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SOME INTEGRAL INEQUALITIES INVOLVING TAYLOR'S REMAINDER. II

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ABSTRACT. In this paper, using Grüss' and Chebyshev's inequalities we prove several inequalities involving Taylor's remainder.

Key words and phrases: Taylor's remainder, Grüss' inequality, Chebyshev's inequality.

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1. INTRODUCTION AND LEMMA

This paper is a continuation of our paper [4]. As in [4], our goal is to prove several integral inequalities involving Taylor's remainder. Our method is similar to that used in [4]. However, while in [4] we deduced our inequalities from Steffensen's inequality, in the present paper we use Grüss' and Chebyshev's inequalities. We are thankful to Professor S.S. Dragomir who pointed out that Grüss' and Chebyshev's inequalities were used earlier by G.A. Anastassiou and S.S. Dragomir [2], [3] to obtain results on Taylor's remainder different from but related to the results of this paper. The main results of this paper are Theorems 2.1 and 3.1.

In what follows n denotes a non-negative integer. We will denote by $R_{n,f}(c,x)$ the nth Taylor's remainder of function $f(x)$ with center c, i.e.

$$R_{n,f}(c,x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k.$$

Lemma 1.1. *Let f be a function defined on $[a,b]$. Assume that $f \in C^{n+1}([a,b])$. Then one has the representations*

$$\int_a^b \frac{(b-x)^{n+1}}{(n+1)!} f^{(n+1)}(x)dx = \int_a^b R_{n,f}(a,x)dx, \tag{1.1}$$

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and

$$\int_a^b \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(x)dx = (-1)^{n+1} \int_a^b R_{n,f}(b,x)dx. \tag{1.2}$$

Proof. Observe that:

$$\begin{aligned}
&\int_a^b \frac{(b-x)^{n+1}}{(n+1)!} f^{(n+1)}(x)dx \\
&= \int_a^b \frac{(b-x)^{n+1}}{(n+1)!} df^{(n)}_{(x)} \\
&= f^{(n)}(x)\ \frac{(b-x)^{n+1}}{(n+1)!}\Bigg|_{x=a}^{x=b} + \int_a^b \frac{(b-x)^n}{n!} f^{(n)}(x)dx \\
&= -f^{(n)}(a)\frac{(b-a)^{n+1}}{(n+1)!} + \int_a^b \frac{(b-x)^n}{n!} f^{(n)}(x)dx \\
&= -f^{(n)}(a)\frac{(b-a)^{n+1}}{(n+1)!} - f^{(n-1)}(a)\frac{(b-a)^n}{n!} + \int_a^b \frac{(b-x)^{n-1}}{(n-1)!} f^{n-1}(x)dx \\
&= \cdots \\
&= -f^{(n)}(a)\frac{(b-a)^{n+1}}{(n+1)!} - f^{(n-1)}(a)\frac{(b-a)^n}{n!} - \cdots - f(a)\frac{b-a}{1!} + \int_a^b f(x)dx \\
&= \int_a^b \left[f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \right] dx \\
&= \int_a^b R_{n,f}(a,x)dx.
\end{aligned}$$

The proof of (1.2) is similar to the proof of (1.1) and we omit it. □

2. APPLICATIONS OF GRÜSS' INEQUALITY

The following inequality is called Grüss' inequality [5]:
Let $F(x)$ and $G(x)$ be two functions defined and integrable on $[a,b]$. Further let

$$m \le F(x) \le M \quad \textit{and} \quad \varphi \le G(x) \le \Phi$$

for each $x \in [a,b]$, where m, M, φ, Φ are constants. Then

$$\left| \int_a^b F(x)G(x)dx - \frac{1}{b-a}\int_a^b F(x)dx \cdot \int_a^b G(x)dx \right| \le \frac{b-a}{4}(M-m)(\Phi-\varphi).$$

Theorem 2.1. *Let $f(x)$ be a function defined on $[a,b]$ such that $f(x) \in C^{n+1}([a,b])$ and $m \le f^{(n+1)}(x) \le M$ for each $x \in [a,b]$, where m and M are constants. Then*

$$\left| \int_a^b R_{n,f}(a,x)dx - \frac{f^{(n)}(b) - f^{(n)}(a)}{(n+2)!}(b-a)^{n+1} \right| \le \frac{(b-a)^{n+2}}{4(n+1)!}(M-m) \tag{2.1}$$

and

$$\left| (-1)^{n+1}\int_a^b R_{n,f}(b,x)dx - \frac{f^{(n)}(b) - f^{(n)}(a)}{(n+2)!}(b-a)^{n+1} \right| \le \frac{(b-a)^{n+2}}{4(n+1)!}(M-m). \tag{2.2}$$

