

# LOCAL ESTIMATES FOR JACOBI POLYNOMIALS

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*Key words:* Jacobi polynomials, Jacobi weights, Local estimates.

*Abstract:* It is shown that if  $\alpha, \beta \geq -\frac{1}{2}$ , then the orthonormal Jacobi polynomials  $p_n^{(\alpha, \beta)}$  fulfill the local estimate

$$|p_n^{(\alpha, \beta)}(t)| \leq \frac{C(\alpha, \beta)}{(\sqrt{1-x+\frac{1}{n}})^{\alpha+\frac{1}{2}}(\sqrt{1+x+\frac{1}{n}})^{\beta+\frac{1}{2}}}$$

for all  $t \in U_n(x)$  and each  $x \in [-1, 1]$ , where  $U_n(x)$  are subintervals of  $[-1, 1]$  defined by  $U_n(x) = [x - \frac{\varphi_n(x)}{n}, x + \frac{\varphi_n(x)}{n}] \cap [-1, 1]$  for  $n \in \mathbb{N}$  and  $x \in [-1, 1]$  with  $\varphi_n(x) = \sqrt{1-x^2} + \frac{1}{n}$ . Applications of the local estimate are given at the end of the paper.

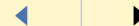
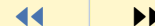
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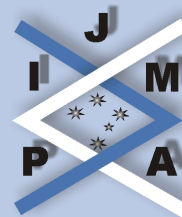
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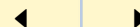
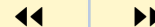
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## 1. Introduction

Let  $w^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$ ,  $x \in [-1, 1]$ , be a Jacobi weight with  $\alpha, \beta > -1$ . Let  $p_n(x) = p_n^{(\alpha,\beta)}(x) = \gamma_n^{(\alpha,\beta)}x^n + \dots$ ,  $n \in \mathbb{N}_0$ , denote the unique *Jacobi polynomials* of precise degree  $n$ , with leading coefficients  $\gamma_n^{(\alpha,\beta)} > 0$ , fulfilling the orthonormal condition  $\int_{-1}^1 p_n(x)p_m(x)w^{(\alpha,\beta)}(x) dx = \delta_{n,m}$ ,  $n, m \in \mathbb{N}_0$ .

This paper is concerned with local estimates of Jacobi polynomials by means of modified Jacobi weights. By the *modified Jacobi weights* we understand the functions

$$(1.1) \quad w_n^{(\alpha,\beta)}(x) := \left( \sqrt{1-x} + \frac{1}{n} \right)^{2\alpha} \left( \sqrt{1+x} + \frac{1}{n} \right)^{2\beta}, \quad x \in [-1, 1], \quad n \in \mathbb{N}.$$

We observe that all modified Jacobi weights  $w_n^{(\alpha,\beta)}$  are finite and positive. This is in contrast to the fact that the Jacobi weight  $w^{(\alpha,\beta)}$  may have singularities and roots in  $\pm 1$ , depending on whether  $\alpha$  and  $\beta$  are negative or positive. The Jacobi polynomials can be estimated by means of modified Jacobi weights as follows (see [3] and Theorem 2.1 below):

$$|p_n^{(\alpha,\beta)}(x)| \leq C \frac{1}{w_n^{(\frac{\alpha}{2} + \frac{1}{4}, \frac{\beta}{2} + \frac{1}{4})}(x)}$$

for all  $x \in [-1, 1]$ . If  $\alpha, \beta \geq -\frac{1}{2}$ , then we will show that this estimate can be further extended, namely

$$|p_n^{(\alpha,\beta)}(t)| \leq C \frac{1}{w_n^{(\frac{\alpha}{2} + \frac{1}{4}, \frac{\beta}{2} + \frac{1}{4})}(x)}$$

for all  $t \in U_n(x)$  and each  $x \in [-1, 1]$ , where  $U_n(x)$  are subintervals of  $[-1, 1]$

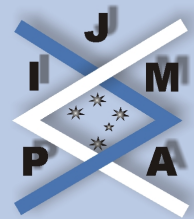
defined by

$$(1.2) \quad U_n(x) := \left\{ t \in [-1, 1] \mid |t - x| \leq \frac{\varphi_n(x)}{n} \right\} \\ = \left[ x - \frac{\varphi_n(x)}{n}, x + \frac{\varphi_n(x)}{n} \right] \cap [-1, 1]$$

for  $n \in \mathbb{N}$  and  $x \in [-1, 1]$  with

$$(1.3) \quad \varphi_n(x) := \sqrt{1 - x^2} + \frac{1}{n}.$$

Thus  $U_n(x)$  is located around  $x$  and is *small*, i.e.,  $|U_n(x)| = O(1/n)$ . In our case of Jacobi weights on  $[-1, 1]$  we need intervals around  $x$  with radius  $\frac{\varphi_n(x)}{n}$  instead of  $\frac{1}{n}$ . In this case the radius varies together with  $x$  and becomes smaller if  $x$  tends to 1 or  $-1$ .



