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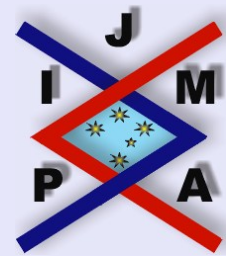
ASYMPTOTIC FORMULAE

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Abstract

Let $t_{s,n}$ be the n -th positive integer number which can be written as a power p^t , $t \geq s$, of a prime p ($s \geq 1$ is fixed). Let $\pi_s(x)$ denote the number of prime powers p^t , $t \geq s$, not exceeding x . We study the asymptotic behaviour of the sequence $t_{s,n}$ and of the function $\pi_s(x)$. We prove that the sequence $t_{s,n}$ has an asymptotic expansion comparable to that of p_n (the Cipolla's expansion).

2000 Mathematics Subject Classification: 11N05, 11N37.

Key words: Primes, Powers of primes, Cipolla's expansion.

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1. Introduction

Let p_n be the n -th prime. M. Cipolla [1] proved the following theorem:

There exists a unique sequence $P_j(X)$ ($j \geq 1$) of polynomials with rational coefficients such that, for every nonnegative integer m ,

$$(1.1) \quad p_n = n \log n + n \log \log n - n + \sum_{j=1}^m \frac{(-1)^{j-1} n P_j(\log \log n)}{\log^j n} + o\left(\frac{n}{\log^m n}\right).$$

The polynomials $P_j(X)$ have degree j and leading coefficient $\frac{1}{j}$.

$$P_1(X) = X - 2, \quad P_2(X) = \frac{X^2 - 6X + 11}{2}, \quad \dots$$

If $m = 0$ equation (1.1) is:

$$(1.2) \quad p_n = n \log n + n \log \log n - n + o(n).$$

Let $\pi(x)$ denote the number of prime numbers not exceeding x , then

$$(1.3) \quad \pi(x) = \left(\sum_{i=1}^m \frac{(i-1)!x}{\log^i x} \right) + \varepsilon(x) \frac{(m-1)!x}{\log^m x} \quad (m \geq 1),$$

where $\lim_{x \rightarrow \infty} \varepsilon(x) = 0$.

Lemma 1.1. *There exists a positive number M such that in the interval $[2, \infty)$, $|\varepsilon(x)| \leq M$.*



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Proof. Let us consider the closed interval $[2, a]$. In this interval, $\pi(x) \leq x$, so $\pi(x)$ is bounded. The functions $\frac{(i-1)!x}{\log^i x}$, $i = 1, \dots, m$ and $\frac{\log^m x}{(m-1)!x}$ are continuous on the compact $[2, a]$, so they are also bounded.

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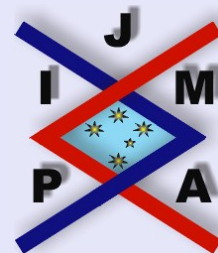
$$\varepsilon(x) = \left[\pi(x) - \left(\sum_{i=1}^m \frac{(i-1)!x}{\log^i x} \right) \right] \frac{\log^m x}{(m-1)!x},$$

$\varepsilon(x)$ is in its turn bounded on $[2, a]$.

Since a is arbitrary and $\lim_{x \rightarrow \infty} \varepsilon(x) = 0$, the lemma is proved. \square

Let us consider the sequence of positive integer numbers which can be written as a power p^t of a prime p ($t \geq 1$ is fixed). The number of prime powers p^t not exceeding x will be (in view of (1.3))

$$\begin{aligned} (1.4) \quad \pi(x^{\frac{1}{t}}) &= \left(\sum_{i=1}^m \frac{(i-1)!x^{\frac{1}{t}}}{\log^i x^{\frac{1}{t}}} \right) + \varepsilon\left(x^{\frac{1}{t}}\right) \frac{(m-1)!x^{\frac{1}{t}}}{\log^m x^{\frac{1}{t}}} \\ &= \left(\sum_{i=1}^m \frac{t^i(i-1)!x^{\frac{1}{t}}}{\log^i x} \right) + \varepsilon\left(x^{\frac{1}{t}}\right) \frac{t^m(m-1)!x^{\frac{1}{t}}}{\log^m x} \\ &= \left(\sum_{i=1}^m \frac{t^i(i-1)!x^{\frac{1}{t}}}{\log^i x} \right) + o\left(\frac{x^{\frac{1}{t}}}{\log^m x}\right). \end{aligned}$$



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2. The Function $\pi_s(x)$

Let $t_{s,n}$ be the n -th positive integer number (in increasing order) which can be written as a power p^t , $t \geq s$, of a prime p ($s \geq 1$ is fixed). Let $\pi_s(x)$ denote the number of prime powers p^t , $t \geq s$, not exceeding x .

