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FURTHER REVERSE RESULTS FOR JENSEN'S DISCRETE INEQUALITY AND APPLICATIONS IN INFORMATION THEORY

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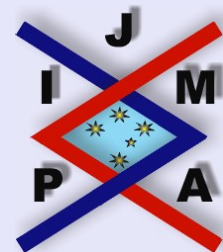
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Abstract

Contents

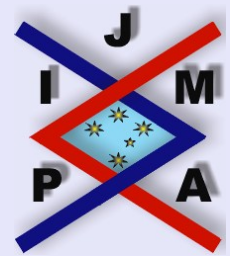


Home Page

Go Back

Close

Quit



Abstract

Some new inequalities which counterpart Jensen's discrete inequality and improve the recent results from [4] and [5] are given. A related result for generalized means is established. Applications in Information Theory are also provided.

2000 Mathematics Subject Classification: 26D15, 94Xxx.

Key words: Convex functions, Jensen's Inequality, Entropy Mappings.

Contents

1	Introduction	3
2	Some New Counterparts for Jensen's Discrete Inequality ...	5
3	A Converse Inequality for Convex Mappings Defined on \mathbb{R}^n	12
4	Some Related Results	18
5	Applications in Information Theory	23
References		

Further Reverse Results for Jensen's Discrete Inequality and Applications in Information Theory

I. Budimir, S.S. Dragomir and
J. Pečarić

Title Page

Contents



Go Back

Close

Quit

Page 2 of 31

1. Introduction

Let $f : X \rightarrow \mathbb{R}$ be a convex mapping defined on the linear space X and $x_i \in X$, $p_i \geq 0$ ($i = 1, \dots, m$) with $P_m := \sum_{i=1}^m p_i > 0$.

The following inequality is well known in the literature as Jensen's inequality

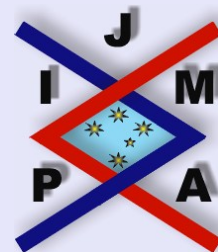
$$(1.1) \quad f\left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i\right) \leq \frac{1}{P_m} \sum_{i=1}^m p_i f(x_i).$$

There are many well known inequalities which are particular cases of Jensen's inequality, such as the weighted arithmetic mean-geometric mean-harmonic mean inequality, the Ky-Fan inequality, the Hölder inequality, etc. For a comprehensive list of recent results on Jensen's inequality, see the book [25] and the papers [9]-[15] where further references are given.

In 1994, Dragomir and Ionescu [18] proved the following inequality which counterparts (1.1) for real mappings of a real variable.

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable convex mapping on $\overset{\circ}{I}$ ($\overset{\circ}{I}$ is the interior of I), $x_i \in \overset{\circ}{I}$, $p_i \geq 0$ ($i = 1, \dots, n$) and $\sum_{i=1}^n p_i = 1$. Then we have the inequality*

$$(1.2) \quad \begin{aligned} 0 &\leq \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right) \\ &\leq \sum_{i=1}^n p_i x_i f'(x_i) - \sum_{i=1}^n p_i x_i \sum_{i=1}^n p_i f'(x_i), \end{aligned}$$



Further Reverse Results for
Jensen's Discrete Inequality
and Applications in Information
Theory

I. Budimir, S.S. Dragomir and
J. Pečarić

Title Page

Contents



Go Back

Close

Quit

Page 3 of 31

where f' is the derivative of f on $\overset{\circ}{I}$.

Using this result and the discrete version of the Grüss inequality for weighted sums, S.S. Dragomir obtained the following simple counterpart of Jensen's inequality [5]:

Theorem 1.2. *With the above assumptions for f and if $m, M \in \overset{\circ}{I}$ and $m \leq x_i \leq M$ ($i = 1, \dots, n$), then we have*

$$(1.3) \quad 0 \leq \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right) \leq \frac{1}{4} (M - m) (f'(M) - f'(m)).$$

This was subsequently applied in Information Theory for Shannon's and Rényi's entropy.

In this paper we point out some other counterparts of Jensen's inequality that are similar to (1.3), some of which are better than the above inequalities.



**Further Reverse Results for
Jensen's Discrete Inequality
and Applications in Information
Theory**

I. Budimir, S.S. Dragomir and
J. Pečarić

Title Page

Contents



Go Back

Close

Quit

Page 4 of 31

2. Some New Counterparts for Jensen's Discrete Inequality

The following result holds.

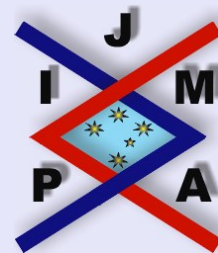
Theorem 2.1. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable convex mapping on $\overset{\circ}{I}$ and $x_i \in \overset{\circ}{I}$ with $x_1 \leq x_2 \leq \dots \leq x_n$ and $p_i \geq 0$ ($i = 1, \dots, n$) with $\sum_{i=1}^n p_i = 1$. Then we have

$$\begin{aligned}
 (2.1) \quad 0 &\leq \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right) \\
 &\leq (x_n - x_1) (f'(x_n) - f'(x_1)) \max_{1 \leq k \leq n-1} \{P_k \bar{P}_{k+1}\} \\
 &\leq \frac{1}{4} (x_n - x_1) (f'(x_n) - f'(x_1)),
 \end{aligned}$$

where $P_k := \sum_{i=1}^k p_i$ and $\bar{P}_{k+1} := 1 - P_k$.

Proof. We use the following Grüss type inequality due to J. E. Pečarić (see for example [25]):

$$\begin{aligned}
 (2.2) \quad \left| \frac{1}{Q_n} \sum_{i=1}^n q_i a_i b_i - \frac{1}{Q_n} \sum_{i=1}^n q_i a_i \cdot \frac{1}{Q_n} \sum_{i=1}^n q_i b_i \right| \\
 \leq |a_n - a_1| |b_n - b_1| \max_{1 \leq k \leq n-1} \left[\frac{Q_k \bar{Q}_{k+1}}{Q_n^2} \right],
 \end{aligned}$$



Further Reverse Results for
Jensen's Discrete Inequality
and Applications in Information
Theory

I. Budimir, S.S. Dragomir and
J. Pečarić

Title Page

Contents



Go Back

Close

Quit

Page 5 of 31

provided that a, b are two monotonic n -tuples, q is a positive one, $Q_n := \sum_{i=1}^n q_i > 0$, $Q_k := \sum_{i=1}^k q_i$ and $\bar{Q}_{k+1} = Q_n - Q_{k+1}$.