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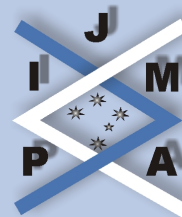
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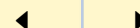
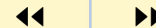
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## 2. Theorems

The following theorem provides a useful local estimate of the orthonormal Jacobi polynomials by means of the modified weights  $w_n$ . The estimate can also be found in the paper [3] by Lubinsky and Totik. Here we will give an explicit proof. The proof is essentially based on an estimate taken from Szegő [4].

**Theorem 2.1.** *Let  $\alpha, \beta > -1$  and  $n \in \mathbb{N}$ . Then*

$$(2.1) \quad |p_n^{(\alpha, \beta)}(x)| \leq C \frac{1}{w_n^{(\frac{\alpha}{2} + \frac{1}{4}, \frac{\beta}{2} + \frac{1}{4})}(x)}$$

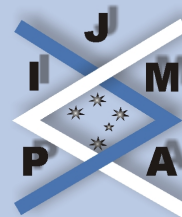
for all  $x \in [-1, 1]$  with a positive constant  $C = C(\alpha, \beta)$  being independent of  $n$  and  $x$ .

*Proof.* First let  $x \in [0, 1]$ , and let  $t \in [0, \frac{\pi}{2}]$  such that  $x = \cos t$ . Moreover, let  $P_n = P_n^{(\alpha, \beta)} = (h_n^{(\alpha, \beta)})^{\frac{1}{2}} p_n^{(\alpha, \beta)}(x)$ ,  $n \in \mathbb{N}$ , be the polynomials normalized by the factor  $(h_n^{(\alpha, \beta)})^{\frac{1}{2}}$ , namely  $P_n^{(\alpha, \beta)} = (h_n^{(\alpha, \beta)})^{\frac{1}{2}} p_n^{(\alpha, \beta)}(x)$ , as can be found in Szegő [4, eq. (4.3.4)]. According to Szegő's book [4, Theorem 7.32.2] the estimate

$$(2.2) \quad |P_n^{(\alpha, \beta)}(\cos t)| \leq C \begin{cases} n^\alpha, & \text{if } 0 \leq t \leq \frac{c}{n} \\ t^{-(\alpha + \frac{1}{2})} n^{-\frac{1}{2}}, & \text{if } \frac{c}{n} \leq t \leq \frac{\pi}{2} \end{cases}$$

is valid, where  $c$  and  $C$  are fixed positive constants being independent of  $n$  and  $t$ . We substitute  $t = \arccos x \in [0, \frac{\pi}{2}]$  and  $P_n^{(\alpha, \beta)}(x) = (h_n^{(\alpha, \beta)})^{\frac{1}{2}} p_n^{(\alpha, \beta)}(x)$  in (2.2) and obtain, using  $(h_n^{(\alpha, \beta)})^{-\frac{1}{2}} \leq \tilde{C} \cdot n^{\frac{1}{2}}$  (resulting from [4, eq. (4.3.4)]),

$$(2.3) \quad |p_n^{(\alpha, \beta)}(x)| \leq C_1 \begin{cases} n^{\alpha + \frac{1}{2}}, & \text{if } 0 \leq \arccos x \leq \frac{c}{n} \\ (\arccos x)^{-(\alpha + \frac{1}{2})}, & \text{if } \frac{c}{n} \leq \arccos x \leq \frac{\pi}{2} \end{cases}$$



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with  $C_1 = C_1(\alpha, \beta) > 0$  independent of  $n$  and  $x$ . Below we will make use of the estimates

$$(2.4) \quad \begin{aligned} \frac{\pi}{2} \sqrt{1-x} &= \frac{\pi}{\sqrt{2}} \sqrt{\frac{1-x}{2}} = \frac{\pi}{\sqrt{2}} \sin \frac{t}{2} \\ &\geq \frac{\pi}{\sqrt{2}} \left( \frac{2}{\pi} \cdot \frac{t}{\sqrt{2}} \right) = t = \arccos x \end{aligned}$$

and

$$(2.5) \quad \sqrt{2} \sqrt{1-x} = 2 \sqrt{\frac{1-x}{2}} = 2 \sin \frac{t}{2} \leq 2 \cdot \frac{t}{2} = t = \arccos x.$$

The cases  $-1 < \alpha \leq -\frac{1}{2}$  and  $\alpha > -\frac{1}{2}$  are considered separately in the following.

