad m\left(r, \frac{1}{f}\right) \leq \sum_{j=1}^k m\left(r, \frac{f^{(j)}}{f}\right) + \sum_{j=1}^k m(r, A_{k-j}) + m\left(r, \frac{1}{F}\right) + O(1).$$

By (3.17) and (3.19), we get for $|z| = r$ outside a set E_3 of finite linear measure,

$$(3.20) \quad \begin{aligned} T(r, f) &= T\left(r, \frac{1}{f}\right) + O(1) \\ &\leq k\bar{N}\left(r, \frac{1}{f}\right) + \sum_{j=1}^k T(r, A_{k-j}) \\ &\quad + T(r, F) + O(\log(rT(r, f))). \end{aligned}$$



**Growth Of Solutions Of Certain
Non-Homogeneous Linear
Differential Equations With
Entire Coefficients**

Benharrat Belaïdi

Title Page

Contents



Go Back

Close

Quit

Page 14 of 21

For sufficiently large r , we have

$$(3.21) \quad O(\log r + \log T(r, f)) \leq \frac{1}{2}T(r, f)$$

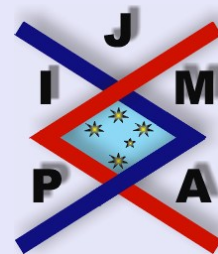
$$(3.22) \quad T(r, A_0) + \cdots + T(r, A_{k-1}) \leq k r^{\sigma+\varepsilon}$$

$$(3.23) \quad T(r, F) \leq r^{\sigma(F)+\varepsilon}.$$

Thus, by (3.20) – (3.23), we have

$$(3.24) \quad T(r, f) \leq 2k \bar{N}\left(r, \frac{1}{f}\right) + 2k r^{\sigma+\varepsilon} + 2r^{\sigma(F)+\varepsilon} \quad (|z| = r \notin E_3).$$

Hence for any f with $\sigma_2(f) = \sigma$, by (3.24), we have $\sigma_2(f) \leq \bar{\lambda}_2(f)$. Therefore, $\bar{\lambda}_2(f) = \sigma_2(f) = \sigma$.



**Growth Of Solutions Of Certain
Non-Homogeneous Linear
Differential Equations With
Entire Coefficients**

Benharrat Belaïdi

Title Page

Contents



Go Back

Close

Quit

Page 15 of 21

4. Proof of Theorem 1.2

Assume that $f(z)$ is an entire solution of (1.8). For case (i), we assume $\sigma(A_1) > 0$ (when $\sigma(A_1) = 0$, Theorem 1.2 clearly holds). By (1.8) we get

$$(4.1) \quad \begin{aligned} A_1 &= \frac{F'}{f'} - \frac{f^{(k)}}{f'} - A_{k-1} \frac{f^{(k-1)}}{f'} - \dots - A_2 \frac{f''}{f'} - A_0 \frac{f}{f'} \\ &= \frac{F'}{f} \frac{f}{f'} - \frac{f^{(k)}}{f'} - A_{k-1} \frac{f^{(k-1)}}{f'} - \dots - A_2 \frac{f''}{f'} - A_0 \frac{f}{f'}. \end{aligned}$$

By Lemma 2.3, we see that there exists a set $E_4 \subset (1, +\infty)$ with finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_4$, we have

$$(4.2) \quad \left| \frac{f^{(j)}(z)}{f'(z)} \right| \leq Br [T(2r, f)]^k \quad (j = 2, \dots, k).$$

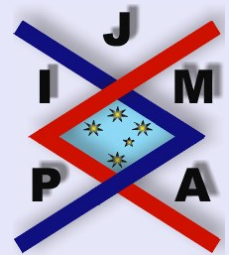
Now set $b = \max\{\sigma(A_j) : j = 0, 2, \dots, k-1; \sigma(F)\}$, and we choose real numbers α, β such that

$$(4.3) \quad b < \alpha < \beta < \sigma(A_1).$$

Then for sufficiently large r , we have

$$(4.4) \quad |A_j(z)| \leq \exp(r^\alpha) \quad (j = 0, 2, \dots, k-1),$$

$$(4.5) \quad |F(z)| \leq \exp(r^\alpha).$$



Growth Of Solutions Of Certain
Non-Homogeneous Linear
Differential Equations With
Entire Coefficients