If in (2.2) we choose $q_i = p_i$, $a_i = x_i$, $b_i = f'(x_i)$ (and a_i, b_i will be monotonic nondecreasing), then we may state that

$$(2.3) \quad \sum_{i=1}^n p_i x_i f'(x_i) - \sum_{i=1}^n p_i x_i \sum_{i=1}^n p_i f'(x_i) \leq (x_n - x_1) (f'(x_n) - f'(x_1)) \max_{1 \leq k \leq n-1} \{P_k \bar{P}_{k+1}\}.$$

Now, using (1.2) and (2.3) we obtain the first inequality in (2.1).

For the second inequality, we observe that

$$P_k \bar{P}_{k+1} = P_k (1 - P_k) \leq \frac{1}{4} (P_k + 1 - P_k)^2 = \frac{1}{4}$$

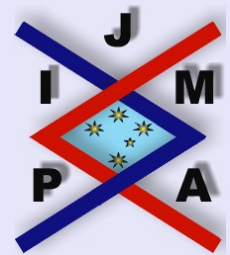
for all $k \in \{1, \dots, n-1\}$ and then

$$\max_{1 \leq k \leq n-1} \{P_k \bar{P}_{k+1}\} \leq \frac{1}{4},$$

which proves the last part of (2.1). □

Remark 2.1. It is obvious that the inequality (2.1) is an improvement of (1.3) if we assume that the order for x_i is as in the statement of Theorem 2.1.

Another result is embodied in the following theorem.



Further Reverse Results for Jensen's Discrete Inequality and Applications in Information Theory

I. Budimir, S.S. Dragomir and J. Pečarić

Title Page

Contents

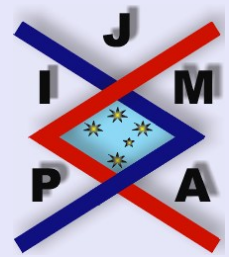


Go Back

Close

Quit

Page 6 of 31



Further Reverse Results for Jensen's Discrete Inequality and Applications in Information Theory

I. Budimir, S.S. Dragomir and J. Pečarić

Title Page

Contents



Go Back

Close

Quit

Page 7 of 31

Theorem 2.2. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable convex mapping on $\overset{\circ}{I}$ and $m, M \in \overset{\circ}{I}$ with $m \leq x_i \leq M$ ($i = 1, \dots, n$) and $p_i \geq 0$ ($i = 1, \dots, n$) with $\sum_{i=1}^n p_i = 1$. If S is a subset of the set $\{1, \dots, n\}$ minimizing the expression

$$(2.4) \quad \left| \sum_{i \in S} p_i - \frac{1}{2} \right|,$$

then we have the inequality

$$(2.5) \quad \begin{aligned} 0 &\leq \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right) \\ &\leq Q(M-m)(f'(M) - f'(m)) \\ &\leq \frac{1}{4}(M-m)(f'(M) - f'(m)), \end{aligned}$$

where

$$Q = \sum_{i \in S} p_i \left(1 - \sum_{i \in S} p_i\right).$$

Proof. We use the following Grüss type inequality due the Andrica and Badea [2]:

$$(2.6) \quad \left| Q_n \sum_{i=1}^n q_i a_i b_i - \sum_{i=1}^n q_i a_i \cdot \sum_{i=1}^n q_i b_i \right| \leq (M_1 - m_1)(M_2 - m_2) \sum_{i \in S} q_i \left(Q_n - \sum_{i \in S} q_i \right)$$

provided that $m_1 \leq a_i \leq M_1$, $m_2 \leq b_i \leq M_2$ for $i = 1, \dots, n$, and S is the subset of $\{1, \dots, n\}$ which minimises the expression

$$\left| \sum_{i \in S} q_i - \frac{1}{2} Q_n \right|.$$

Choosing $q_i = p_i$, $a_i = x_i$, $b_i = f'(x_i)$, then we may state that

$$(2.7) \quad 0 \leq \sum_{i=1}^n p_i x_i f'(x_i) - \sum_{i=1}^n p_i x_i \sum_{i=1}^n p_i f'(x_i) \\ \leq (M - m) (f'(M) - f'(m)) \sum_{i \in S} p_i \left(1 - \sum_{i \in S} p_i \right).$$

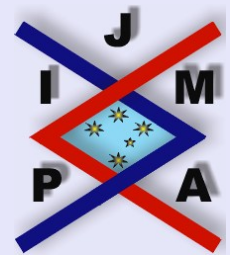
Now, using (1.2) and (2.7), we obtain the first inequality in (2.5). For the last part, we observe that

$$Q \leq \frac{1}{4} \left(\sum_{i \in S} p_i + 1 - \sum_{i \in S} p_i \right)^2 = \frac{1}{4}$$

and the theorem is thus proved. □

The following inequality is well known in the literature as the arithmetic mean-geometric mean-harmonic-mean inequality:

$$(2.8) \quad A_n(p, x) \geq G_n(p, x) \geq H_n(p, x),$$



Further Reverse Results for Jensen's Discrete Inequality and Applications in Information Theory

I. Budimir, S.S. Dragomir and J. Pečarić

Title Page

Contents



Go Back

Close

Quit

Page 8 of 31

where

$$A_n(p, x) : = \sum_{i=1}^n p_i x_i - \text{the arithmetic mean,}$$

$$G_n(p, x) : = \prod_{i=1}^n x_i^{p_i} - \text{the geometric mean,}$$

$$H_n(p, x) : = \frac{1}{\sum_{i=1}^n \frac{p_i}{x_i}} - \text{the harmonic mean,}$$

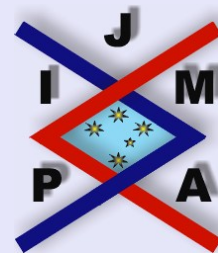
and $\sum_{i=1}^n p_i = 1$ ($p_i \geq 0, i = \overline{1, n}$).

Using the above two theorems, we are able to point out the following reverse of the AGH - inequality.

