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COEFFICIENT INEQUALITIES FOR CERTAIN CLASSES OF MEROMORPHICALLY STARLIKE AND MEROMORPHICALLY CONVEX FUNCTIONS

SHIGEYOSHI OWA AND NICOLAE N. PASCU

Department of Mathematics,
Kinki University
Higashi-Osaka, Osaka 577-8502
JAPAN.
EMail: owa@math.kindai.ac.jp

Department of Mathematics
Transilvania University of Brasov
R-2200 Brasov
ROMANIA.
EMail: pascu@info.unitbv.ro

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Abstract

Let Σ_r be the class of meromorphic functions $f(z)$ in \mathbb{D}_r with a simple pole at the origin. Two subclasses $T_r^*(\alpha)$ and $\mathcal{C}_r(\alpha)$ of Σ_r are considered. Some coefficient properties of functions $f(z)$ to be in the classes $T_r^*(\alpha)$ and $\mathcal{C}_r(\alpha)$ of Σ_r are discussed. Also, the starlikeness and the convexity of functions $f(z)$ in Σ_r are discussed.

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Key words: Meromorphic functions, Univalent functions, Starlike functions, Convex functions.

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1. Introduction

Let Σ_r denote the class of functions $f(z)$ of the form:

$$(1.1) \quad f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$$

which are analytic in the punctured disk $\mathbb{D}_r = \{z \in \mathbb{C} : 0 < |z| < r \leq 1\}$. A function $f(z) \in \Sigma_r$ is said to be starlike of order α if it satisfies the inequality:

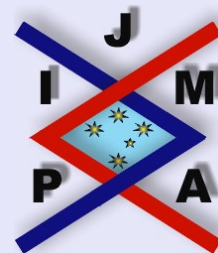
$$(1.2) \quad \operatorname{Re} \left(-\frac{z f'(z)}{f(z)} \right) > \alpha \quad (z \in \mathbb{D}_r)$$

for some $\alpha (0 \leq \alpha < 1)$. We say that $f(z)$ is in the class $\mathcal{T}_r^*(\alpha)$ for such functions. A function $f(z) \in \Sigma_r$ is said to be convex of order α if it satisfies the inequality:

$$(1.3) \quad \operatorname{Re} \left\{ -\left(1 + \frac{z f''(z)}{f'(z)} \right) \right\} > \alpha \quad (z \in \mathbb{D}_r)$$

for some $\alpha (0 \leq \alpha < 1)$. We say that $f(z)$ is in the class $\mathcal{C}_r(\alpha)$ if it is convex of order α in \mathbb{D}_r . We note that $f(z) \in \mathcal{C}_r(\alpha)$ if and only if $-z f'(z) \in \mathcal{T}_r^*(\alpha)$. There are many papers discussing various properties of classes consisting of univalent, starlike, convex, multivalent, and meromorphic functions in the book by Srivastava and Owa [3].

Ozaki [2] has shown that the necessary and sufficient condition that $f(z) \in \Sigma_r$ with $a_n \geq 0 (n = 1, 2, 3, \dots)$ is meromorphic and univalent in \mathbb{D}_r is that



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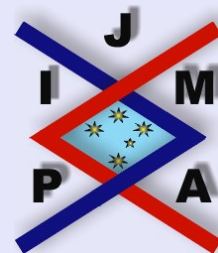
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there should exist the relation:

$$\sum_{n=1}^{\infty} na_n r^{n+1} \leq 1$$

between its coefficients.

Our results in the present paper are an improvement and extension of the above theorem by Ozaki [2].



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2. Coefficient Inequalities for Functions

Our first result for the functions $f(z) \in \Sigma_r$ is contained in

Theorem 2.1. *If $f(z) \in \Sigma_r$ satisfies*

$$(2.1) \quad \sum_{n=0}^{\infty} (n+k+|2\alpha+n-k|)|a_n|r^{n+1} \leq 2(1-\alpha)$$

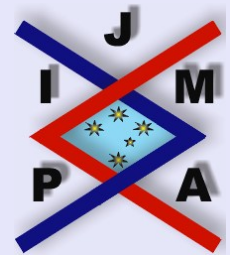
for some α ($0 \leq \alpha < 1$) and k ($\alpha < k \leq 1$), then $f(z) \in \mathcal{T}_r^*(\alpha)$.

