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ITERATIVE ALGORITHM FOR A NEW SYSTEM OF NONLINEAR SET-VALUED VARIATIONAL INCLUSIONS INVOLVING (H, η)-MONOTONE MAPPINGS

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Abstract

In this paper, a new system of nonlinear set-valued variational inclusions involving (H,η)-monotone mappings in Hilbert spaces is introduced and studied. By using the resolvent operator method associated with (H,η)-monotone mappings, an existence theorem of solutions for this kind of system of nonlinear set-valued variational inclusion is established and a new iterative algorithm is suggested and discussed. The results presented in this paper improve and generalize some recent results in this field.

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Contents

1. Introduction

Variational inclusions are an important generalization of classical variational inequalities and thus, have wide applications to many fields including, for example, mechanics, physics, optimization and control, nonlinear programming, economics, and the engineering sciences. For these reasons, various variational inclusions have been intensively studied in recent years. For details, we refer the reader to [1] – [21], [23] – [31] and the references therein.

Verma [24, 25] introduced and studied some systems of variational inequalities and developed some iterative algorithms for approximating the solutions of a system of variational inequalities in Hilbert spaces. Recently, Kim and Kim [21] introduced a new system of generalized nonlinear mixed variational inequalities and obtained some existence and uniqueness results for solutions of the system of generalized nonlinear mixed variational inequalities in Hilbert spaces. Very recently, Fang, Huang and Thompson [9] introduced a system of variational inclusions and developed a Mann iterative algorithm to approximate the unique solution of the system.

On the other hand, monotonicity techniques were extended and applied in recent years because of their importance in the theory of variational inequalities, complementarity problems, and variational inclusions. In 2003, Huang and Fang [16] introduced a class of generalized monotone mappings, maximal η-monotone mappings, and defined an associated resolvent operator. Using resolvent operator methods, they developed some iterative algorithms to approximate the solution of a class of general variational inclusions involving maximal η-monotone operators. Huang and Fang's method extended the resolvent operator method associated with an η-subdifferential operator due to Ding and Luo

[6]. In [7], Fang and Huang introduced another class of generalized monotone operators, H-monotone operators, and defined an associated resolvent operator. They also established the Lipschitz continuity of the resolvent operator and studied a class of variational inclusions in Hilbert spaces using the resolvent operator associated with H-monotone operators. In a recent paper [9], Fang, Huang and Thompson further introduced a new class of generalized monotone operators, (H, η)-monotone operators, which provide a unifying framework for classes of maximal monotone operators, maximal η-monotone operators, and H-monotone operators. They also studied a system of variational inclusions using the resolvent operator associated with (H, η)-monotone operators.

Inspired and motivated by recent research works in this field, in this paper, we shall introduce and study a new system of nonlinear set-valued variational inclusions involving (H, η)-monotone mappings in Hilbert spaces. By using the resolvent operator method associated with (H, η)-monotone mappings, an existence theorem for solutions for this type of system of nonlinear set-valued variational inclusion is established and a new iterative algorithm is suggested and discussed. The results presented in this paper improve and generalize some recent results in this field.

2. Preliminaries

Let X be a real Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$, respectively. 2^X and $C(X)$ denote the family of all the nonempty subsets of X and the family of all closed subsets of X, respectively. Let us recall the following definitions and some known results.

Definition 2.1. *Let $T, H : X \to X$ be two single-valued mappings. T is said to be:*

(i) *monotone, if*

$$\langle Tx - Ty, x - y\rangle \geq 0 \qquad \text{for all } x, y \in X;$$

(ii) *strictly monotone, if T is monotone and*

$$\langle Tx - Ty, x - y\rangle = 0$$

if and only if $x = y$;

(iii) *r-strongly monotone, if there exists a constant $r > 0$ such that*

$$\langle T(x) - T(y), x - y\rangle \geq r\|x - y\|^2 \qquad \text{for all } x, y \in X;$$

(iv) *s-strongly monotone with respect to H, if there exists a constant $s > 0$ such that*

$$\langle T(x) - T(y), H(x) - H(y)\rangle \geq S\|x - y\|^2 \qquad \text{for all } x, y \in X;$$