Proposition 2.3. *Let $x_i > 0$ ($i = 1, \dots, n$) and $p_i \geq 0$ with $\sum_{i=1}^n p_i = 1$.*

(i) *If $x_1 \leq x_2 \leq \dots \leq x_{n-1} \leq x_n$, then we have*

$$\begin{aligned} (2.9) \quad 1 &\leq \frac{A_n(p, x)}{G_n(p, x)} \\ &\leq \exp \left[\frac{(x_n - x_1)^2}{x_1 x_n} \max_{1 \leq k \leq n-1} \{P_k \bar{P}_{k+1}\} \right] \\ &\leq \exp \left[\frac{1}{4} \cdot \frac{(x_n - x_1)^2}{x_1 x_n} \right]. \end{aligned}$$



Further Reverse Results for
Jensen's Discrete Inequality
and Applications in Information
Theory

I. Budimir, S.S. Dragomir and
J. Pečarić

Title Page

Contents

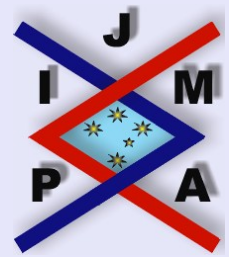


Go Back

Close

Quit

Page 9 of 31



Further Reverse Results for Jensen's Discrete Inequality and Applications in Information Theory

I. Budimir, S.S. Dragomir and J. Pečarić

Title Page

Contents



Go Back

Close

Quit

Page 10 of 31

(ii) If the set $S \subseteq \{1, \dots, n\}$ minimizes the expression (2.4), and $0 < m \leq x_i \leq M < \infty$ ($i = 1, \dots, n$), then

$$(2.10) \quad 1 \leq \frac{A_n(p, x)}{G_n(p, x)} \leq \exp \left[Q \cdot \frac{(M - m)^2}{mM} \right] \leq \exp \left[\frac{1}{4} \cdot \frac{(M - m)^2}{mM} \right].$$

The proof goes by the inequalities (2.1) and (2.5), choosing $f(x) = -\ln x$. A similar result can be stated for G_n and H_n .

Proposition 2.4. Let $p \geq 1$ and $x_i > 0$, $p_i \geq 0$ ($i = 1, \dots, n$) with $\sum_{i=1}^n p_i = 1$.

(i) If $x_1 \leq x_2 \leq \dots \leq x_{n-1} \leq x_n$, then we have

$$(2.11) \quad 0 \leq \sum_{i=1}^n p_i x_i^p - \left(\sum_{i=1}^n p_i x_i \right)^p \leq p(x_n - x_1) (x_n^{p-1} - x_1^{p-1}) \max_{1 \leq k \leq n-1} \{P_k \bar{P}_{k+1}\} \leq \frac{p}{4} (x_n - x_1) (x_n^{p-1} - x_1^{p-1}).$$

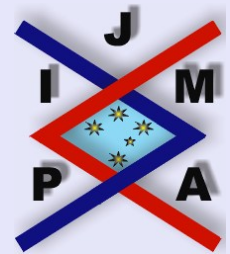
(ii) If the set $S \subseteq \{1, \dots, n\}$ minimizes the expression (2.4), and $0 < m \leq x_i \leq M < \infty$ ($i = 1, \dots, n$), then

$$(2.12) \quad 0 \leq \sum_{i=1}^n p_i x_i^p - \left(\sum_{i=1}^n p_i x_i \right)^p$$

$$\begin{aligned} &\leq pQ(M - m)(M^{p-1} - m^{p-1}) \\ &\leq \frac{1}{4}p(M - m)(M^{p-1} - m^{p-1}). \end{aligned}$$

Remark 2.2. The above results are improvements of the corresponding inequalities obtained in [5].

Remark 2.3. Similar inequalities can be stated if we choose other convex functions such as: $f(x) = x \ln x$, $x > 0$ or $f(x) = \exp(x)$, $x \in \mathbb{R}$. We omit the details.



Further Reverse Results for Jensen's Discrete Inequality and Applications in Information Theory

I. Budimir, S.S. Dragomir and J. Pečarić

Title Page

Contents



Go Back

Close

Quit

Page 11 of 31

3. A Converse Inequality for Convex Mappings Defined on \mathbb{R}^n

In 1996, Dragomir and Goh [15] proved the following converse of Jensen's inequality for convex mappings on \mathbb{R}^n .

Theorem 3.1. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a differentiable convex mapping on \mathbb{R}^n and*

$$(\nabla f)(x) := \left(\frac{\partial f(x)}{\partial x^1}, \dots, \frac{\partial f(x)}{\partial x^n} \right),$$

the vector of the partial derivatives, $x = (x^1, \dots, x^n) \in \mathbb{R}^n$.

If $x_i \in \mathbb{R}^m$ ($i = 1, \dots, m$), $p_i \geq 0$, $i = 1, \dots, m$, with $P_m := \sum_{i=1}^m p_i > 0$, then

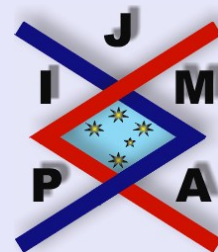
$$(3.1) \quad 0 \leq \frac{1}{P_m} \sum_{i=1}^m p_i f(x_i) - f \left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i \right) \\ \leq \frac{1}{P_m} \sum_{i=1}^m p_i \langle \nabla f(x_i), x_i \rangle - \left\langle \frac{1}{P_m} \sum_{i=1}^m p_i \nabla f(x_i), \frac{1}{P_m} \sum_{i=1}^m p_i x_i \right\rangle.$$

The result was applied to different problems in Information Theory by providing different counterpart inequalities for Shannon's entropy, conditional entropy, mutual information, conditional mutual information, etc.

For generalizations of (3.1) in Normed Spaces and other applications in Information Theory, see Matic's Ph.D dissertation [23].

Recently, Dragomir [4] provided an upper bound for Jensen's difference

$$(3.2) \quad \Delta(f, p, x) := \frac{1}{P_m} \sum_{i=1}^m p_i f(x_i) - f \left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i \right),$$



Further Reverse Results for
Jensen's Discrete Inequality
and Applications in Information
Theory

I. Budimir, S.S. Dragomir and
J. Pečarić

Title Page

Contents



Go Back

Close

Quit

Page 12 of 31

which, even though it is not as sharp as (3.1), provides a simpler way, and for applications, a better way, of estimating the Jensen's differences Δ . His result is embodied in the following theorem.