Theorem 2.1.

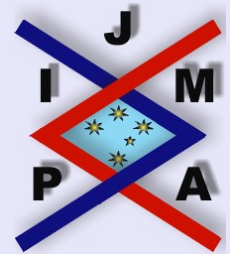
$$(2.1) \quad \pi_s(x) = \left(\sum_{i=1}^m \frac{s^i (i-1)! x^{\frac{1}{s}}}{\log^i x} \right) + o\left(\frac{x^{\frac{1}{s}}}{\log^m x} \right).$$

Proof. If $x \in [2^{s+k}, 2^{s+k+1})$ ($k \geq 1$), then

$$\pi_s(x) = \pi\left(x^{\frac{1}{s}}\right) + \sum_{i=1}^k \pi\left(x^{\frac{1}{s+i}}\right).$$

Using (1.4), we obtain

$$(2.2) \quad \begin{aligned} \pi_s(x) &= \left(\sum_{i=1}^m \frac{s^i (i-1)! x^{\frac{1}{s}}}{\log^i x} \right) + \varepsilon\left(x^{\frac{1}{s}}\right) \frac{s^m (m-1)! x^{\frac{1}{s}}}{\log^m x} \\ &\quad + \sum_{j=1}^k \left(\left(\sum_{i=1}^m \frac{(s+j)^i (i-1)! x^{\frac{1}{s+j}}}{\log^i x} \right) \right. \\ &\quad \left. + \varepsilon\left(x^{\frac{1}{s+j}}\right) \frac{(s+j)^m (m-1)! x^{\frac{1}{s+j}}}{\log^m x} \right) \end{aligned}$$



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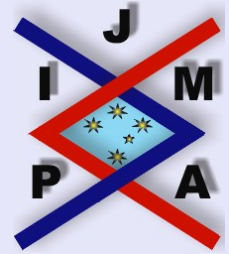
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$$\begin{aligned}
&= \left(\sum_{i=1}^m \frac{s^i (i-1)! x^{\frac{1}{s}}}{\log^i x} \right) + \varepsilon \left(x^{\frac{1}{s}} \right) \frac{s^m (m-1)! x^{\frac{1}{s}}}{\log^m x} \\
&\quad + \sum_{i=1}^m \left(\sum_{j=1}^k \frac{(s+j)^i (i-1)! x^{\frac{1}{s+j}}}{\log^i x} \right) \\
&\quad + \sum_{j=1}^k \varepsilon \left(x^{\frac{1}{s+j}} \right) \frac{(s+j)^m (m-1)! x^{\frac{1}{s+j}}}{\log^m x}.
\end{aligned}$$

In the given conditions, the following inequalities hold for x :

$$\begin{aligned}
\frac{\sum_{j=1}^k \frac{(s+j)^i (i-1)! x^{\frac{1}{s+j}}}{\log^i x}}{\frac{s^m (m-1)! x^{\frac{1}{s}}}{\log^m x}} &= \frac{\sum_{j=1}^k \frac{(s+j)^i}{s^m} \cdot \frac{(i-1)!}{(m-1)!} x^{\frac{1}{s+j}} \log^{m-i} x}{x^{\frac{1}{s}}} \\
&\leq \sum_{j=1}^k \frac{\frac{(s+j)^i}{s^m} \cdot \frac{(i-1)!}{(m-1)!} \log^{m-i} (2^{s+k+1})}{2^{\frac{(s+k)j-s}{s(s+j)}}} \\
&\leq \sum_{j=1}^k \frac{(s+k)^i (s+k+1)^{m-i}}{\left(2^{\frac{1}{s(s+1)}}\right)^k} \\
&= \frac{k (s+k)^i (s+k+1)^{m-i}}{\left(2^{\frac{1}{s(s+1)}}\right)^k} \quad (i = 1, \dots, m).
\end{aligned}$$



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Now, since

$$\lim_{k \rightarrow \infty} \frac{k(s+k)^i (s+k+1)^{m-i}}{\left(2^{\frac{1}{s+1}}\right)^k} = 0 \quad (i = 1, \dots, m),$$

we find that

$$(2.3) \quad \lim_{x \rightarrow \infty} \frac{\sum_{j=1}^k \frac{(s+j)^i (i-1)! x^{\frac{1}{s+j}}}{\log^i x}}{\frac{s^m (m-1)! x^{\frac{1}{s}}}{\log^m x}} = 0 \quad (i = 1, \dots, m).$$