(v) *t-Lipschitz continuous, if there exists a constant $t > 0$ such that*

$$\|T(x) - T(y)\| \leq t\|x - y\| \qquad \text{for all } x, y \in X.$$

Definition 2.2. *A single-valued mapping $\eta : X \times X \to X$ is said to be:*

(i) *monotone, if*

$$\langle x - y, \eta(x, y)\rangle \geq 0 \qquad \text{for all } x, y \in X;$$

(ii) *strictly monotone, if*

$$\langle x - y, \eta(x, y)\rangle \geq 0 \qquad \text{for all } x, y \in X$$

and equality holds if and only if $x = y$;

(iii) *δ-srongly monotone, if there exists a constant $\delta > 0$ such that*

$$\langle x - y, \eta(x, y)\rangle \geq \delta\|x - y\|^2 \qquad \text{for all } x, y \in X;$$

(iv) *τ-Lipschitz continuous, if there exists a constant $\tau > 0$ such that*

$$\|\eta(x, y)\| \leq \tau\|x - y\|, \qquad \text{for all } x, y \in X.$$

Definition 2.3. *Let $\eta : X \times X \to X$ and $H : X \to X$ be two single-valued mappings. A set-valued mapping $M : X \to 2^X$ is said to be:*

(i) *monotone, if*

$$\langle u - v, x - y\rangle \geq 0, \qquad \forall x, y \in X, \quad u \in Mx, v \in My;$$

(ii) η-monotone, if

$$\langle u - v, \eta(x, y)\rangle \geq 0 \qquad \forall x, y \in X, u \in Mx, v \in My;$$

(iii) strictly η-monotone, if M is η-monotone and equality holds if and only if $x = y$;

(iv) r-strongly η-monotone, if there exists a constant $r > 0$ such that

$$\langle u - v, \eta(x, y)\rangle \geq r\|x - y\|^2 \qquad \forall x, y \in X, u \in Mx, v \in My;$$

(v) maximal monotone, if M is monotone and $(I + \lambda M)(X) = X$, for all $\lambda > 0$, where I denotes the identity mapping on X;

(vi) maximal η-monotone, if M is η-monotone and $(I + \lambda M)(X) = X$, for all $\lambda > 0$;

(vii) H-monotone, if M is monotone and $(H + \lambda M)(X) = X$, for all $\lambda > 0$;

(viii) (H, η)-monotone, if M is η-monotone and $(H + \lambda M)(X) = X$, for all $\lambda > 0$.

Remark 1. *Maximal η-monotone mappings, H-monotone mappings, and (H, η)-monotone mappings were first introduced in Huang and Fang [16], Fang and Huang [7, 9], respectively. Obviously, the class of (H, η)- monotone mappings provides a unifying framework for classes of maximal monotone mappings, maximal η-monotone mappings, and H-monotone mappings. For details about these mappings, we refer the reader to [6, 7, 9, 16] and the references therein.*

Lemma 2.1 ([9]). *Let $\eta : X \times X \to X$ be a single-valued mapping, $H : X \to X$ be a strictly η-monotone mapping and $M : X \to 2^X$ an (H, η)-monotone mapping. Then the mapping $(H + \lambda M)^{-1}$ is single-valued.*

By Lemma 2.1, we can define the resolvent operator $R^{H,\eta}_{M,\lambda}$ as follows.

Definition 2.4 ([9]). *Let $\eta : X \times X \to X$ be a single-valued mapping, $H : X \to X$ a strictly η-monotone mapping and $M : X \to 2^X$ an (H, η)-monotone mapping. The resolvent operator $R^{H,\eta}_{M,\lambda} : X \to X$ is defined by*

$$R^{H,\eta}_{M,\lambda}(z) = (H + \lambda M)^{-1}(z) \qquad \text{for all } z \in X,$$

where $\lambda > 0$ is a constant.

Remark 2.

(i) When $H = I$, Definition 2.4 reduces to the definition of the resolvent operator of a maximal η-monotone mapping, see [16].

(ii) When $\eta(x, y) = x - y$ for all $x, y \in X$, Definition 2.4 reduces to the definition of the resolvent operator of a H-monotone mapping, see [7].