Theorem 3.2. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a differentiable convex mapping and $x_i \in \mathbb{R}^n$, $i = 1, \dots, m$. Suppose that there exists the vectors $\phi, \Phi \in \mathbb{R}^n$ such that*

$$(3.3) \quad \phi \leq x_i \leq \Phi \quad (\text{the order is considered on the co-ordinates})$$

and $m, M \in \mathbb{R}^n$ are such that

$$(3.4) \quad m \leq \nabla f(x_i) \leq M$$

for all $i \in \{1, \dots, m\}$. Then for all $p_i \geq 0$ ($i = 1, \dots, m$) with $P_m > 0$, we have the inequality

$$(3.5) \quad 0 \leq \frac{1}{P_m} \sum_{i=1}^m p_i f(x_i) - f\left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i\right) \leq \frac{1}{4} \|\Phi - \phi\| \|M - m\|,$$

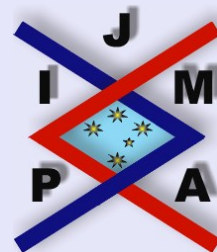
where $\|\cdot\|$ is the usual Euclidean norm on \mathbb{R}^n .

He applied this inequality to obtain different upper bounds for Shannon's and Rényi's entropies.

In this section, we point out another counterpart for Jensen's difference, assuming that the ∇ -operator is of Hölder's type, as follows.

Theorem 3.3. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a differentiable convex mapping and $x_i \in \mathbb{R}^n$, $p_i \geq 0$ ($i = 1, \dots, m$) with $P_m > 0$. Suppose that the ∇ -operator satisfies a condition of r -H-Hölder type, i.e.,*

$$(3.6) \quad \|\nabla f(x) - \nabla f(y)\| \leq H \|x - y\|^r, \quad \text{for all } x, y \in \mathbb{R}^n,$$



Further Reverse Results for
Jensen's Discrete Inequality
and Applications in Information
Theory

I. Budimir, S.S. Dragomir and
J. Pečarić

Title Page

Contents



Go Back

Close

Quit

Page 13 of 31

where $H > 0$, $r \in (0, 1]$ and $\|\cdot\|$ is the Euclidean norm.
Then we have the inequality:

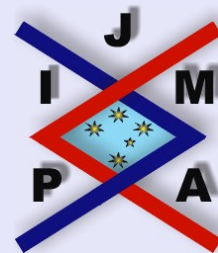
$$(3.7) \quad 0 \leq \frac{1}{P_m} \sum_{i=1}^m p_i f(x_i) - f\left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i\right) \\ \leq \frac{H}{P_m^2} \sum_{1 \leq i < j \leq m} p_i p_j \|x_i - x_j\|^{r+1}.$$

Proof. We recall Korkine's identity,

$$\frac{1}{P_m} \sum_{i=1}^m p_i \langle y_i, x_i \rangle - \left\langle \frac{1}{P_m} \sum_{i=1}^m p_i y_i, \frac{1}{P_m} \sum_{i=1}^m p_i x_i \right\rangle \\ = \frac{1}{2P_m^2} \sum_{i,j=1}^n p_i p_j \langle y_i - y_j, x_i - x_j \rangle, \quad x, y \in \mathbb{R}^n,$$

and simply write

$$\frac{1}{P_m} \sum_{i=1}^m p_i \langle \nabla f(x_i), x_i \rangle - \left\langle \frac{1}{P_m} \sum_{i=1}^m p_i \nabla f(x_i), \frac{1}{P_m} \sum_{i=1}^m p_i x_i \right\rangle \\ = \frac{1}{2P_m^2} \sum_{i,j=1}^n p_i p_j \langle \nabla f(x_i) - \nabla f(x_j), x_i - x_j \rangle.$$



**Further Reverse Results for
Jensen's Discrete Inequality
and Applications in Information
Theory**

I. Budimir, S.S. Dragomir and
J. Pečarić

Title Page

Contents



Go Back

Close

Quit

Page 14 of 31

Using (3.1) and the properties of the modulus, we have

$$\begin{aligned}
 0 &\leq \frac{1}{P_m} \sum_{i=1}^m p_i f(x_i) - f\left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i\right) \\
 &\leq \frac{1}{2P_m^2} \sum_{i,j=1}^m p_i p_j |\langle \nabla f(x_i) - \nabla f(x_j), x_i - x_j \rangle| \\
 &\leq \frac{1}{2P_m^2} \sum_{i,j=1}^m p_i p_j \|\nabla f(x_i) - \nabla f(x_j)\| \|x_i - x_j\| \\
 &\leq \frac{H}{P_m^2} \sum_{i,j=1}^m p_i p_j \|x_i - x_j\|^{r+1}
 \end{aligned}$$

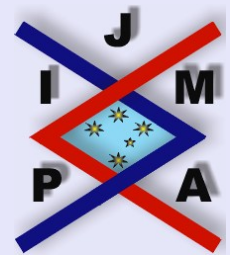
and the inequality (3.7) is proved. □

Corollary 3.4. *With the assumptions of Theorem 3.3 and if $\Delta = \max_{1 \leq i < j \leq m} \|x_i - x_j\|$, then we have the inequality*

$$(3.8) \quad 0 \leq \frac{1}{P_m} \sum_{i=1}^m p_i f(x_i) - f\left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i\right) \leq \frac{H\Delta^{r+1}}{2P_m^2} \left(1 - \sum_{i=1}^m p_i^2\right).$$

Proof. Indeed, as

$$\sum_{1 \leq i < j \leq m} p_i p_j \|x_i - x_j\|^{r+1} \leq \Delta^{r+1} \sum_{1 \leq i < j \leq m} p_i p_j.$$



Further Reverse Results for Jensen's Discrete Inequality and Applications in Information Theory

I. Budimir, S.S. Dragomir and J. Pečarić

Title Page

Contents



Go Back

Close

Quit

Page 15 of 31

However,

$$\sum_{1 \leq i < j \leq m} p_i p_j = \frac{1}{2} \left(\sum_{i,j=1}^m p_i p_j - \sum_{i=j} p_i p_j \right) = \frac{1}{2} \left(1 - \sum_{i=1}^m p_i^2 \right),$$

and the inequality (3.8) is proved. \square

The case of Lipschitzian mappings is embodied in the following corollary.