On the other hand, from the lemma we have the following inequality

$$\left| \sum_{j=1}^k \varepsilon \left(x^{\frac{1}{s+j}}\right) \frac{(s+j)^m (m-1)! x^{\frac{1}{s+j}}}{\log^m x} \right| \leq M \sum_{j=1}^k \frac{(s+j)^m (m-1)! x^{\frac{1}{s+j}}}{\log^m x}.$$

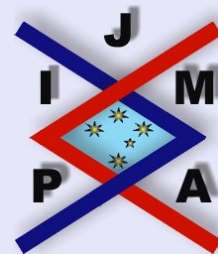
This inequality and (2.3) with $i = m$ give

$$(2.4) \quad \lim_{x \rightarrow \infty} \frac{\sum_{j=1}^k \varepsilon \left(x^{\frac{1}{s+j}}\right) \frac{(s+j)^m (m-1)! x^{\frac{1}{s+j}}}{\log^m x}}{\frac{s^m (m-1)! x^{\frac{1}{s}}}{\log^m x}} = 0.$$

Finally, from (2.2), (2.3) and (2.4) we find that

$$\pi_s(x) = \left(\sum_{i=1}^m \frac{s^i (i-1)! x^{\frac{1}{s}}}{\log^i x} \right) + \varepsilon'(x) \frac{s^m (m-1)! x^{\frac{1}{s}}}{\log^m x},$$

where $\lim_{x \rightarrow \infty} \varepsilon'(x) = 0$. The theorem is proved. \square



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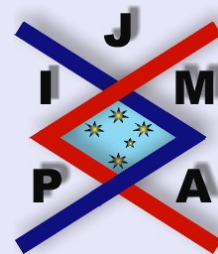
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From (1.4) and (2.1) we obtain the following corollary

Corollary 2.2. *The functions $\pi_s(x)$ and $\pi\left(x^{\frac{1}{s}}\right)$ have the same asymptotic behaviour ($s \geq 1$).*



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3. The Sequences $(t_{s,n})^{\frac{1}{s}}$ and p_n

Theorem 3.1.

$$(3.1) \quad (t_{s,n})^{\frac{1}{s}} = p_n + o\left(\frac{n}{\log^r n}\right) \quad (r \geq 0).$$

Proof. We proceed by mathematical induction on r .

Equation (2.1) gives ($m = 1$)

$$(3.2) \quad \lim_{x \rightarrow \infty} \frac{\pi_s(x)}{\frac{sx^{\frac{1}{s}}}{\log x}} = 1.$$

If we put $x = t_{s,n}$, we get

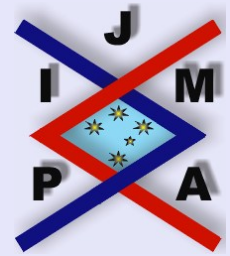
$$(3.3) \quad \lim_{n \rightarrow \infty} \frac{(t_{s,n})^{\frac{1}{s}}}{n \log (t_{s,n})^{\frac{1}{s}}} = 1.$$

From (3.3) we find that

$$(3.4) \quad \lim_{n \rightarrow \infty} \left(\log s + \log (t_{s,n})^{\frac{1}{s}} - \log n - \log \log t_{s,n} \right) = 0.$$

Now, since

$$(3.5) \quad \lim_{n \rightarrow \infty} \frac{\log (t_{s,n})^{\frac{1}{s}}}{\log n} = 1,$$



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we obtain

$$\lim_{n \rightarrow \infty} \frac{(t_{s,n})^{\frac{1}{s}}}{n \log (t_{s,n})^{\frac{1}{s}}} = 1$$

if and only if

$$\lim_{n \rightarrow \infty} \frac{(t_{s,n})^{\frac{1}{s}}}{n \log n} = 1.$$

We also derive

$$\lim_{n \rightarrow \infty} \frac{t_{s,n}}{n^s \log^s n} = 1.$$

From (3.5) we find that

$$(3.6) \quad \lim_{n \rightarrow \infty} (-\log s + \log \log t_{s,n} - \log \log n) = 0.$$