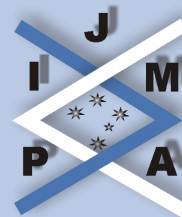
**Case  $-1 < \alpha \leq -\frac{1}{2}$ :** In this case it follows that  $-(\alpha + \frac{1}{2}) \geq 0$ . If  $0 \leq \arccos x \leq \frac{c}{n}$ , then

$$|p_n^{(\alpha, \beta)}(x)| \stackrel{(2.3)}{\leq} C_1 n^{\alpha + \frac{1}{2}} = C_1 \left( \frac{1}{n} \right)^{-(\alpha + \frac{1}{2})} \leq C_1 \left( \sqrt{1-x} + \frac{1}{n} \right)^{-(\alpha + \frac{1}{2})}.$$

If  $\frac{c}{n} \leq \arccos x \leq \frac{\pi}{2}$ , then

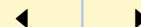
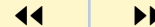
$$\begin{aligned} |p_n^{(\alpha, \beta)}(x)| &\stackrel{(2.3)}{\leq} C_1 (\arccos x)^{-(\alpha + \frac{1}{2})} \stackrel{(2.4)}{\leq} C_2 (\sqrt{1-x})^{-(\alpha + \frac{1}{2})} \\ &\leq C_2 \left( \sqrt{1-x} + \frac{1}{n} \right)^{-(\alpha + \frac{1}{2})}. \end{aligned}$$

**Case  $\alpha > -\frac{1}{2}$ :** In this case we obtain  $-(\alpha + \frac{1}{2}) < 0$ . If  $0 \leq \arccos x \leq \frac{c}{n}$ , then



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from (2.5) we obtain  $\frac{c}{n} \geq \sqrt{2}\sqrt{1-x}$  and hence

$$\begin{aligned} |p_n^{(\alpha,\beta)}(x)| &\stackrel{(2.3)}{\leq} C_1 n^{\alpha+\frac{1}{2}} = C_2 \left(\frac{c}{n} + \frac{c}{n}\right)^{-(\alpha+\frac{1}{2})} \\ &\leq C_3 \left(\sqrt{1-x} + \frac{1}{n}\right)^{-(\alpha+\frac{1}{2})}. \end{aligned}$$

If  $\frac{c}{n} \leq \arccos x \leq \frac{\pi}{2}$ , then

$$\begin{aligned} |p_n^{(\alpha,\beta)}(x)| &\stackrel{(2.3)}{\leq} C_1 (\arccos x)^{-(\alpha+\frac{1}{2})} = C_4 (\arccos x + \underbrace{\arccos x}_{\geq \frac{c}{n}})^{-(\alpha+\frac{1}{2})} \\ &\stackrel{(2.5)}{\leq} C_5 \left(\sqrt{1-x} + \frac{1}{n}\right)^{-(\alpha+\frac{1}{2})}. \end{aligned}$$

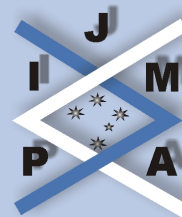
With both previous cases we have proved

$$|p_n^{(\alpha,\beta)}(x)| \leq C_6(\alpha, \beta) \left(\sqrt{1-x} + \frac{1}{n}\right)^{-(\alpha+\frac{1}{2})} \cdot \left(\sqrt{1+x} + \frac{1}{n}\right)^{-(\beta+\frac{1}{2})}$$

for all  $x \in [0, 1]$ ,  $n \in \mathbb{N}$  and  $\alpha, \beta > -1$ . Since  $p_n^{(\alpha,\beta)}(x) = (-1)^n p_n^{(\beta,\alpha)}(-x)$ , we obtain

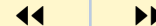
$$|p_n^{(\alpha,\beta)}(x)| \leq C_6(\beta, \alpha) \left(\sqrt{1+x} + \frac{1}{n}\right)^{-(\beta+\frac{1}{2})} \cdot \left(\sqrt{1-x} + \frac{1}{n}\right)^{-(\alpha+\frac{1}{2})}$$

for all  $x \in [-1, 0)$ ,  $n \in \mathbb{N}$  and  $\alpha, \beta > -1$ . This furnishes the validity of (2.1). ■