Proof. Set $F(x) = f^{(n+1)}(x)$, $G(x) = \frac{(b-x)^{n+1}}{(n+1)!}$. Then $m \le F(x) \le M$ and $0 \le G(x) \le \frac{(b-a)^{n+1}}{(n+1)!}$. By Grüss' inequality,

$$\left|\int_a^b \frac{(b-x)^{n+1}}{(n+1)!} f^{(n+1)}(x)dx - \frac{1}{b-a}\int_a^b f^{(n+1)}(x)dx \cdot \int_a^b \frac{(b-x)^{n+1}}{(n+1)!}dx\right| \le \frac{b-a}{4}\cdot\frac{(b-a)^{n+1}}{(n+1)!}(M-m).$$

Using Lemma 1.1, we obtain

$$\left|\int_a^b R_{n,f}(a,x) - \frac{1}{b-a}\left[f^{(n)}(b) - f^{(n)}(a)\right]\cdot\frac{(b-a)^{n+2}}{(n+2)!}\right| \le \frac{(b-a)^{n+2}}{4(n+1)!}(M-m).$$

That proves (2.1).

To prove (2.2), we set $F(x) = f^{(n+1)}(x)$, $G(x) = \frac{(a-x)^{n+1}}{(n+1)!}$, and continue as in the proof of (2.1). □

Now we consider the simplest cases of Theorem 2.1, namely the cases when $n = 0$ or 1.

Corollary 2.2. *Let $f(x)$ be a function defined on $[a,b]$ such that $f(x) \in C^2([a,b])$ and $m \le f''(x) \le M$ for each $x \in [a,b]$, where m and M are constants. Then*

$$\left|\int_a^b f(x)dx - f(a)(b-a) - \frac{2f'(a)+f'(b)}{6}(b-a)^2\right| \le \frac{(b-a)^3}{8}(M-m), \tag{2.3}$$

$$\left|\int_a^b f(x)dx - f(b)(b-a) + \frac{2f'(b)+f'(a)}{6}(b-a)^2\right| \le \frac{(b-a)^3}{8}(M-m), \tag{2.4}$$

$$\left|\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}(b-a) + \frac{f'(b)-f'(a)}{12}(b-a)^2\right| \le \frac{(b-a)^3}{8}(M-m). \tag{2.5}$$

Proof. To obtain (2.3) and (2.4) we take $n = 1$ in (2.1) and (2.2) of Theorem 2.1. Taking half the sum of (2.3) and (2.4), we obtain (2.5). □

Remark 2.3. Taking $n = 0$ in (2.1) and (2.2), we obtain that if $m \le f'(x) \le M$ on $[a,b]$, then

$$\left|\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}(b-a)\right| \le \frac{(b-a)^2}{4}(M-m).$$

This inequality is weaker than a modification of Iyengar's inequality due to Agarwal and Dragomir [1].

3. APPLICATIONS OF CHEBYSHEV'S INEQUALITY

The following is Chebyshev's inequality [5]:

Let $F, G : [a,b] \to \mathbb{R}$ be integrable functions, both increasing or both decreasing. Then

$$\int_a^b F(x)G(x)dx \ge \frac{1}{b-a}\int_a^b F(x)dx \cdot \int_a^b G(x)dx.$$

If one of the functions is increasing and the other decreasing, then the above inequality is reversed.

Theorem 3.1. *Let $f(x)$ be a function defined on $[a,b]$ such that $f(x) \in C^{(n+1)}([a,b])$.*
If $f^{(n+1)}(x)$ is increasing on $[a,b]$, then,

$$
\begin{aligned}
(3.1)\qquad & -\frac{f^{(n+1)}(b)-f^{(n+1)}(a)}{4(n+1)!}(b-a)^{n+2} \\
&\leq \int_a^b R_{n,f}(a,x)dx - \frac{f^{(n)}(b)-f^{(n)}(a)}{(n+2)!}(b-a)^{n+1} \\
&\leq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
(3.2)\qquad & 0 \leq (-1)^{(n+1)} \int_a^b R_{n,f}(b,x)dx - \frac{f^{(n)}(b)-f^{(n)}(a)}{(n+2)!}(b-a)^{n+1} \\
&\leq \frac{f^{(n+1)}(b)-f^{(n+1)}(a)}{4(n+1)!}(b-a)^{n+2}.
\end{aligned}
$$