(3.4) and (3.6) give

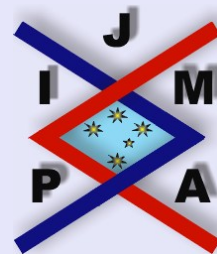
$$(3.7) \quad \log (t_{s,n})^{\frac{1}{s}} = \log n + \log \log n + o(1).$$

Equation (2.1) gives ($m = 2$)

$$\pi_s(x) = \frac{x^{\frac{1}{s}}}{\log x^{\frac{1}{s}}} + \frac{x^{\frac{1}{s}}}{\log^2 \left(x^{\frac{1}{s}}\right)} + o\left(\frac{x^{\frac{1}{s}}}{\log^2 \left(x^{\frac{1}{s}}\right)}\right),$$

so

$$x^{\frac{1}{s}} = \pi_s(x) \log x^{\frac{1}{s}} - \frac{x^{\frac{1}{s}}}{\log x^{\frac{1}{s}}} + o\left(\frac{x^{\frac{1}{s}}}{\log x^{\frac{1}{s}}}\right).$$



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If we put $x = t_{s,n}$, we get

$$(3.8) \quad (t_{s,n})^{\frac{1}{s}} = n \log (t_{s,n})^{\frac{1}{s}} - \frac{(t_{s,n})^{\frac{1}{s}}}{\log (t_{s,n})^{\frac{1}{s}}} + o\left(\frac{(t_{s,n})^{\frac{1}{s}}}{\log (t_{s,n})^{\frac{1}{s}}}\right).$$

Finally, from (3.8), (3.3) and (3.7) we find that

$$(3.9) \quad (t_{s,n})^{\frac{1}{s}} = n \log n + n \log \log n - n + o(n).$$

Therefore, for $r = 0$ the theorem is true because of (1.2) and (3.9).

Let $r \geq 0$ be given, and assume that the theorem holds for r , we will prove it is also true for $r + 1$.

From the inductive hypothesis we have (in view of (1.1))

$$(3.10) \quad p_n = n \log n + n \log \log n - n + \sum_{j=1}^r \frac{(-1)^{j-1} n P_j(\log \log n)}{\log^j n} + o\left(\frac{n}{\log^r n}\right),$$

and

$$(3.11) \quad (t_{s,n})^{\frac{1}{s}} = n \log n + n \log \log n - n + \sum_{j=1}^r \frac{(-1)^{j-1} n P_j(\log \log n)}{\log^j n} + o\left(\frac{n}{\log^r n}\right).$$



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From (3.10) we find that

$$(3.12) \quad \log p_n = \log n + \log \log n + \log \left[1 + \frac{\log \log n - 1}{\log n} + \sum_{j=1}^r \frac{(-1)^{j-1} P_j(\log \log n)}{\log^{j+1} n} + o\left(\frac{1}{\log^{r+1} n}\right) \right].$$

Let us write (1.3) in the form

$$(3.13) \quad \pi(x) = \left(\sum_{i=1}^{r+3} \frac{(i-1)!x}{\log^i x} \right) + o\left(\frac{x}{\log^{r+3} x}\right).$$

If we put $x = p_n$ and use the prime number theorem, we get

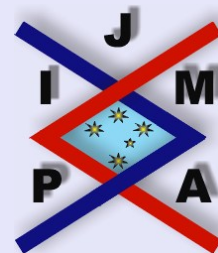
$$(3.14) \quad \frac{n}{p_n} = \left(\sum_{i=1}^{r+3} \frac{(i-1)!}{\log^i p_n} \right) + o\left(\frac{1}{\log^{r+3} n}\right).$$

Similarly, from (3.11) we find that

$$(3.15) \quad \log(t_{s,m})^{\frac{1}{s}} = \log n + \log \log n + \log \left[1 + \frac{\log \log n - 1}{\log n} + \sum_{j=1}^r \frac{(-1)^{j-1} P_j(\log \log n)}{\log^{j+1} n} + o\left(\frac{1}{\log^{r+1} n}\right) \right].$$

Let us write (2.1) in the form

$$(3.16) \quad \pi_s(x) = \left(\sum_{i=1}^{r+3} \frac{(i-1)!x^{\frac{1}{s}}}{\log^i(x^{\frac{1}{s}})} \right) + o\left(\frac{x^{\frac{1}{s}}}{\log^{r+3}(x^{\frac{1}{s}})}\right).$$



