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AN ALGEBRAIC INEQUALITY

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ABSTRACT. In this short note, an algebraic inequality related to those of Alzer, Minc and Sathre is proved by using analytic arguments and Cauchy's mean-value theorem. An open problem is proposed.

Key words and phrases: Algebraic Inequality, Cauchy's Mean-Value Theorem, Alzer's Inequality.

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1. AN ALGEBRAIC INEQUALITY

In this note, we prove the following algebraic inequality

Theorem 1.1. *Let $b > a > 0$ and $\delta > 0$ be real numbers. Then for any given positive $r \in \mathbb{R}$, we have*

$$\left(\frac{b+\delta-a}{b-a} \cdot \frac{b^{r+1}-a^{r+1}}{(b+\delta)^{r+1}-a^{r+1}}\right)^{1/r} > \frac{b}{b+\delta}. \tag{1.1}$$

The lower bound in (1.1) *is best possible.*

Proof. The inequality (1.1) is equivalent to

$$\frac{b^{r+1}-a^{r+1}}{b-a} \Big/ \frac{(b+\delta)^{r+1}-a^{r+1}}{b+\delta-a} > \left(\frac{b}{b+\delta}\right)^r,$$

that is,

$$\frac{b^{r+1}-a^{r+1}}{b^r(b-a)} > \frac{(b+\delta)^{r+1}-a^{r+1}}{(b+\delta)^r(b+\delta-a)}. \tag{1.2}$$

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Therefore, it is sufficient to prove that the function $(s^{r+1}-a^{r+1})/s^r(s-a)$ is decreasing for $s>a$. By direct computation, we have

$$\left(\frac{s^{r+1}-a^{r+1}}{s^r(s-a)}\right)_s' = \frac{(r+1)(s-a)s^{2r}-s^{r-1}(s^{r+1}-a^{r+1})[(r+1)s-ra]}{[s^r(s-a)]^2}.$$

So, it suffices to prove

$$(r+1)(s-a)s^{r+1}-[(r+1)s-ra](s^{r+1}-a^{r+1})\leqslant 0. \tag{1.3}$$

A straightforward calculation shows that the inequality (1.3) reduces to

$$\frac{s^r-a^r}{r(s-a)} > \frac{a^r}{s}. \tag{1.4}$$

From Cauchy's mean-value theorem, there exists a point $\xi\in(a,s)$ such that

$$\frac{s^r-a^r}{r(s-a)} = \xi^{r-1} = \frac{\xi^r}{\xi} > \frac{a^r}{\xi} > \frac{a^r}{s}.$$

Hence, the inequality (1.4) holds.

The L'Hospital rule yields

$$\lim_{r\to+\infty}\left(\frac{b+\delta-a}{b-a}\cdot\frac{b^{r+1}-a^{r+1}}{(b+\delta)^{r+1}-a^{r+1}}\right)^{1/r} = \frac{b}{b+\delta}, \tag{1.5}$$

so the lower bound in (1.1) is best possible. The proof is complete. □

Remark 1.2. The inequality (1.1) can be rewritten as

$$\frac{b}{b+\delta} < \left(\frac{1}{b-a}\int_a^b x^r dx\Big/\frac{1}{b+\delta-a}\int_a^{b+\delta} x^r dx\right)^{1/r}. \tag{1.6}$$

It is easy to see that inequality (1.6) is indeed an integral analogue of the following inequality

$$\frac{n+k}{n+m+k} < \left(\frac{1}{n}\sum_{i=k+1}^{n+k} i^r\Big/\frac{1}{n+m}\sum_{i=k+1}^{n+m+k} i^r\right)^{1/r}, \tag{1.7}$$

where r is a given positive real number, n and m are natural numbers, and k is a nonnegative integer. The lower bound in (1.7) is best possible.

The inequality (1.7) was presented in [5] by the author using Cauchy's mean-value theorem and mathematical induction. It generalizes the inequality of Alzer in [1].

Using the same method as in [5], the author in [9] further generalized the inequality of Alzer and obtained that, if $a=(a_1,a_2,\dots)$ is a positive and increasing sequence satisfying

$$a_{k+1}^2 \geqslant a_k a_{k+2}, \tag{1.8}$$

$$\frac{a_{k+1}-a_k}{a_{k+1}^2-a_k a_{k+2}} \geqslant \max\left\{\frac{k+1}{a_{k+1}}, \frac{k+2}{a_{k+2}}\right\} \tag{1.9}$$

for $k\in\mathbb{N}$, then we have

$$\frac{a_n}{a_{n+m}} < \left(\frac{1}{n}\sum_{i=1}^{n} a_i^r\Big/\frac{1}{n+m}\sum_{i=1}^{n+m} a_i^r\right)^{1/r}, \tag{1.10}$$

where n and m are natural numbers. The lower bound in (1.10) is best possible.

Recently, some new inequalities related to those of Alzer, Minc and Sathre were obtained by many mathematician. These inequalities involve ratios for the sum of powers of positive numbers (see [2, 12]) and for the geometric mean of natural numbers (see [4, 6, 7, 10, 11]). Many of them can be deduced from monotonicity and convexity considerations (see [8]). Moreover, inequality (1.1) has been generalised to an inequality for linear positive functionals in [3].

Here L'Hospital's rule yields

$$\lim_{r\to 0^+}\left(\frac{b+\delta-a}{b-a}\cdot\frac{b^{r+1}-a^{r+1}}{(b+\delta)^{r+1}-a^{r+1}}\right)^{1/r}=\frac{[b^b/a^a]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^a]^{1/(b+\delta-a)}}. \tag{1.11}$$

Hence, we propose the following

Open Problem. *Let $b>a>0$ and $\delta>0$ be real numbers. Then for any positive $r\in\mathbb{R}$, we have*

$$\left(\frac{b+\delta-a}{b-a}\cdot\frac{b^{r+1}-a^{r+1}}{(b+\delta)^{r+1}-a^{r+1}}\right)^{1/r}<\frac{[b^b/a^a]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^a]^{1/(b+\delta-a)}}. \tag{1.12}$$

The upper bound in (1.12) *is best possible.*

Remark 1.3. The inequalities in this paper are related to the study of monotonicity of the ratios and differences of mean values.

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