Proof. For $f(z) \in \Sigma_r$, we know that

$$\begin{aligned} & |zf'(z) + kf(z)| - |zf'(z) + (2\alpha - k)f(z)| \\ &= \left| (k-1)\frac{1}{z} + \sum_{n=0}^{\infty} (n+k)a_n z^n \right| \\ &\quad - \left| (2\alpha - k - 1)\frac{1}{z} + \sum_{n=0}^{\infty} (2\alpha + n - k)a_n z^n \right|. \end{aligned}$$

Therefore, applying the condition of the theorem, we have

$$\begin{aligned} & r |zf'(z) + kf(z)| - r |zf'(z) + (2\alpha - k)f(z)| \\ &\leq (k-1) + \sum_{n=0}^{\infty} (n+k)|a_n|r^{n+1} - (k+1-2\alpha) \\ &\quad + \sum_{n=0}^{\infty} |2\alpha + n - k||a_n|r^{n+1} \end{aligned}$$



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$$= 2(\alpha - 1) + \sum_{n=0}^{\infty} (n + k + |2\alpha + n - k|) |a_n| r^{n+1} \leq 0,$$

which shows that

$$\sum_{n=0}^{\infty} (n + k + |2\alpha + n - k|) |a_n| r^{n+1} \leq 2(1 - \alpha).$$

It follows from the above that

$$\left| \frac{zf'(z) + kf(z)}{zf'(z) + (2\alpha - k)f(z)} \right| \leq 1,$$

so that

$$\operatorname{Re} \left(-\frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathbb{D}_r).$$

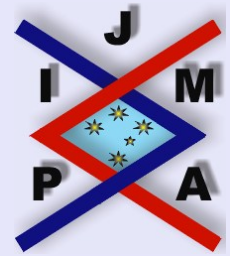
□

Letting $k = 0$ in Theorem 2.1, we have

Corollary 2.2. *If $f(z) \in \Sigma_r$ satisfies*

$$(2.2) \quad \sum_{n=0}^{\infty} (n + \alpha) |a_n| r^{n+1} \leq 1 - \alpha$$

for some α ($\frac{1}{2} \leq \alpha < 1$), then $f(z) \in \mathcal{T}_r^(\alpha)$.*



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Theorem 2.1 gives us the following results.

Corollary 2.3. Let the function $f(z) \in \Sigma_r$ be given by (1.1) with $a_n = |a_n|e^{-\frac{n+1}{2\pi}i}$, then $f(z) \in \mathcal{T}_r^*(\alpha)$ if and only if

$$(2.3) \quad \sum_{n=0}^{\infty} (n + \alpha) |a_n| r^{n+1} \leq 1 - \alpha$$

for some α ($\frac{1}{2} \leq \alpha < 1$).

Proof. In view of Theorem 2.1, we see that if the coefficient inequality (2.3) holds true for some α ($\frac{1}{2} \leq \alpha < 1$), then $f(z) \in \mathcal{T}_r^*(\alpha)$.

Conversely, let $f(z)$ be in the class $\mathcal{T}_r^*(\alpha)$, then

$$\operatorname{Re} \left(-\frac{z f'(z)}{f(z)} \right) = \operatorname{Re} \left(\frac{1 - \sum_{n=0}^{\infty} n a_n z^{n+1}}{1 + \sum_{n=0}^{\infty} a_n z^{n+1}} \right) > \alpha$$

for all $z \in \mathbb{D}_r$. Letting $z = r e^{\frac{1}{2\pi}i}$, we have that $a_n z^{n+1} = |a_n| r^{n+1}$. This implies that

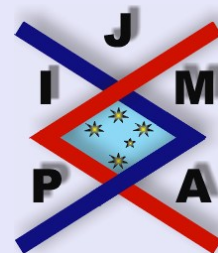
$$1 - \sum_{n=0}^{\infty} n |a_n| r^{n+1} \geq \alpha \left(1 + \sum_{n=0}^{\infty} |a_n| r^{n+1} \right),$$

which is equivalent to (2.3). □

Example 2.1. The function $f(z)$ given by

$$f(z) = \frac{1}{z} + a_0 + \left(\frac{1 - \alpha - \alpha |a_0|}{n + \alpha} \right) e^{i\theta} z^n$$

belongs to the class $\mathcal{T}_r^*(\alpha)$ for some real θ with $\frac{1}{2} \leq \alpha \leq \frac{1}{1+|a_0|} < 1$.



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Remark 2.1. If $f(z) \in \Sigma_r$ with $a_0 = 0$, then Corollary 2.3 holds true for some α ($0 \leq \alpha < 1$).