Corollary 3.5. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a differentiable convex mapping and $x_i \in \mathbb{R}^n$, $p_i \geq 0$ ($i = 1, \dots, n$) with $P_m > 0$. Suppose that the ∇ -operator is Lipschitzian with the constant $L > 0$, i.e.,*

$$(3.9) \quad \|\nabla f(x) - \nabla f(y)\| \leq L \|x - y\|, \text{ for all } x, y \in \mathbb{R}^n,$$

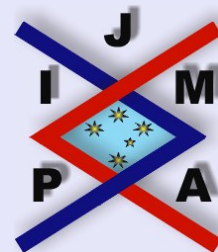
where $\|\cdot\|$ is the Euclidean norm. Then

$$(3.10) \quad 0 \leq \frac{1}{P_m} \sum_{i=1}^m p_i f(x_i) - f\left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i\right) \leq L \left[\frac{1}{P_m} \sum_{i=1}^m p_i \|x_i\|^2 - \left\| \frac{1}{P_m} \sum_{i=1}^m p_i x_i \right\|^2 \right].$$

Proof. The argument is obvious by Theorem 3.3, taking into account that for $r = 1$,

$$\sum_{1 \leq i < j \leq m} p_i p_j \|x_i - x_j\|^2 = P_m \sum_{i=1}^m p_i \|x_i\|^2 - \left\| \sum_{i=1}^m p_i x_i \right\|^2,$$

and $\|\cdot\|$ is the Euclidean norm. \square



Further Reverse Results for
Jensen's Discrete Inequality
and Applications in Information
Theory

I. Budimir, S.S. Dragomir and
J. Pečarić

Title Page

Contents



Go Back

Close

Quit

Page 16 of 31

Moreover, if we assume more about the vectors $(x_i)_{i=1, \dots, n}$, we can obtain a simpler result that is similar to the one in [4].

Corollary 3.6. Assume that f is as in Corollary 3.5. If

$$(3.11) \quad \phi \leq x_i \leq \Phi \quad (\text{on the co-ordinates}), \phi, \Phi \in \mathbb{R}^n \quad (i = 1, \dots, m),$$

then we have the inequality

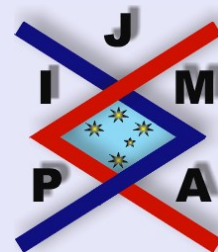
$$(3.12) \quad \begin{aligned} 0 &\leq \frac{1}{P_m} \sum_{i=1}^m p_i f(x_i) - f\left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i\right) \\ &\leq \frac{1}{4} \cdot L \cdot \|\Phi - \phi\|^2. \end{aligned}$$

Proof. It follows by the fact that in \mathbb{R}^n , we have the following Grüss type inequality (as proved in [4])

$$(3.13) \quad \frac{1}{P_m} \sum_{i=1}^m p_i \|x_i\|^2 - \left\| \frac{1}{P_m} \sum_{i=1}^m p_i x_i \right\|^2 \leq \frac{1}{4} \|\Phi - \phi\|^2,$$

provided that (3.11) holds. □

Remark 3.1. For some Grüss type inequalities in Inner Product Spaces, see [7].



Further Reverse Results for
Jensen's Discrete Inequality
and Applications in Information
Theory

I. Budimir, S.S. Dragomir and
J. Pečarić

Title Page

Contents



Go Back

Close

Quit

Page 17 of 31

4. Some Related Results

Start with the following definitions from [3].

Definition 4.1. Let $-\infty < a < b < \infty$. Then $CM[a, b]$ denotes the set of all functions with domain $[a, b]$ that are continuous and strictly monotonic there.

Definition 4.2. Let $-\infty < a < b < \infty$, and let $f \in CM[a, b]$. Then, for each positive integer n , each n -tuple $x = (x_1, \dots, x_n)$, where $a \leq x_j \leq b$ ($j = 1, 2, \dots, n$), and each n -tuple $p = (p_1, p_2, \dots, p_n)$, where $p_j > 0$ ($j = 1, 2, \dots, n$) and $\sum_{j=1}^n p_j = 1$, let $M_f(x, y)$ denote the (weighted) mean

$$f^{-1} \left\{ \sum_{j=1}^n p_j f(x_j) \right\}.$$

We may state now the following result.

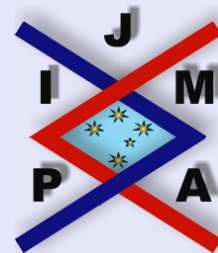
Theorem 4.1. Let S be the subset of $\{1, \dots, n\}$ which minimizes the expression $|\sum_{i \in S} p_i - 1/2|$. If $f, g \in CM[a, b]$, then

$$\sup_x \{|M_f(x, p) - M_g(x, p)|\} \leq Q \cdot \left\| (f^{-1})' \right\|_{\infty} \cdot \left\| (f \circ g^{-1})'' \right\|_{\infty} \cdot |g(b) - g(a)|^2,$$

provided that the right-hand side of the inequality is finite, where, as above,

$$Q = \left(\sum_{i \in S} p_i \right) \left(1 - \sum_{i \in S} p_i \right),$$

and $\|\cdot\|_{\infty}$ is the usual sup-norm.



Further Reverse Results for
Jensen's Discrete Inequality
and Applications in Information
Theory

I. Budimir, S.S. Dragomir and
J. Pečarić

Title Page

Contents



Go Back

Close

Quit

Page 18 of 31

Proof. Let, as in [3], $h = f \circ g^{-1}$, $n > 1$,

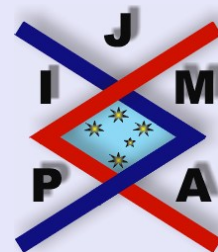
$$x = (x_1, x_2, \dots, x_n) \text{ and } p = (p_1, p_2, \dots, p_n)$$

be as in the Definition 4.2, and $y_j = g(x_j)$ ($j = 1, 2, \dots, n$). By the mean-value theorem, for some α in the open interval joining $f(a)$ to $f(b)$, we have

$$\begin{aligned} M_f(x, p) - M_g(x, p) &= f^{-1} \left\{ \sum_{j=1}^n p_j f(x_j) \right\} - f^{-1} \left[h \left\{ \sum_{j=1}^n p_j g(x_j) \right\} \right] \\ &= (f^{-1})'(\alpha) \left[\sum_{j=1}^n p_j f(x_j) - h \left\{ \sum_{j=1}^n p_j g(x_j) \right\} \right] \\ &= (f^{-1})'(\alpha) \left[\sum_{j=1}^n p_j h(y_j) - h \left\{ \sum_{j=1}^n p_j y_j \right\} \right] \\ &= (f^{-1})'(\alpha) \left[\sum_{j=1}^n p_j \left\{ h(y_j) - h \left(\sum_{k=1}^n p_k y_k \right) \right\} \right]. \end{aligned}$$