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Estimate (2.1) of Theorem 2.1 cannot hold true for  $n = 0$  since the modified weight  $w_n$  is not defined for  $n = 0$ . However, if  $n = 0$ , then

$$(2.6) \quad \left| p_0^{(\alpha, \beta)}(x) \right| \leq C(\alpha, \beta) \frac{1}{w_1^{(\frac{\alpha}{2} + \frac{1}{4}, \frac{\beta}{2} + \frac{1}{4})}(x)},$$

since  $p_0^{(\alpha, \beta)}(x)$  is a constant and  $C_1(\alpha, \beta) \leq w_1^{(\frac{\alpha}{2} + \frac{1}{4}, \frac{\beta}{2} + \frac{1}{4})}(x) \leq C_2(\alpha, \beta)$  with positive constants  $C_1(\alpha, \beta)$  and  $C_2(\alpha, \beta)$ .

Next, we will see that the local estimate of Theorem 2.1 can be further extended. We will show that  $\left| p_n^{(\alpha, \beta)}(x) \right|$  in (2.1) can be replaced by  $\left| p_n^{(\alpha, \beta)}(t) \right|$ , whenever  $t$  is not too far away from  $x$ , namely if  $t$  is in the interval  $U_n(x) = \left[ x - \frac{\varphi_n(x)}{n}, x + \frac{\varphi_n(x)}{n} \right] \cap [-1, 1]$ . However, for this estimate we will need the assumption  $\alpha, \beta \geq -\frac{1}{2}$ . The result is stated in the following

**Theorem 2.2.** *Let  $\alpha, \beta \geq -\frac{1}{2}$  and  $n \in \mathbb{N}$ . Then*

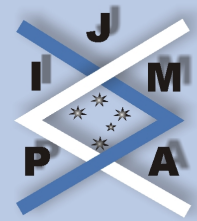
$$(2.7) \quad \left| p_n^{(\alpha, \beta)}(t) \right| \leq C \frac{1}{w_n^{(\frac{\alpha}{2} + \frac{1}{4}, \frac{\beta}{2} + \frac{1}{4})}(x)}$$

for all  $t \in U_n(x)$  and each  $x \in [-1, 1]$ , where the interval  $U_n(x)$  has been given in (1.2) and  $C = C(\alpha, \beta)$  is a positive constant independent of  $n$ ,  $t$  and  $x$ .

It must be mentioned that Theorem 2.2 cannot be extended to hold true even for all  $\alpha, \beta > -1$ . This is due to the fact that  $1/w_n^{(\frac{\alpha}{2} + \frac{1}{4}, \frac{\beta}{2} + \frac{1}{4})}(x) \rightarrow 0$  as  $n \rightarrow \infty$ , if  $x$  is a boundary point  $x = 1$  or  $x = -1$  and  $\frac{\alpha}{2} + \frac{1}{4} < 0$  or  $\frac{\beta}{2} + \frac{1}{4} < 0$  respectively.

First, we need an auxiliary lemma.





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**Lemma 2.3.** Let  $a, b \leq 0$ ,  $n \in \mathbb{N}$  and  $x \in [-1, 1]$ . Then

$$(2.8) \quad w_n^{(a,b)}(t) \leq 16^{-(a+b)} w_n^{(a,b)}(x)$$

for all  $t \in U_n(x)$ .

*Proof.* First, let  $a \leq 0$ . We will prove that

$$(2.9) \quad 16^a \left( \sqrt{1-t} + \frac{1}{n} \right)^{2a} \leq \left( \sqrt{1-x} + \frac{1}{n} \right)^{2a}$$

holds true for all  $t \in U_n(x)$  with  $x \in [-1, 1]$  and  $n \in \mathbb{N}$ . There is nothing to prove for  $a = 0$ . Let  $a < 0$ . Then inequality (2.9) is equivalent to

$$4 \left( \sqrt{1-t} + \frac{1}{n} \right) \geq \sqrt{1-x} + \frac{1}{n}$$

and

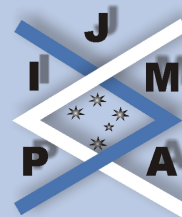
$$(2.10) \quad 4\sqrt{1-t} \geq \sqrt{1-x} - \frac{3}{n}$$

respectively. In order to prove (2.10) for  $t \in U_n(x)$  we will discuss below the cases  $x \in [1 - \frac{9}{n^2}, 1]$  and  $x \in [-1, 1 - \frac{9}{n^2}]$  separately. We must note that the latter interval is empty for  $n = 1, 2, 3$ .