Corollary 2.4. Let the function $f(z) \in \Sigma_r$ be given by (1.1) with $a_n \geq 0$, then $f(z) \in \mathcal{T}_r^*(\alpha)$ if and only if

$$\sum_{n=0}^{\infty} (n + \alpha) a_n r^{n+1} \leq 1 - \alpha$$

for some α ($\frac{1}{2} \leq \alpha < 1$).

Remark 2.2. If $f(z) \in \Sigma_r$ with $a_0 = 0$, then Corollary 2.4 holds true for $0 \leq \alpha < 1$.

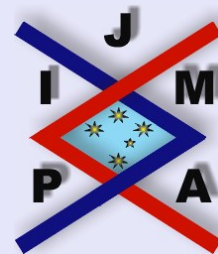
Remark 2.3. Juneja and Reddy [1] have given that $f(z) \in \Sigma_1$ with $a_0 = 0$ and $a_n \geq 0$ belongs to the class $\mathcal{T}_1^*(\alpha)$ if and only if

$$\sum_{n=1}^{\infty} (n + \alpha) a_n \leq 1 - \alpha.$$

Theorem 2.5. If $f(z) \in \Sigma_r$ satisfies

$$(2.4) \quad \sum_{n=1}^{\infty} n(n + \alpha) |a_n| r^{n+1} \leq 1 - \alpha$$

for some α ($0 \leq \alpha < 1$), then $f(z)$ belongs to the class $\mathcal{C}_r(\alpha)$.



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Proof. Noting that $f(z) \in \mathcal{C}_r(\alpha)$ if and only if $-zf'(z) \in \mathcal{T}_r^*(\alpha)$, and that

$$-zf'(z) = \frac{1}{z} - \sum_{n=1}^{\infty} na_n z^n,$$

we complete the proof of the theorem with the aid of Theorem 2.1. □

Corollary 2.6. *Let the function $f(z) \in \Sigma_r$ be given by (1.1) with $a_n = |a_n|e^{-\frac{n+1}{2\pi}i}$, then $f(z) \in \mathcal{C}_r(\alpha)$ if and only if the inequality (2.4) holds true for some $\alpha (0 \leq \alpha < 1)$.*

Example 2.2. *The function $f(z)$ given by*

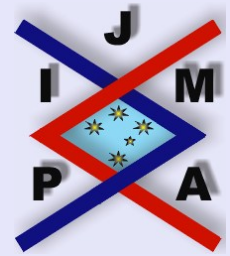
$$f(z) = \frac{1}{z} + a_0 + \left(\frac{1 - \alpha}{n(n + \alpha)} \right) e^{i\theta} z^n$$

belongs to the class $\mathcal{C}_r(\alpha)$ for some real θ with $0 \leq \alpha < 1$.

Corollary 2.7. *Let the function $f(z) \in \Sigma_r$ be given by (1.1) with $a_n \geq 0$, then $f(z) \in \mathcal{C}_r(\alpha)$ if and only if*

$$(2.5) \quad \sum_{n=1}^{\infty} n(n + \alpha)a_n r^{n+1} \leq 1 - \alpha$$

for some $\alpha (0 \leq \alpha < 1)$.



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3. Starlikeness and Convexity of Functions

We consider the radius problems for starlikeness and convexity of functions $f(z)$ belonging to the class Σ_r .

Theorem 3.1. A function $f(z) \in \Sigma_r$ belongs to the class $\mathcal{T}_r^*(\alpha)$ for $0 \leq r < r_0$, where r_0 is the smallest positive root of the equation

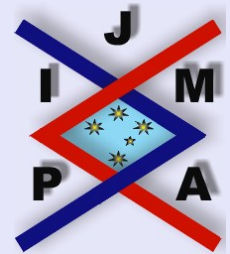
$$(3.1) \quad \alpha|a_0|r^3 - (\delta + 1 - \alpha)r^2 - \alpha|a_0|r + 1 - \alpha = 0,$$

and

$$(3.2) \quad \delta = \sqrt{\sum_{n=1}^{\infty} n|a_n|^2} + \alpha \sqrt{\sum_{n=1}^{\infty} \frac{1}{n}|a_n|^2}.$$