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If we put $x = t_{s,n}$ and use (3.5), we get

$$(3.17) \quad \frac{n}{(t_{s,n})^{\frac{1}{s}}} = \left(\sum_{i=1}^{r+3} \frac{(i-1)!}{\log^i \left((t_{s,n})^{\frac{1}{s}} \right)} \right) + o \left(\frac{1}{\log^{r+3} n} \right).$$

If $x \geq 1$ and $y \geq 1$, Lagrange's theorem gives us the inequality

$$|\log y - \log x| \leq |y - x|$$

with (3.12) and (3.15), it leads to

$$(3.18) \quad \log (t_{s,n})^{\frac{1}{s}} - \log p_n = o \left(\frac{1}{\log^{r+1} n} \right).$$

From (3.18) we find that

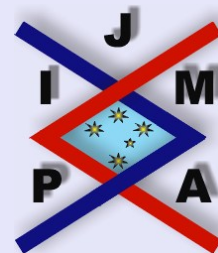
$$(3.19) \quad \frac{1}{\log^k p_n} - \frac{1}{\log^k (t_{s,n})^{\frac{1}{s}}} = o \left(\frac{1}{\log^{r+k+2} n} \right) \\ = o \left(\frac{1}{\log^{r+3} n} \right) \quad (k = 1, \dots, r+3).$$

(3.14), (3.17) and (3.19) give

$$\frac{n}{p_n} - \frac{n}{(t_{s,n})^{\frac{1}{s}}} = o \left(\frac{1}{\log^{r+3} n} \right),$$

that is

$$(3.20) \quad (t_{s,n})^{\frac{1}{s}} - p_n = (t_{s,n})^{\frac{1}{s}} \frac{1}{\log^{r+2} n} o(1).$$



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If we write

$$(3.21) \quad (t_{s,n})^{\frac{1}{s}} = p_n + f(n)$$

substituting (3.21) into (3.20) we find that

$$f(n) = \frac{p_n}{\log^{r+2} n + o(1)},$$

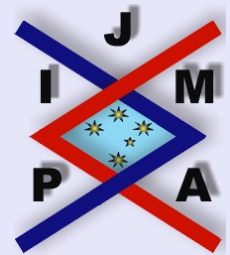
so

$$(3.22) \quad f(n) = o\left(\frac{n}{\log^{r+1} n}\right)$$

(3.21) and (3.22) give

$$(t_{s,n})^{\frac{1}{s}} = p_n + o\left(\frac{n}{\log^{r+1} n}\right).$$

The theorem is thus proved. □



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4. The Asymptotic Behaviour of $t_{s,n}$

Theorem 4.1. *There exists a unique sequence $P_{s,j}(X)$ ($j \geq 1$) of polynomials with rational coefficients such that, for every nonnegative integer m*

$$(4.1) \quad t_{s,n} = \sum c_i f_i(n) + \sum_{j=1}^m \frac{(-1)^{j-1} n^s P_{s,j}(\log \log n)}{\log^j n} + o\left(\frac{n^s}{\log^m n}\right).$$

The polynomials $P_{s,j}(X)$ have degree $j + s - 1$ and leading coefficient $\frac{1}{\binom{j+s-1}{s}}$.

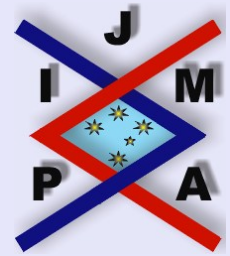
The $f_i(n)$ are sequences of the form $n^s \log^r n (\log \log n)^u$ and the c_i are constants. $f_1(n) = n^s \log^s n$ and $c_1 = 1$, if $i \neq 1$ then $f_i(n) = o(f_1(n))$.

If $m = 0$ equation (4.1) is

$$(4.2) \quad t_{s,n} = \sum c_i f_i(n) + o(n^s).$$

Proof. From (1.1) and (3.1) we obtain (4.1)

$$\begin{aligned} t_{s,n} &= \left[n \log n + n \log \log n - n + \sum_{j=1}^{m+s-1} \frac{(-1)^{j-1} n P_j(\log \log n)}{\log^j n} \right. \\ &\quad \left. + o\left(\frac{n}{\log^{m+s-1} n}\right) \right]^s \\ &= \sum c_i f_i(n) + \sum_{j=1}^m \frac{(-1)^{j-1} n^s P_{s,j}(\log \log n)}{\log^j n} + o\left(\frac{n^s}{\log^m n}\right) \end{aligned}$$



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if we write

$$(4.3) \quad P_{s,j}(X) = \sum_{(r,k)} \sum_{j_1+\dots+j_t=j+r} (-1)^{r-t+1} \binom{s}{r} \binom{s-r}{k} (X-1)^k P_{j_1}(X) \cdots P_{j_t}(X),$$

where $r + k + t = s$.

