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AN APPLICATION OF ALMOST INCREASING AND δ-QUASI-MONOTONE SEQUENCES

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ABSTRACT. In this paper a general theorem on absolute weighted mean summability factors has been proved under weaker conditions by using an almost increasing and δ-quasi-monotone sequences.

Key words and phrases: Almost Increasing Sequences, Quasi-monotone Sequences, Absolute Summability Factors, Infinite Series.

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1. INTRODUCTION

A sequence (b_n) of positive numbers is said to be δ-quasi-monotone, if $b_n > 0$ ultimately and $\Delta b_n \geq -\delta_n$, where (δ_n) is a sequence of positive numbers (see [2]). Let $\sum a_n$ be a given infinite series with (s_n) as the sequence of its n-th partial sums. By u_n and t_n we denote the n-th $(C, 1)$ means of the sequence (s_n) and (na_n), respectively. The series $\sum a_n$ is said to be summable $|C, 1|_k$, $k \geq 1$, if (see [5])

$$\sum_{n=1}^{\infty} n^{k-1} |u_n - u_{n-1}|^k = \sum_{n=1}^{\infty} \frac{1}{n} |t_n|^k < \infty.$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \text{ as } n \to \infty, \quad (P_{-i} = p_{-i} = 0, \ i \geq 1).$$

The sequence-to-sequence transformation

$$z_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

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defines the sequence (z_n) of the $\left(\bar{N}, p_n\right)$ mean of the sequence (s_n) generated by the sequence of coefficients (p_n) (see [6]). The series $\sum a_n$ is said to be summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$, if (see [3])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \left|z_n - z_{n-1}\right|^k < \infty.$$

In the special case $p_n = 1$ for all values of n (resp. $k = 1$), then $\left|\bar{N}, p_n\right|_k$ summability is the same as $|C, 1|_k$ (resp. $\left|\bar{N}, p_n\right|$) summability. Also if we take $p_n = \frac{1}{n+1}$, then $\left|\bar{N}, p_n\right|_k$ summability reduces to $\left|\bar{N}, \frac{1}{n+1}\right|_k$ summability.

Mazhar [7] has proved the following theorem for summability factors by using δ-quasi-monotone sequences.

Theorem 1.1. *Let $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers (A_n) such that it is δ-quasi-monotone with $\sum n\delta_n \log n < \infty$, $\sum A_n \log n$ is convergent and $|\Delta\lambda_n| \leq |A_n|$ for all n. If*

$$\sum_{n=1}^{m} \frac{1}{n} |t_n|^k = O\left(\log m\right) \quad as \quad m \to \infty,$$

then the series $\sum a_n \lambda_n$ is summable $|C, 1|_k$, $k \geq 1$.

Later on Bor [4] generalized Theorem 1.1 for a $\left|\bar{N}, p_n\right|_k$ summability method in the following form.

Theorem 1.2. *Let $\lambda_n \to 0$ as $n \to \infty$ and let (p_n) be a sequence of positive numbers such that*

$$P_n = O\left(np_n\right) \quad as \quad n \to \infty.$$

Suppose that there exists a sequence of numbers (A_n) such that it is δ-quasi-monotone with $\sum n\delta_n X_n < \infty$, $\sum A_n X_n$ is convergent and $|\Delta\lambda_n| \leq |A_n|$ for all n. If

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O\left(X_m\right) \quad as \quad m \to \infty,$$

where (X_n) is a positive increasing sequence, then the series $\sum a_n \lambda_n$ is summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$.

It should be noted that if we take $X_n = \log n$ and $p_n = 1$ for all values of n in Theorem 1.2, then we get Theorem 1.1.

