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A GENERALIZATION OF AN INEQUALITY OF JIA AND CAU

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ABSTRACT. Let L, H_r, and A_s stand for the logarithmic mean, the Heronian mean of order r, and the power mean of order s, of two positive variables. A generalization of the inequality

$$L \le H_r \le A_s$$

$(1/2 \le r \le 3s/2)$, of G. Jia and J. Cao ([3]), is obtained.

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1. INTRODUCTION AND DEFINITIONS

Let x and y be positive numbers. The Heronian mean of order $a \in \mathbb{R}$ of x and y, denoted by $H_a \equiv H_a(x, y)$, is defined as

$$H_a = \begin{cases} \left(\dfrac{x^a + (xy)^{a/2} + y^a}{3}\right)^{\frac{1}{a}}, & a \neq 0 \\ G, & a = 0, \end{cases}$$

where $G = \sqrt{xy}$ is the geometric mean of x and y. When $a = 1$, we will write H instead of H_1. Let us note that $H = (2A + G)/3$, where $A = (x + y)/2$ is the arithmetic mean of x and y. The logarithmic mean L of x and y and the power mean A_a of order a of x and y are defined as

$$L = \begin{cases} \dfrac{x - y}{\ln x - \ln y}, & x \neq y \\ x, & x = y, \end{cases}$$

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and

$$A_a = \begin{cases} \left(\dfrac{x^a + y^a}{2}\right)^{\frac{1}{a}}, & a \neq 0 \\ G, & a = 0, \end{cases}$$

respectively. Throughout the sequel the means of order one will be denoted by a single letter with the subscript 1 being omitted.

In the recent paper [3] the authors have established the following result. Let $\frac{1}{2} \le r \le \frac{3}{2}s$. Then

$$L \le H_r \le A_s. \tag{1.1}$$

All the means mentioned earlier in this section belong to the large family of means introduced by K.B. Stolarsky in [8]. This two-parameter class of means, denoted by $\mathcal{D}_{a,b}$, is defined as follows

$$\mathcal{D}_{a,b} = \begin{cases} \left(\dfrac{b}{a} \cdot \dfrac{x^a - y^a}{x^b - y^b}\right)^{\frac{1}{(a-b)}}, & ab(a-b) \neq 0 \\ \exp\left(-\dfrac{1}{a} + \dfrac{x^a \ln x - y^a \ln y}{x^a - y^a}\right), & a = b \neq 0 \\ \left[\dfrac{x^a - y^a}{a(\ln x - \ln y)}\right]^{\frac{1}{a}}, & a \neq 0,\ b = 0 \\ G, & a = b = 0. \end{cases} \tag{1.2}$$

For later use let us record some formulas which follow from (1.2). We have

$$H_r = \mathcal{D}_{3r/2,r/2},\ A_s = \mathcal{D}_{2s,s},\ L_p = \mathcal{D}_{p,0},\ I_t = \mathcal{D}_{t,t}. \tag{1.3}$$

Here L_p is the logarithmic mean of order p and I_t is called the identric mean of order t.

The inequalities (1.1) can be written in terms of the Stolarsky means as

$$\mathcal{D}_{1,0} \le \mathcal{D}_{3r/2,r/2} \le \mathcal{D}_{2s,s}. \tag{1.1$'$}$$

The goal of this note is to provide a short proof of a general inequality (see (2.1)) which contains (1.1) as a special case.

2. MAIN RESULT

For the reader's convenience, we recall the Comparison Theorem for the Stolarsky means. Two functions

$$k(p,q) = \begin{cases} \dfrac{|p| - |q|}{p - q}, & p \neq q \\ \operatorname{sign}(p), & p = q \end{cases}$$

and

$$l(p,q) = \begin{cases} L(p,q), & p > 0,\ q > 0 \\ 0, & p \cdot q = 0 \end{cases}$$

play a crucial role in the Comparison Theorem which has been established by E.B. Leach and M.C. Sholander [4] and also by Zs. Páles [6].

Theorem 2.1 (Comparison Theorem)**.** *Let $a, b, c, d \in \mathbb{R}$. Then the comparison inequality*

$$\mathcal{D}_{a,b} \le \mathcal{D}_{c,d}$$

holds true if and only if $a+b \le c+d$ *and*

$$\begin{aligned} l(a,b) &\le l(c,d) && \text{if } 0 \le \min(a,b,c,d), \\ k(a,b) &\le k(c,d) && \text{if } \min(a,b,c,d) < 0 < \max(a,b,c,d), \\ -l(-a,-b) &\le -l(-c,-d) && \text{if } \max(a,b,c,d) \le 0. \end{aligned}$$

In what follows the symbols $\mathbb{R}_+$ and $\mathbb{R}_-$ will stand for the nonnegative semi-axis and the nonpositive semi-axis, respectively.