The first sum runs through the vectors (r, k) ($r \geq 0, k \geq 0, r + k \in \{0, 1, \dots, s-1\}$), such that the set of vectors (j_1, j_2, \dots, j_t) whose coordinates are positive integers which satisfy $j_1 + j_2 + \dots + j_t = j + r$ is nonempty. The second sum runs through the former nonempty set of vectors (j_1, j_2, \dots, j_t) (this set depends on the vector (r, k)).

If $m = 0$ we obtain (4.2).

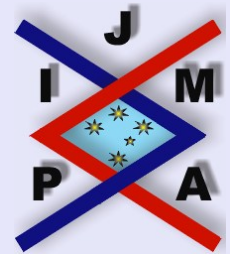
Let us consider a vector (r, k) . The degree of each polynomial

$$(-1)^{r-t+1} \binom{s}{r} \binom{s-r}{k} (X-1)^k P_{j_1}(X) \cdot P_{j_2}(X) \cdots P_{j_t}(X)$$

is $j + r + k$. Hence the degree of the polynomial

$$(4.4) \quad \sum_{j_1+j_2+\dots+j_t=j+r} (-1)^{r-t+1} \binom{s}{r} \binom{s-r}{k} \times (X-1)^k P_{j_1}(X) \cdot P_{j_2}(X) \cdots P_{j_t}(X)$$

does not exceed $j + r + k$. Since $r + k \in \{0, 1, \dots, s-1\}$, the greatest degree of the polynomials (4.4) does not exceed $j + s - 1$. On the other hand, in (4.3)



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there are s polynomials (4.4) of degree $j + s - 1$. Since in this case $t = 1$, these s polynomials are

$$(-1)^r \binom{s}{r} \binom{s-r}{k} (X-1)^k P_{j+r}(X) \quad (r+k=s-1)$$

and their sum is

$$(4.5) \quad \sum_{r=0}^{s-1} (-1)^r \binom{s}{r} \binom{s-r}{s-r-1} (X-1)^{s-r-1} P_{j+r}(X) \\ = \sum_{r=0}^{s-1} (-1)^r \binom{s}{r} (s-r) (X-1)^{s-r-1} P_{j+r}(X).$$

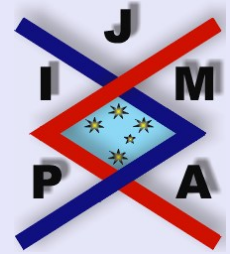
Since the leading coefficient of the polynomial $P_{j+r}(X)$ is $\frac{1}{j+r}$, the leading coefficient of the polynomial (4.5) will be

$$\sum_{r=0}^{s-1} (-1)^r \binom{s}{r} \frac{s-r}{j+r} = \frac{1}{\binom{j+s-1}{s}}.$$

Hence the degree of the polynomial (4.3) is $j + s - 1$ and its leading coefficient is $\frac{1}{\binom{j+s-1}{s}}$. The theorem is thus proved. \square

Examples.

$$t_{1,n} = n \log n + n \log \log n - n + \sum_{j=1}^m \frac{(-1)^{j-1} n P_j(\log \log n)}{\log^j n} + o\left(\frac{n}{\log^m n}\right),$$



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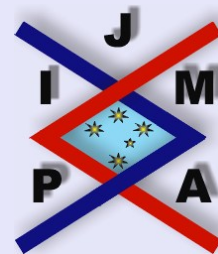
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$$t_{2,n} = n^2 \log^2 n + 2n^2 \log n \log \log n - 2n^2 \log n + n^2 (\log \log n)^2 - 3n^2 + \sum_{j=1}^m \frac{(-1)^{j-1} n^2 P_{2,j}(\log \log n)}{\log^j n} + o\left(\frac{n^2}{\log^m n}\right).$$

Corollary 4.2. *The sequences $t_{s,n}$ and p_n^s ($s \geq 1$) have the same asymptotic expansion, namely (4.1).*

Note. G. Mincu [2] proved Theorem 3.1 and Theorem 4.1 when $s = 2$.



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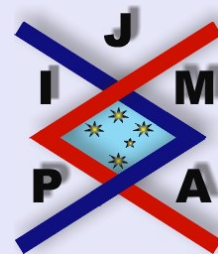
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