Using the mean-value theorem a second time, we conclude that there exists points z_1, z_2, \dots, z_n in the open interval joining $g(a)$ to $g(b)$, such that

$$\begin{aligned} M_f(x, p) - M_g(x, p) &= (f^{-1})'(\alpha) [p_1 \{(1 - p_1) y_1 - p_2 y_2 - \dots - p_n y_n\} h'(z_1) \\ &\quad + p_2 \{-p_1 y_1 + (1 - p_2) y_2 - \dots - p_n y_n\} h'(z_2) \\ &\quad + \dots \\ &\quad + p_n \{-p_1 y_1 - p_2 y_2 - \dots + (1 - p_n) y_n\} h'(z_n)] \end{aligned}$$



**Further Reverse Results for
Jensen's Discrete Inequality
and Applications in Information
Theory**

I. Budimir, S.S. Dragomir and
J. Pečarić

Title Page

Contents



Go Back

Close

Quit

Page 19 of 31

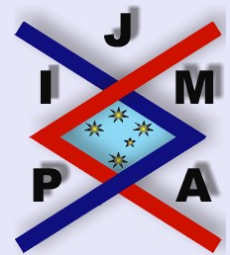
$$\begin{aligned}
&= (f^{-1})'(\alpha) [p_1 \{p_2 (y_1 - y_2) + \cdots + p_n (y_1 - y_n)\} h'(z_1) \\
&\quad + p_2 \{p_1 (y_2 - y_1) + \cdots + p_n (y_2 - y_n)\} h'(z_2) \\
&\quad + \cdots \\
&\quad + p_n \{p_1 (y_n - y_1) + \cdots + p_{n-1} (y_n - y_{n-1})\} h'(z_n)] \\
&= (f^{-1})'(\alpha) \sum_{1 \leq i < j \leq n} p_i p_j (y_i - y_j) \{h'(z_i) - h'(z_j)\}.
\end{aligned}$$

Using the mean value theorem a third time, we conclude that there exists points ω_{ij} ($1 \leq i < j \leq n$) in the open interval joining $g(a)$ to $g(b)$, such that

$$\begin{aligned}
(f^{-1})'(\alpha) \sum_{1 \leq i < j \leq n} p_i p_j (y_i - y_j) \{h'(z_i) - h'(z_j)\} \\
= (f^{-1})'(\alpha) \sum_{1 \leq i < j \leq n} p_i p_j (y_i - y_j) (z_i - z_j) h''(\omega_{ij}).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&|M_f(x, p) - M_g(x, p)| \\
&\leq \left| (f^{-1})'(\alpha) \right| \sum_{1 \leq i < j \leq n} p_i p_j |y_i - y_j| \cdot |z_i - z_j| \cdot |h''(\omega_{ij})| \\
&\leq \left\| (f^{-1})' \right\|_{\infty} \cdot \|h''\|_{\infty} \cdot \sum_{1 \leq i < j \leq n} p_i p_j |y_i - y_j| \cdot |z_i - z_j| \\
&\leq (\text{by the Cauchy-Buniakowski-Schwartz inequality})
\end{aligned}$$



Further Reverse Results for Jensen's Discrete Inequality and Applications in Information Theory

I. Budimir, S.S. Dragomir and J. Pečarić

Title Page

Contents



Go Back

Close

Quit

Page 20 of 31

$$\begin{aligned}
&\leq \left\| (f^{-1})' \right\|_{\infty} \cdot \left\| (f \circ g^{-1})'' \right\|_{\infty} \cdot \sqrt{\sum_{1 \leq i < j \leq n} p_i p_j |y_i - y_j|^2} \\
&\quad \cdot \sqrt{\sum_{1 \leq i < j \leq n} p_i p_j |z_i - z_j|^2} \\
&\leq \text{(by the Andrica and Badea result)} \\
&\leq \left\| (f^{-1})' \right\|_{\infty} \cdot \left\| (f \circ g^{-1})'' \right\|_{\infty} \cdot \sqrt{\left(\sum_{i \in S} p_i \right) \left(1 - \sum_{i \in S} p_i \right) |g(b) - g(a)|^2} \\
&\quad \cdot \sqrt{\left(\sum_{i \in S} p_i \right) \left(1 - \sum_{i \in S} p_i \right) |g(b) - g(a)|^2} \\
&= Q \left\| (f^{-1})' \right\|_{\infty} \cdot \left\| (f \circ g^{-1})'' \right\|_{\infty} \cdot |g(b) - g(a)|^2,
\end{aligned}$$

and the theorem is proved. \square

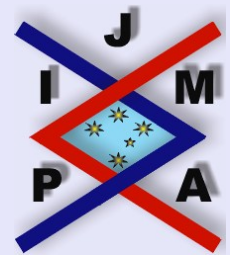
Corollary 4.2. *If $f, g \in CM[a, b]$, then*

$$\sup_x \{ |M_f(x, p) - M_g(x, p)| \} \leq Q \cdot \left\| \frac{1}{f'} \right\|_{\infty} \cdot \left\| \frac{1}{g'} \left(\frac{f'}{g'} \right)' \right\|_{\infty} \cdot |g(b) - g(a)|^2,$$

provided that the right hand side of the inequality exists.

Proof. This follows at once from the fact that

$$(f^{-1})' = \frac{1}{f' \circ f^{-1}}$$



**Further Reverse Results for
Jensen's Discrete Inequality
and Applications in Information
Theory**

I. Budimir, S.S. Dragomir and
J. Pečarić

Title Page

Contents



Go Back

Close

Quit

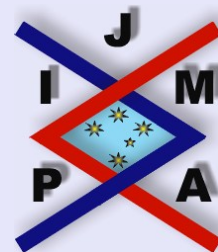
Page 21 of 31

and

$$(f \circ g^{-1})'' = \frac{(g' \circ g^{-1})(f'' \circ g^{-1}) - (f' \circ g^{-1})(g'' \circ g^{-1})}{(g' \circ g^{-1})^3} = \left[\frac{1}{g'} \left(\frac{f'}{g'} \right)' \right] \circ g^{-1}.$$

□

Remark 4.1. This establishes Theorem 4.3 from [3] and replaces the multiplicative factor $\frac{1}{4}$ by Q . In Corollary 4.2, we also replaced the multiplicative factor $\frac{1}{4}$ by Q .