**Case  $x \in [1 - \frac{9}{n^2}, 1]$ :** In this case we obtain  $\sqrt{1-x} - \frac{3}{n} \leq \frac{3}{n} - \frac{3}{n} = 0$ , which immediately gives (2.10).

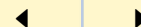
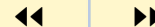
**Case  $x \in [-1, 1 - \frac{9}{n^2}]$ :** In this case we obtain  $\sqrt{1-x} - \frac{3}{n} > 0$ . Therefore inequality (2.10) is equivalent to (squaring both sides of (2.10))

$$16(1-t) \geq 1-x - \frac{6}{n}\sqrt{1-x} + \frac{9}{n^2}$$



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or, rewritten,

$$(2.11) \quad 15 + x + \frac{6}{n}\sqrt{1-x} - \frac{9}{n^2} \geq 16t.$$

Since  $t \in U_n(x) \subset \left[ x - \frac{\varphi_n(x)}{n}, x + \frac{\varphi_n(x)}{n} \right]$ , we obtain

$$\begin{aligned} x + \frac{6}{n}\sqrt{1-x} - \frac{9}{n^2} &= \left( x + \frac{2}{n}\sqrt{1-x} + \frac{1}{n^2} \right) + \left( \frac{4}{n}\sqrt{1-x} - \frac{10}{n^2} \right) \\ &\geq x + \frac{\varphi_n(x)}{n} + \frac{4}{n}\sqrt{1-x} - \frac{10}{n^2} \\ &\geq t + \frac{4}{n}\sqrt{1-x} - \frac{10}{n^2}. \end{aligned}$$

Hence, inequality (2.11) holds true if

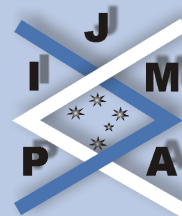
$$15 + \frac{4}{n} \underbrace{\sqrt{1-x}}_{\geq \frac{3}{n}} - \frac{10}{n^2} \geq 15t$$

or if

$$(2.12) \quad 15 + \frac{2}{n^2} \geq 15t.$$

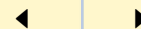
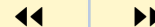
Since  $t \leq 1$ , inequality (2.12) is fulfilled. Hence inequality (2.10) is also proved. This completes the proof of (2.9) for all  $x \in [-1, 1]$  and  $t \in U_n(x)$ .

Now, let  $b \leq 0$ ,  $x \in [-1, 1]$  and  $t \in U_n(x)$ . Then  $-t \in U_n(-x)$ . From (2.9) we



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obtain

$$\begin{aligned} 16^b \left( \sqrt{1+t} + \frac{1}{n} \right)^{2b} &= 16^b \left( \sqrt{1-(-t)} + \frac{1}{n} \right)^{2b} \\ &\stackrel{(2.9)}{\leq} \left( \sqrt{1-(-x)} + \frac{1}{n} \right)^{2b} = \left( \sqrt{1+x} + \frac{1}{n} \right)^{2b}, \end{aligned}$$

which proves the validity of (2.8). ■

*Proof of Theorem 2.2.* Since  $\alpha, \beta \geq -\frac{1}{2}$ , it follows that  $\frac{\alpha}{2} + \frac{1}{4}, \frac{\beta}{2} + \frac{1}{4} \geq 0$ . Therefore we can apply Lemma 2.3 with  $a = -\frac{\alpha}{2} - \frac{1}{4}$  and  $b = -\frac{\beta}{2} - \frac{1}{4}$ , obtaining

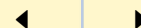
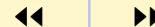
$$\frac{1}{w_n^{(\frac{\alpha}{2} + \frac{1}{4}, \frac{\beta}{2} + \frac{1}{4})}(t)}} = w_n^{(-\frac{\alpha}{2} - \frac{1}{4}, -\frac{\beta}{2} - \frac{1}{4})}(t) \stackrel{\text{Lem. 2.3}}{\leq} \frac{4^{\alpha+\beta+1}}{w_n^{(\frac{\alpha}{2} + \frac{1}{4}, \frac{\beta}{2} + \frac{1}{4})}(x)}}$$

for all  $t \in U_n(x)$ . Application of Theorem 2.1 therefore yields inequality (2.2) for all  $t \in U_n(x)$  as claimed. ■



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### 3. Applications

In this section we will give some applications of the local estimates of the Jacobi polynomials.