Benharrat Belaïdi

Title Page

Contents



Go Back

Close

Quit

Page 16 of 21

By Lemma 2.6 (if $\mu(A_1) = \sigma(A_1)$) or Lemma 2.7 (if $\mu(A_1) < \sigma(A_1)$) there exists a subset $E_5 \subset (1, +\infty)$ with logarithmic measure $lm(E_5) = \infty$ such that for all z satisfying $|z| = r \in E_5$, we have

$$(4.6) \quad |A_1(z)| > \exp(r^\beta).$$

Since $M(r, f) > 1$ for sufficiently large r , we have by (4.5)

$$(4.7) \quad \frac{|F(z)|}{M(r, f)} \leq \exp(r^\alpha).$$

On the other hand, by Lemma 2.5, there exists a set $E_6 \subset (1, +\infty)$ of finite logarithmic measure such that (2.7) holds for some point z satisfying $|z| = r \notin [0, 1] \cup E_6$ and $|f(z)| = M(r, f)$. By (2.7), we get

$$\left| \frac{f'(z)}{f(z)} \right| \geq \frac{1}{2} \left| \frac{\nu_f(r)}{z} \right| > \frac{1}{2r}$$

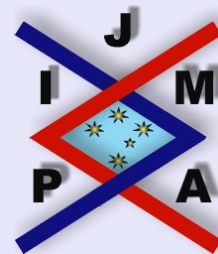
or

$$(4.8) \quad \left| \frac{f(z)}{f'(z)} \right| < 2r.$$

Now by (4.1), (4.2), (4.4), and (4.6) – (4.8), we get

$$\exp(r^\beta) \leq Lr [T(2r, f)]^k 2 \exp(r^\alpha) 2r$$

for $|z| = r \in E_5 \setminus ([0, 1] \cup E_4 \cup E_6)$ and $|f(z)| = M(r, f)$, where $L(> 0)$ is some constant. From this and since β is arbitrary, we get $\sigma(f) = +\infty$ and $\sigma_2(f) \geq \sigma(A_1)$.



**Growth Of Solutions Of Certain
Non-Homogeneous Linear
Differential Equations With
Entire Coefficients**

Benharrat Belaïdi

Title Page

Contents



Go Back

Close

Quit

Page 17 of 21

On the other hand, for any given $\varepsilon > 0$, if r is sufficiently large, we have

$$(4.9) \quad |A_j(z)| \leq \exp(r^{\sigma(A_1)+\varepsilon}) \quad (j = 0, 1, \dots, k-1),$$

$$(4.10) \quad |F(z)| \leq \exp(r^{\sigma(A_1)+\varepsilon}).$$

Since $M(r, f) > 1$ for sufficiently large r , we have by (4.10)

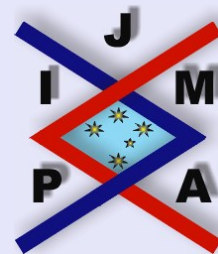
$$(4.11) \quad \frac{|F(z)|}{M(r, f)} \leq \exp(r^{\sigma(A_1)+\varepsilon}).$$

Substituting (2.7), (4.9) and (4.11) into (1.8), we obtain

$$(4.12) \quad \left(\frac{\nu_f(r)}{|z|}\right)^k |1 + o(1)| \leq \exp(r^{\sigma(A_1)+\varepsilon}) \left(\frac{\nu_f(r)}{|z|}\right)^{k-1} |1 + o(1)| \\ + \exp(r^{\sigma(A_1)+\varepsilon}) \left(\frac{\nu_f(r)}{|z|}\right)^{k-2} |1 + o(1)| + \dots \\ + \exp(r^{\sigma(A_1)+\varepsilon}) \left(\frac{\nu_f(r)}{|z|}\right) |1 + o(1)| + 2 \exp(r^{\sigma(A_1)+\varepsilon}),$$

where z satisfies $|z| = r \notin [0, 1] \cup E_6$ and $|f(z)| = M(r, f)$. By (4.12), we get

$$(4.13) \quad \overline{\lim}_{r \rightarrow +\infty} \frac{\log \log \nu_f(r)}{\log r} \leq \sigma(A_1) + \varepsilon.$$