Proof. Using the Cauchy inequality, we see that

$$\begin{aligned} & \sum_{n=0}^{\infty} (n + \alpha)|a_n|r^{n+1} \\ &= \alpha|a_0|r + \sum_{n=1}^{\infty} |a_n|r^{n+1} \\ &\leq \alpha|a_0|r + \sqrt{\sum_{n=1}^{\infty} n|a_n|^2} \sqrt{\sum_{n=1}^{\infty} nr^{2n+2}} + \alpha \sqrt{\sum_{n=1}^{\infty} \frac{1}{n}|a_n|^2} \sqrt{\sum_{n=1}^{\infty} nr^{2n+2}} \end{aligned}$$



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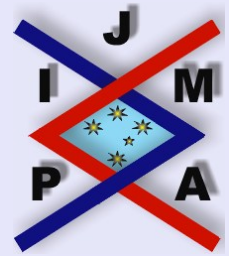


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$$\begin{aligned}
 &= \alpha|a_0|r + \sqrt{\frac{r^4}{(1-r^2)^2}} \left(\sqrt{\sum_{n=1}^{\infty} n|a_n|^2} + \alpha \sqrt{\sum_{n=1}^{\infty} \frac{1}{n}|a_n|^2} \right) \\
 &= \alpha|a_0|r + \frac{r^2}{1-r^2} \delta < 1 - \alpha,
 \end{aligned}$$

where δ is given by (3.2). Therefore, an application of Corollary 2.2 gives us that $f(z) \in \mathcal{T}_r^*(\alpha)$ for $0 \leq r < r_0$. \square

Letting $a_0 = 0$ in Theorem 3.1, we have

Corollary 3.2. A function $f(z) \in \Sigma_r$ with $a_0 = 0$ belongs to the class $\mathcal{T}_r^*(\alpha)$ for $0 \leq r < r_0$, where

$$r_0 = \sqrt{1 - \frac{\delta}{\delta + 1 - \alpha}}$$

and δ is given by (3.2).

Example 3.1. If we consider the function $f(z)$ given by

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} e^{i\theta_n} z^n \quad (\theta_n \text{ is real}),$$

then $f(z) \in \mathcal{T}_r^*(\alpha)$ for $0 \leq r < r_0$ with

$$\begin{aligned}
 \delta &= \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^2}} + \alpha \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^4}} \\
 &= \sqrt{\zeta(2)} + \alpha \sqrt{\zeta(4)} = \pi \left(\frac{1}{\sqrt{6}} + \frac{\pi\alpha}{3\sqrt{10}} \right).
 \end{aligned}$$

Further, letting $\alpha = 0$, we have that

$$\delta = \frac{\pi}{\sqrt{6}} \approx 1.282550$$

and

$$r_0 = \sqrt{\frac{\sqrt{6}}{\sqrt{6} + \pi}} \approx 0.661896.$$

Finally, for convexity of functions $f(z)$, we derive

Theorem 3.3. A function $f(z) \in \Sigma_r$ belongs to the class $\mathcal{C}_r(\alpha)$ for $0 \leq r < r_1$, where

$$r_1 = \sqrt{1 - \frac{\sigma}{\sigma + 1 - \alpha}}$$

and

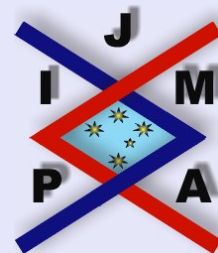
$$\sigma = \sqrt{\sum_{n=1}^{\infty} n^3 |a_n|^2} + \alpha \sqrt{\sum_{n=1}^{\infty} n |a_n|^2}.$$

Example 3.2. Let us consider the function $f(z)$ given by

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{n^2 \sqrt{n}} e^{i\theta_n} z^n \quad (\theta_n \text{ is real}).$$

We see that $f(z) \in \mathcal{C}_r(\alpha)$ for $0 \leq r < r_0$ with

$$\delta = \pi \left(\frac{1}{\sqrt{6}} + \frac{\pi\alpha}{3\sqrt{10}} \right).$$



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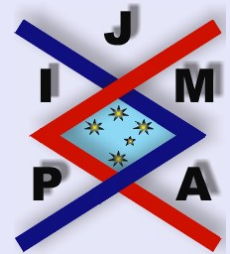
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Taking $\alpha = 0$, we obtain

$$\delta = \frac{\pi}{\sqrt{6}} \approx 1.282550$$

and

$$r_0 = \sqrt{\frac{\sqrt{6}}{\sqrt{6} + \pi}} \approx 0.661896.$$



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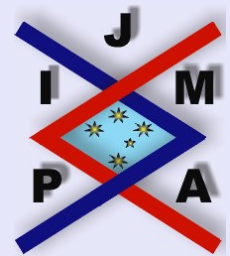
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