We apply Theorem 2.2 and obtain

$$\int_{U_n(x)} |p_n^{(\alpha,\beta)}(t)|^2 w^{(\alpha,\beta)}(t) dt \leq C \frac{1}{w_n^{(\alpha+\frac{1}{2},\beta+\frac{1}{2})}(x)} \int_{U_n(x)} w^{(\alpha,\beta)}(t) dt.$$

Using

$$\int_{U_n(x)} w^{(\alpha,\beta)}(t) dt \leq C \frac{1}{n} w_n^{(\alpha+\frac{1}{2},\beta+\frac{1}{2})}(x)$$

(see [2]) we find that

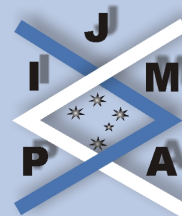
$$(3.1) \quad \int_{U_n(x)} |p_n^{(\alpha,\beta)}(t)|^2 w^{(\alpha,\beta)}(t) dt \leq C(\alpha, \beta) \frac{1}{n}, \quad x \in [-1, 1],$$

is valid for all  $n \in \mathbb{N}$  with  $\alpha, \beta \geq -\frac{1}{2}$ . Estimate (3.1) shows that the intervals  $U_n(x)$  are appropriate for measuring the growth of the orthonormal polynomials on subintervals of  $[-1, 1]$ :  $U_n(x)$  is located around  $x$ ,  $|U_n(x)| = O(1/n)$ , the radius  $\frac{\varphi_n(x)}{n}$  varies together with  $x$  and becomes smaller if  $x$  tends to 1 or  $-1$  and the weighted integration of  $(p_n^{(\alpha,\beta)}(t))^2$  on  $U_n(x)$  is  $O(1/n)$ , whereas the weighted integral on  $[-1, 1]$  equals 1, i.e.,

$$\int_{-1}^1 |p_n^{(\alpha,\beta)}(t)|^2 w^{(\alpha,\beta)}(t) dt = 1, \quad x \in [-1, 1].$$

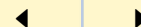
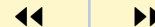
Let  $a, b > -\frac{1}{2}$  and  $C_1, C_2 > 0$ . Let  $m: [1, \infty) \rightarrow \mathbb{R}$  be a differentiable function fulfilling the Hormander conditions

$$0 \leq m(t) \leq C_1 \quad \text{and} \quad |m'(t)| \leq C_2 t^{-1}$$



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for  $t \geq 1$ . It was proved in [1] that

$$(3.2) \quad \sum_{k=1}^n \frac{m(k)}{w_k^{(a,b)}(x)} \leq C \frac{n}{w_n^{(a,b)}(x)}$$

for all  $x \in [-1, 1]$  and  $n \in \mathbb{N}$  with a positive constant  $C = C(a, b, C_1, C_2)$  being independent of  $n$  and  $x$ .

Let  $\alpha, \beta \geq -\frac{1}{2}$ . Now, we will apply Theorem 2.2 and the above estimate (3.2) with  $a = \alpha + \frac{1}{2} \geq 0$  and  $b = \beta + \frac{1}{2} \geq 0$ , to obtain

$$(3.3) \quad \sum_{k=1}^n m(k) (p_k^{(\alpha,\beta)}(t))^2 \stackrel{\text{Theorem 2.2}}{\underset{(3.2)}{\leq}} C \frac{n}{w_n^{(\alpha+\frac{1}{2},\beta+\frac{1}{2})}(x)}$$

for all  $t \in U_n(x)$  and each  $x \in [-1, 1]$  with a constant  $C = C(\alpha, \beta, C_1, C_2) > 0$  being independent of  $n$  and  $x$ .

In particular, if we let  $m(k) = 1$ , then estimate (3.3) shows that the Christoffel function, defined by

$$\lambda_n^{(\alpha,\beta)}(t) := \left\{ \sum_{k=1}^n (p_k^{(\alpha,\beta)}(t))^2 \right\}^{-1},$$

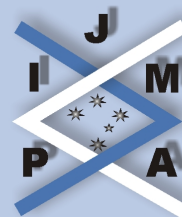
fulfills the estimate

$$(\lambda_n^{(\alpha,\beta)}(t))^{-1} \leq C(\alpha, \beta) \frac{n}{w_n^{(\alpha+\frac{1}{2},\beta+\frac{1}{2})}(x)}$$

for  $t \in U_n(x)$  and  $x \in [-1, 1]$  and  $n \in \mathbb{N}$ .

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Polynomials

Michael Felten

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