2. The Main Result.

Due to the restriction $P_n = O\left(np_n\right)$ on (p_n), no result for $p_n = \frac{1}{n+1}$ can be deduced from Theorem 1.2. Therefore the aim of this paper is to prove Theorem 1.2 under weaker conditions and in a more general form without this condition. For this we need the concept of almost increasing sequence. A positive sequence (d_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq d_n \leq Bc_n$ (see [1]). Obviously, every increasing sequence is almost increasing but the converse need not be true as can be seen from the example $d_n = ne^{(-1)^n}$. Since (X_n) is increasing in Theorem 1.2, we are weakening the hypotheses of the theorem by replacing the increasing sequence with an almost increasing sequence.

Now, we shall prove the following theorem.

Theorem 2.1. *Let (X_n) be an almost increasing sequence and $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers (A_n) such that it is δ-quasi-monotone with $\sum n\delta_n X_n < \infty$, $\sum A_n X_n$ is convergent and $|\Delta\lambda_n| \le |A_n|$ for all n. If*

$$\sum_{n=1}^{m} \frac{1}{n} |\lambda_n| = O(1),$$
$$\sum_{n=1}^{m} \frac{1}{n} |t_n|^k = O(X_m) \quad \text{as} \quad m \to \infty$$

and

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O(X_m) \quad \text{as} \quad m \to \infty,$$

then the series $\sum a_n\lambda_n$ is summable $\left|\bar{N}, p_n\right|_k$, $k \ge 1$.

We need the following lemmas for the proof of our theorem.

Lemma 2.2. *Under the conditions of the theorem, we have*

$$|\lambda_n| X_n = O(1) \quad \text{as} \quad n \to \infty.$$

Proof. Since $\lambda_n \to 0$ as $n \to \infty$, we have that

$$|\lambda_n| X_n = X_n \left|\sum_{v=n}^{\infty} \Delta\lambda_v\right| \le X_n \sum_{v=n}^{\infty} |\Delta\lambda_v| \le \sum_{v=0}^{\infty} X_v |\Delta\lambda_v| \le \sum_{v=0}^{\infty} X_v |A_v| < \infty.$$

Hence $|\lambda_n| X_n = O(1)$ as $n \to \infty$. □

Lemma 2.3. *Let (X_n) be an almost increasing sequence. If (A_n) is δ-quasi-monotone with $\sum n\delta_n X_n < \infty$, $\sum A_n X_n$ is convergent, then*

$$nA_nX_n = O(1),$$
$$\sum_{n=1}^{\infty} nX_n |\Delta A_n| < \infty.$$

The proof of Lemma 2.3 is similar to the proof of Theorem 1 and Theorem 2 of Boas [2, case $\gamma = 1$], and hence is omitted.

3. PROOF OF THE THEOREM

Proof of Theorem 2.1. Let (T_n) denote the $\left(\bar{N}, p_n\right)$ mean of the series $\sum a_n\lambda_n$. Then, by definition and changing the order of summation, we have

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{i=0}^{v} a_i\lambda_i = \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1}) a_v\lambda_v.$$

Then, for $n \ge 1$, we have

$$T_n - T_{n-1} = \frac{p_n}{P_nP_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \lambda_v = \frac{p_n}{P_nP_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1}\lambda_v}{v} v a_v.$$

By Abel's transformation, we get

$$T_n - T_{n-1} = \frac{n+1}{nP_n} p_n t_n \lambda_n - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v t_v \lambda_v \frac{v+1}{v} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v t_v \Delta\lambda_v \frac{v+1}{v}$$
$$+ \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v t_v \lambda_{v+1} \frac{1}{v} = T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \qquad \text{say.}$$

Since

$$|T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}|^k \leq 4^k \left(|T_{n,1}|^k + |T_{n,2}|^k + |T_{n,3}|^k + |T_{n,4}|^k \right),$$

to complete the proof of the theorem, it is enough to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,r}|^k < \infty \text{ for } r = 1, 2, 3, 4.$$