**Further Reverse Results for
Jensen's Discrete Inequality
and Applications in Information
Theory**

I. Budimir, S.S. Dragomir and
J. Pečarić

Title Page

Contents



Go Back

Close

Quit

Page 22 of 31

5. Applications in Information Theory

We give some new applications for Shannon's entropy

$$H_b(X) := \sum_{i=1}^r p_i \log_b \frac{1}{p_i},$$

where X is a random variable with the probability distribution $(p_i)_{i=\overline{1,r}}$.

Theorem 5.1. *Let X be as above and assume that $p_1 \geq p_2 \geq \dots \geq p_r$ or $p_1 \leq p_2 \leq \dots \leq p_r$. Then we have the inequality*

$$(5.1) \quad 0 \leq \log_b r - H_b(X) \leq \frac{(p_1 - p_r)^2}{p_1 p_r} \max_{1 \leq k \leq r} \{P_k \bar{P}_{k+1}\}.$$

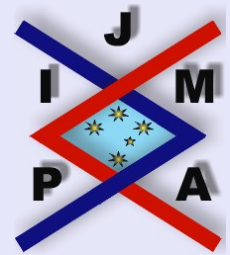
Proof. We choose in Theorem 2.1, $f(x) = -\log_b x$, $x > 0$, $x_i = \frac{1}{p_i}$ ($i = 1, \dots, r$). Then we have $x_1 \leq x_2 \leq \dots \leq x_r$ and by (2.1) we obtain

$$0 \leq \log_b r - H_b(X) \leq \left(\frac{1}{p_r} - \frac{1}{p_1} \right) \left(\frac{1}{-\frac{1}{p_r}} + \frac{1}{\frac{1}{p_1}} \right) \max_{1 \leq k \leq r} \{P_k \bar{P}_{k+1}\},$$

which is equivalent to (5.1). The same inequality is obtained if $p_1 \leq p_2 \leq \dots \leq p_r$. □

Theorem 5.2. *Let X be as above and suppose that*

$$\begin{aligned} p_M & : = \max \{p_i | i = 1, \dots, r\}, \\ p_m & : = \min \{p_i | i = 1, \dots, r\}. \end{aligned}$$



Further Reverse Results for Jensen's Discrete Inequality and Applications in Information Theory

I. Budimir, S.S. Dragomir and J. Pečarić

Title Page

Contents



Go Back

Close

Quit

Page 23 of 31

If S is a subset of the set $\{1, \dots, r\}$ minimizing the expression $|\sum_{i \in S} p_i - 1/2|$, then we have the estimation

$$(5.2) \quad 0 \leq \log_b r - H_b(X) \leq Q \cdot \frac{(p_M - p_m)^2}{\ln b \cdot p_M p_m}.$$

Proof. We shall choose in Theorem 2.2,

$$f(x) = -\log_b x, \quad x > 0, \quad x_i = \frac{1}{p_i} \quad (i = \overline{1, r}).$$

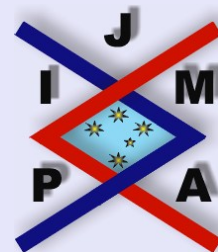
Then $m = \frac{1}{p_M}$, $M = \frac{1}{p_m}$, $f'(x) = -\frac{1}{x \ln b}$ and the inequality (2.3) becomes:

$$\begin{aligned} 0 &\leq \log_b r - \sum_{i=1}^r p_i \log_b \frac{1}{p_i} \\ &\leq Q \frac{1}{\ln b} \left(\frac{1}{p_m} - \frac{1}{p_M} \right) \left(-\frac{1}{p_m} + \frac{1}{p_M} \right) \\ &= Q \cdot \frac{1}{\ln b} \cdot \frac{(p_M - p_m)^2}{p_M p_m}, \end{aligned}$$

hence the estimation (5.2) is proved. \square

Consider the Shannon entropy

$$(5.3) \quad H(X) := H_e(X) = \sum_{i=1}^r p_i \ln \frac{1}{p_i}$$



Further Reverse Results for Jensen's Discrete Inequality and Applications in Information Theory

I. Budimir, S.S. Dragomir and J. Pečarić

Title Page

Contents



Go Back

Close

Quit

Page 24 of 31

and Rényi's entropy of order α ($\alpha \in (0, \infty) \setminus \{1\}$)

$$(5.4) \quad H_{[\alpha]}(X) := \frac{1}{1-\alpha} \ln \left(\sum_{i=1}^r p_i^\alpha \right).$$

Using the classical Jensen's discrete inequality for convex mappings, i.e.,

$$(5.5) \quad f \left(\sum_{i=1}^r p_i x_i \right) \leq \sum_{i=1}^r p_i f(x_i),$$

where $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a convex mapping on I , $x_i \in I$ ($i = 1, \dots, r$) and $(p_i)_{i=1, \dots, r}$ is a probability distribution, for the convex mapping $f(x) = -\ln x$, we have

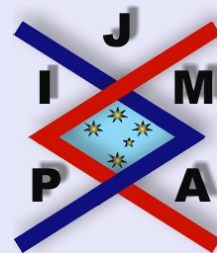
$$(5.6) \quad \ln \left(\sum_{i=1}^r p_i x_i \right) \geq \sum_{i=1}^r p_i \ln x_i.$$

Choose $x_i = p_i^{\alpha-1}$ ($i = 1, \dots, r$) in (5.6) to obtain

$$\ln \left(\sum_{i=1}^r p_i^\alpha \right) \geq (\alpha - 1) \sum_{i=1}^r p_i \ln p_i,$$

which is equivalent to

$$(1 - \alpha) [H_{[\alpha]}(X) - H(X)] \geq 0.$$



**Further Reverse Results for
Jensen's Discrete Inequality
and Applications in Information
Theory**

I. Budimir, S.S. Dragomir and
J. Pečarić

Title Page

Contents



Go Back

Close

Quit

Page 25 of 31

Now, if $\alpha \in (0, 1)$, then $H_{[\alpha]}(X) \leq H(X)$, and if $\alpha > 1$ then $H_{[\alpha]}(X) \geq H(X)$. Equality holds iff $(p_i)_{i=\overline{1,r}}$ is a uniform distribution and this fact follows by the strict convexity of $-\ln(\cdot)$. This inequality also follows as a special case of the following well known fact: $H_{[\alpha]}(X)$ is a nondecreasing function of α . See for example [26] or [22].