**Growth Of Solutions Of Certain
Non-Homogeneous Linear
Differential Equations With
Entire Coefficients**

Benharrat Belaïdi

Title Page

Contents



Go Back

Close

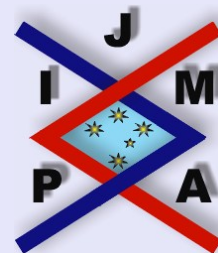
Quit

Page 18 of 21

Since ε is arbitrary, by (4.13) and Lemma 2.4 we have $\sigma_2(f) \leq \sigma(A_1)$. This and the fact that $\sigma_2(f) \geq \sigma(A_1)$ yield $\sigma_2(f) = \sigma(A_1)$.

By a similar argument to that used in the proof of Theorem 1.1, we can get $\overline{\lambda_2}(f) = \sigma_2(f) = \sigma(A_1)$.

Finally, case (ii) can also be obtained by using Lemma 2.8 and an argument similar to that in the proof of Theorem 1.1.



**Growth Of Solutions Of Certain
Non-Homogeneous Linear
Differential Equations With
Entire Coefficients**

Benharrat Belaïdi

Title Page

Contents



Go Back

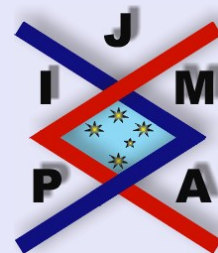
Close

Quit

Page 19 of 21

References

- [1] P.D. BARRY, On a theorem of Besicovitch, *Quart. J. Math. Oxford Ser. (2)*, **14** (1963), 293–302.
- [2] P.D. BARRY, Some theorems related to the $\cos \pi\rho$ -theorem, *Proc. London Math. Soc. (3)*, **21** (1970), 334–360.
- [3] B. BELAÏDI, Estimation of the hyper-order of entire solutions of complex linear ordinary differential equations whose coefficients are entire functions, *E. J. Qualitative Theory of Diff. Equ.*, **5** (2002), 1–8.
- [4] B. BELAÏDI AND K. HAMANI, Order and hyper-order of entire solutions of linear differential equations with entire coefficients, *Electron. J. Diff. Eqns*, **17** (2003), 1–12.
- [5] Z.X. CHEN AND C.C. YANG, Some further results on the zeros and growths of entire solutions of second order linear differential equations, *Kodai Math. J.*, **22** (1999), 273–285.
- [6] Z.X. CHEN AND C.C. YANG, On the zeros and hyper-order of meromorphic solutions of linear differential equations, *Ann. Acad. Sci. Fenn. Math.*, **24** (1999), 215–224.
- [7] G.G. GUNDERSEN, Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates, *J. London Math. Soc. (2)*, **37** (1988), 88–104.
- [8] W.K. HAYMAN, *Meromorphic functions*, Clarendon Press, Oxford, 1964.



Growth Of Solutions Of Certain
Non-Homogeneous Linear
Differential Equations With
Entire Coefficients

Benharrat Belaïdi

Title Page

Contents



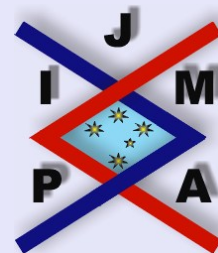
Go Back

Close

Quit

Page 20 of 21

- [9] W.K. HAYMAN, The local growth of power series: a survey of the Wiman-Valiron method, *Canad. Math. Bull.*, **17** (1974), 317–358.
- [10] L. KINNUNEN, Linear differential equations with solutions of finite iterated order, *Southeast Asian Bull. Math.*, **22** (1998), 385–405.
- [11] G. VALIRON, *Lectures on the General Theory of Integral Functions*, translated by E. F. Collingwood, Chelsea, New York, 1949.
- [12] H.X. YI AND C.C. YANG, *The Uniqueness Theory of Meromorphic Functions*, Science Press, Beijing, 1995 (in Chinese).



**Growth Of Solutions Of Certain
Non-Homogeneous Linear
Differential Equations With
Entire Coefficients**

Benharrat Belaïdi

Title Page

Contents



Go Back

Close

Quit

Page 21 of 21