Since λ_n is bounded by the hypothesis, we have that

$$\begin{aligned}
\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,1}|^k &= O(1) \sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k |\lambda_n| |\lambda_n|^{k-1} \\
&= O(1) \sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k |\lambda_n| \\
&= O(1) \sum_{n=1}^{m-1} |\Delta\lambda_n| \sum_{v=1}^{n} \frac{p_v}{P_v} |t_v|^k + O(1) |\lambda_m| \sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k \\
&= O(1) \sum_{n=1}^{m-1} |\Delta\lambda_n| X_n + O(1) |\lambda_m| X_m \\
&= O(1) \sum_{n=1}^{m-1} |A_n| X_n + O(1) |\lambda_m| X_m = O(1) \text{ as } m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of the theorem and Lemma 2.2.

Now, when $k > 1$, applying Hölder's inequality, as in $T_{n,1}$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,2}|^k &= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v |t_v|^k |\lambda_v|^k \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} p_v |t_v|^k |\lambda_v| |\lambda_v|^{k-1} \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} \\
&= O(1) \sum_{v=1}^{m} |\lambda_v| \frac{p_v}{P_v} |t_v|^k = O(1) \text{ as } m \to \infty.
\end{aligned}$$

Again we have that,

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,3}|^k &= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_v|t_v|^k|A_v|\times\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}P_v|A_v|\right\}^{k-1}\\
&= O(1)\sum_{v=1}^{m}P_v|t_v|^k|A_v|\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}\\
&= O(1)\sum_{v=1}^{m}|t_v|^k|A_v|\\
&= O(1)\sum_{v=1}^{m}v|A_v|\frac{1}{v}|t_v|^k\\
&= O(1)\sum_{v=1}^{m-1}\Delta(v|A_v|)\sum_{i=1}^{v}\frac{1}{i}|t_i|^k+O(1)m|A_m|\sum_{v=1}^{m}\frac{1}{v}|t_v|^k\\
&= O(1)\sum_{v=1}^{m-1}|\Delta(v|A_v|)|X_v+O(1)m|A_m|X_m\\
&= O(1)\sum_{v=1}^{m-1}v|\Delta A_v|X_v+O(1)\sum_{v=1}^{m-1}|A_{v+1}|X_{v+1}+O(1)m|A_m|X_m\\
&= O(1) \quad \text{as } m\to\infty,
\end{aligned}$$

in view of the hypotheses of Theorem 2.1 and Lemma 2.3.

Finally, we get that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,4}|^k &\le \sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_v|t_v|^k|\lambda_{v+1}|\frac{1}{v}\times\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}P_v|\lambda_{v+1}|\frac{1}{v}\right\}^{k-1}\\
&= O(1)\sum_{v=1}^{m}P_v|t_v|^k|\lambda_{v+1}|\frac{1}{v}\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}\\
&= O(1)\sum_{v=1}^{m}|\lambda_{v+1}|\frac{1}{v}|t_v|^k\\
&= O(1)\sum_{v=1}^{m-1}\Delta|\lambda_{v+1}|\sum_{i=1}^{v}\frac{1}{i}|t_i|^k+O(1)|\lambda_{m+1}|\sum_{v=1}^{m}\frac{1}{v}|t_v|^k\\
&= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_{v+1}|X_{v+1}+O(1)|\lambda_{m+1}|X_{m+1}\\
&= O(1)\sum_{v=1}^{m-1}|\lambda_{v+1}|X_{v+1}+O(1)|\lambda_{m+1}|X_{m+1}=O(1) \quad \text{as } m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 2.1 and Lemma 2.2.

Therefore we get

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,r}|^k=O(1) \quad \text{as } m\to\infty, \text{ for } r=1,2,3,4.$$

This completes the proof of the theorem. □

If we take $p_n = 1$ for all values of n (resp. $p_n = \frac{1}{n+1}$), then we get a result concerning the $|C,1|_k$ (resp. $\left|\bar{N}, \frac{1}{n+1}\right|_k$) summability factors.

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