Theorem 5.3. *Under the above assumptions, given that $p_m = \min_{i=\overline{1,r}} p_i$, $p_M = \max_{i=\overline{1,r}} p_i$, then we have the inequality*

$$(5.7) \quad 0 \leq (1 - \alpha) [H_{[\alpha]}(X) - H(X)] \leq Q \cdot \frac{(p_M^{\alpha-1} - p_m^{\alpha-1})^2}{p_M^{\alpha-1} p_m^{\alpha-1}},$$

for all $\alpha \in (0, 1) \cup (1, \infty)$.

Proof. If $\alpha \in (0, 1)$, then

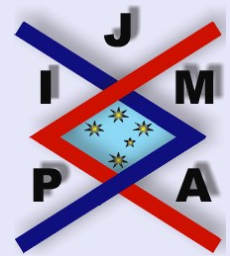
$$x_i := p_i^{\alpha-1} \in [p_M^{\alpha-1}, p_m^{\alpha-1}]$$

and if $\alpha \in (1, \infty)$, then

$$x_i = p_i^{\alpha-1} \in [p_m^{\alpha-1}, p_M^{\alpha-1}], \text{ for } i \in \{1, \dots, n\}.$$

Applying Theorem 2.2 for $x_i := p_i^{\alpha-1}$ and $f(x) = -\ln x$, and taking into account that $f'(x) = -\frac{1}{x}$, we obtain

$$(1 - \alpha) [H_{[\alpha]}(X) - H(X)] \leq \begin{cases} Q (p_m^{\alpha-1} - p_M^{\alpha-1}) \left(-\frac{1}{p_m^{\alpha-1}} + \frac{1}{p_M^{\alpha-1}} \right) & \text{if } \alpha \in (0, 1), \\ Q (p_M^{\alpha-1} - p_m^{\alpha-1}) \left(-\frac{1}{p_M^{\alpha-1}} + \frac{1}{p_m^{\alpha-1}} \right) & \text{if } \alpha \in (1, \infty) \end{cases}$$



Further Reverse Results for Jensen's Discrete Inequality and Applications in Information Theory

I. Budimir, S.S. Dragomir and J. Pečarić

Title Page

Contents



Go Back

Close

Quit

Page 26 of 31

$$\begin{aligned}
&= \begin{cases} Q \cdot \frac{(p_m^{\alpha-1} - p_M^{\alpha-1})^2}{p_m^{\alpha-1} p_M^{\alpha-1}} & \text{if } \alpha \in (0, 1), \\ Q \cdot \frac{(p_M^{\alpha-1} - p_m^{\alpha-1})^2}{p_M^{\alpha-1} p_m^{\alpha-1}} & \text{if } \alpha \in (1, \infty) \end{cases} \\
&= Q \cdot \frac{(p_M^{\alpha-1} - p_m^{\alpha-1})^2}{p_M^{\alpha-1} p_m^{\alpha-1}}
\end{aligned}$$

for all $\alpha \in (0, 1) \cup (1, \infty)$ and the theorem is proved. \square

Using a similar argument to the one in Theorem 5.3, we can state the following direct application of Theorem 2.2.

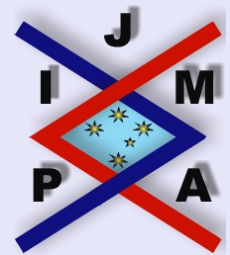
Theorem 5.4. *Let $(p_i)_{i=1,r}$ be as in Theorem 5.3. Then we have the inequality*

$$(5.8) \quad 0 \leq (1 - \alpha) H_{[\alpha]}(X) - \ln r - \alpha \ln G_r(p) \leq Q \cdot \frac{(p_M^{\alpha-1} - p_m^{\alpha-1})^2}{P_M^{\alpha-1} p_m^{\alpha-1}},$$

for all $\alpha \in (0, 1) \cup (1, \infty)$.

Remark 5.1. The above results improve the corresponding results from [5] and [4] with the constant Q which is less than $\frac{1}{4}$.

Acknowledgement 1. The authors would like to thank the anonymous referee for valuable comments and for the references [26] and [22].



Further Reverse Results for
Jensen's Discrete Inequality
and Applications in Information
Theory

I. Budimir, S.S. Dragomir and
J. Pečarić

Title Page

Contents



Go Back

Close

Quit

Page 27 of 31

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Further Reverse Results for
Jensen's Discrete Inequality
and Applications in Information
Theory

I. Budimir, S.S. Dragomir and
J. Pečarić

Title Page

Contents



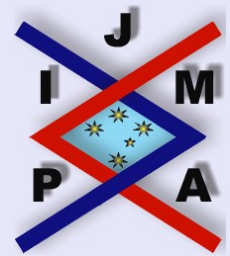
Go Back

Close

Quit

Page 28 of 31

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**Further Reverse Results for
Jensen's Discrete Inequality
and Applications in Information
Theory**

I. Budimir, S.S. Dragomir and
J. Pečarić

Title Page

Contents



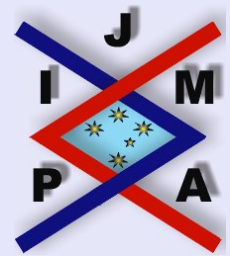
Go Back

Close

Quit

Page 29 of 31

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**Further Reverse Results for
Jensen's Discrete Inequality
and Applications in Information
Theory**

I. Budimir, S.S. Dragomir and
J. Pečarić

Title Page

Contents



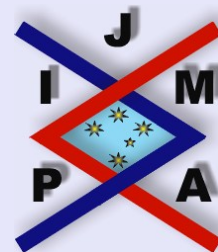
Go Back

Close

Quit

Page 30 of 31

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**Further Reverse Results for
Jensen's Discrete Inequality
and Applications in Information
Theory**

I. Budimir, S.S. Dragomir and
J. Pečarić

Title Page

Contents



Go Back

Close

Quit

Page 31 of 31