If $f^{(n+1)}(x)$ is decreasing on $[a,b]$, then

$$
\begin{aligned}
(3.3)\qquad & 0 \leq \int_a^b R_{n,f}(a,x)dx - \frac{f^{(n)}(b)-f^{(n)}(a)}{(n+2)!}(b-a)^{n+1} \\
&\leq \frac{f^{(n+1)}(a)-f^{(n+1)}(b)}{4(n+1)!}(b-a)^{n+2},
\end{aligned}
$$

and

$$
\begin{aligned}
(3.4)\qquad & -\frac{f^{(n+1)}(a)-f^{(n+1)}(b)}{4(n+1)!}(b-a)^{n+2} \\
&\leq (-1)^{(n+1)} \int_a^b R_{n,f}(b,x)dx - \frac{f^{(n)}(b)-f^{(n)}(a)}{(n+2)!}(b-a)^{n+1} \\
&\leq 0.
\end{aligned}
$$

Proof. Set $F(x) = f^{(n+1)}(x)$ and $G(x) = \frac{(b-x)^{(n+1)}}{(n+1)!}$. Then $F(x)$ is increasing and $G(x)$ decreasing on $[a,b]$. Using Chebyshev's inequality for $F(x)$ and $G(x)$ and (1.1), we obtain right inequality in (3.1). Left inequality in (3.1) follows readily from (2.1), if we take into account that since $f^{(n+1)}(x)$ is increasing on $[a,b]$, $f^{(n+1)}(a) \leq f^{(n+1)}(x) \leq f^{(n+1)}(b)$ for all $x \in [a,b]$.

To prove (3.2), set $F(x) = f^{(n+1)}(x)$ and $G(x) = \frac{(x-a)^{(n+1)}}{(n+1)!}$. The rest of the proof is the same as in the proof of (3.1).

The proofs of (3.3) and (3.4) are similar to those of (3.1) and (3.2) respectively, and we omit them. □

We now consider the simplest cases of Theorem 3.1, namely the cases when $n=0$ or 1.

Corollary 3.2. *Let $f(x)$ be a function defined on $[a,b]$ such that $f(x) \in C^2([a,b])$. If $f''(x)$ is increasing on $[a,b]$, then*

$$
\begin{aligned}
(3.5)\qquad & -\frac{f''(b)-f''(a)}{8}(b-a)^2 \\
&\leq \frac{1}{b-a}\int_a^b f(x)dx - f(a) - \frac{2f'(a)+f'(b)}{6}(b-a) \\
&\leq 0
\end{aligned}
$$

$$0 \le \frac{1}{b-a}\int_a^b f(x)dx - f(b) + \frac{f'(a)+2f'(b)}{6}(b-a) \le \frac{f''(b)-f''(a)}{8}(b-a)^2, \tag{3.6}$$

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{f(a)+f(b)}{2} + \frac{f'(b)-f'(a)}{12}(b-a)\right| \le \frac{f''(b)-f''(a)}{16}(b-a)^2. \tag{3.7}$$

Proof. To obtain (3.5) and (3.6) we take $n = 1$ in (3.1) and (3.2) of Theorem 3.1. We obtain (3.7) taking half the sum of (3.5) and (3.6). □

Remark 3.3. The inequalities similar to (3.5) – (3.7) for the case of decreasing $f''(x)$ can be obtained substituting $-f(x)$ instead of $f(x)$ into inequalities (3.5) – (3.7).

Remark 3.4. Taking $n = 0$ in Theorem 3.1, we obtain that if $f'(x)$ is increasing on $[a, b]$, then

$$\frac{f(a)+f(b)}{2} - \frac{f'(b)-f'(a)}{4}(b-a) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{3.8}$$

Let us compare (3.8) with the following Hermite-Hadamard's inequality [6]:
If $f(x)$ is convex on $[a, b]$ (in particular if $f'(x)$ exists and increasing on $[a, b]$), then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}.$$

We see that the right inequality in (3.8) is the same as the right Hermite-Hadamard's inequality. However, it can be easily proved that the left inequality in (3.8) is weaker than the left Hermite-Hadamard's inequality.

Remark 3.5. Taking the difference of (3.5) and (3.6), we obtain that if $f''(x)$ is increasing on $[a, b]$, then

$$0 \le \frac{f'(a)+f'(b)}{2} - \frac{f(b)-f(a)}{b-a} \le \frac{f''(b)-f''(a)}{4}(b-a).$$

This inequality follows readily if we take $f'(x)$ instead of $f(x)$ in (3.8).

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