The main result of this note reads as follows.

Theorem 2.2. *Let* $p, q, r, s, t \in \mathbb{R}_+$. *Then the inequalities*

$$\mathcal{D}_{p,q} \le H_r \le \mathcal{D}_{s,t} \tag{2.1}$$

hold true if and only if

$$\max\left\{\frac{p+q}{2},\ (\ln 3)l(p,q)\right\} \le r \le \min\left\{\frac{s+t}{2},\ (\ln 3)l(s,t)\right\}. \tag{2.2}$$

If $p, q, r, s, t \in \mathbb{R}_-$*, then the inequalities* (2.1) *are reversed if and only if*

$$\max\left\{\frac{s+t}{2},\ (-\ln 3)l(-s,-t)\right\} \le r \le \min\left\{\frac{p+q}{2},\ (-\ln 3)l(-p,-q)\right\}. \tag{2.3}$$

Proof. We shall establish the first part of the assertion only. Using the Comparison Theorem we see that the inequalities

$$\mathcal{D}_{p,q} \le \mathcal{D}_{3r/2,r/2} \le \mathcal{D}_{s,t} \tag{2.4}$$

hold true if and only if

$$p+q \le 2r \le s+t \tag{2.5}$$

and

$$l(p,q) \le \frac{r}{\ln 3} \le l(s,t). \tag{2.6}$$

Solving the inequalities for r we obtain (2.2). Since the middle term in (2.4) equals to H_r (see (1.3)), the assertion follows. □

Remark 2.3. Letting $p = 1$, $q = 0$, $s := 2s$ and $t = s$ in (2.1) and next using (1.1′) we obtain the inequalities (1.1).

Corollary 2.4. *Let* $p, q, r, s, t \in \mathbb{R}_+$. *Then the inequalities*

$$L_p \le H_r \le A_s \le I_t \tag{2.7}$$

hold true if and only if $p \le 2r \le 3s \le 2t$.

Proof. Letting $q = 0$, $s := 2s$, and $t = s$ in (2.1) and (2.2) we obtain the first two inequalities in (2.7). It is easy to see, using the Comparison Theorem, that the inequality $\mathcal{D}_{2s,s} \le \mathcal{D}_{t,t}$ is valid if and only if $3s \le 2t$. This completes the proof of the third inequality in (2.7) because of (1.3). □

It is worth mentioning that (2.7) contains two known results: $H \le I$ (see [7]) and $\sqrt{AL} \le A_{2/3} \le I$ (see [5]). Indeed, letting $p = 2$, $r = 1$, $s = \frac{3}{2}$ and $t = 1$ in Corollary 2.4 we obtain

$$\sqrt{AL} \le H \le A_{2/3} \le I. \tag{2.8}$$

Here we have used the formula $L_2 = \sqrt{AL}$.

The celebrated Gauss' arithmetic-geometric mean $AGM \equiv AGM(x, y)$ of $x > 0$ and $y > 0$ is the common limit of two sequences $\{x_n\}_0^\infty$ and $\{y_n\}_0^\infty$, i.e.,

$$AGM = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n,$$

where $x_0 = x$, $y_0 = y$, $x_{n+1} = (x_n + y_n)/2$, $y_{n+1} = \sqrt{x_n y_n}$ $(n \geq 0)$. This important mean is used for numerical evaluation of the complete elliptic integral of the first kind [2]

$$R_K(x^2, y^2) = \frac{2}{\pi}\int_0^{\pi/2} (x^2\cos^2\phi + y^2\sin^2\phi)^{-1/2} d\phi.$$

Gauss' famous result states that $R_K(x^2, y^2) = 1/AGM(x, y)$.

Corollary 2.5. *Let* $x > 0$ *and* $y > 0$. *Then*

$$AGM \leq H_{3/4}. \tag{2.9}$$

Proof. J. Borwein and P. Borwein [1, Prop. 2.7] have proven that $AGM \leq L_{3/2}$. On the other hand, using the first inequality in (2.7) with $p = 3/2$ and $r = 3/4$ we obtain $L_{3/2} \leq H_{3/4}$. Hence (2.9) follows. □

Some results of this note can be used to obtain inequalities involving hyperbolic functions. For instance, using (2.7), (1.3), and (1.2), with $x = e$ and $y = e^{-1}$, we obtain

$$\left(\frac{\sinh p}{p}\right)^{\frac{1}{p}} \leq \left(\frac{2\cosh r + 1}{3}\right)^{\frac{1}{r}} \leq (\cosh s)^{\frac{1}{s}} \leq \exp\left(-\frac{1}{t} + \coth t\right)$$

$(0 < p \leq 2r \leq 3s \leq 2t)$.

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