(iii) When $H = I$ and $\eta(x, y) = x - y$ for all $x, y \in X$, Definition 2.4 reduces to the definition of the resolvent operator of a maximal monotone mapping, see [31].

Lemma 2.2 ([9]). *Let $\eta : X \times X \to X$ be a τ-Lipschtiz continuous mapping, $H : X \to X$ be an (r, η)-strongly monotone mapping and $M : X \to 2^X$ be* *an* (H, η)*-monotone mapping. Then the resolvent operator* $R_{M,\lambda}^{H,\eta} : X \to X$ *is* τ/r*-Lipschitz continuous, that is,*

$$\left\| R_{M,\lambda}^{H,\eta}(x) - R_{M,\lambda}^{H,\eta}(y) \right\| \le \frac{\tau}{r} \|x - y\| \qquad \text{for all } x, y \in X.$$

We define a Hausdorff pseudo-metric $D : 2^X \times 2^X \to (-\infty, +\infty) \cup \{+\infty\}$ by

$$D(\cdot,\cdot) = \max\left\{ \sup_{u\in A} \inf_{v\in B} \|u - v\|, \sup_{u\in B} \inf_{v\in A} \|u - v\| \right\}$$

for any given $A, B \in 2^X$. Note that if the domain of D is restricted to closed bounded subsets, then D is the Hausdorff metric.

Definition 2.5. *A set-valued mapping* $A : X \to 2^X$ *is said to be* D*-Lipschitz continuous if there exists a constant* $\eta > 0$ *such that*

$$D(A(u), A(v)) \le \eta \|u - v\|, \qquad \text{for all } u, v \in X.$$

3. System of Variational Inclusions

In this section, we shall introduce a new system of set-valued variational inclusions involving (H,η)-monotone mappings in Hilbert spaces. In what follows, unless other specified, we shall suppose that X_1 and X_2 are two real Hilbert spaces, $K_1 \subset X_1$ and $K_2 \subset X_2$ are two nonempty, closed and convex sets. Let $F : X_1 \times X_2 \to X_1$, $G : X_1 \times X_2 \to X_2$, $H_i : X_i \to X_i$, $\eta_i : X_i \times X_i \to X_i$ $(i = 1,2)$ be nonlinear mappings. Let $A : X_1 \to 2^{X_1}$ and $B : X_2 \to 2^{X_2}$ be set-valued mappings, $M_i : X_i \to 2^{X_i}$ be (H_i,η_i)-monotone mappings $(i = 1,2)$. The system of nonlinear set-valued variational inclusions is formulated as follows. Find $(a,b) \in X_1 \times X_2$, $u \in A(a)$ and $v \in B(b)$ such that

$$\begin{cases} 0 \in F(a,v) + M_1(a) \\ 0 \in G(u,b) + M_2(b) \end{cases} \tag{3.1}$$

Special Cases

Case 1. *If $M_1(x) = \partial\varphi(x)$ and $M_2 = \partial\phi(y)$ for all $x \in X_1$ and $y \in X_2$, where $\varphi : X_1 \to R \cup \{+\infty\}$ and $\phi : X_2 \to R \cup \{+\infty\}$ are two proper, convex and lower semi-continuous functionals, $\partial\varphi$ and $\partial\phi$ denote the subdifferential operators of φ and ϕ, respectively, then problem (3.1) reduces to the following problem: find $(a,b) \in X_1 \times X_2$, $u \in A(a)$, and $v \in B(v)$ such that*

$$\begin{cases} \langle F(a,v), x-a\rangle + \varphi(x) - \varphi(a) \geq 0, \quad \forall x \in X_1, \\ \langle G(u,b), y-a\rangle + \phi(y) - \phi(b) \geq 0, \quad \forall y \in X_2, \end{cases} \tag{3.2}$$

which is called a system of set-valued mixed variational inequalities. Some special cases of problem (3.2) can be found in [26].

Case 2. *If A and B are both identity mappings, then problem (3.2) reduces to the following problem: find $(a,b) \in X_1 \times X_2$ such that*

$$(3.3)\qquad \begin{cases} \langle F(a,b), x-a\rangle + \varphi(x) - \varphi(a) \geq 0, & \forall x \in X_1, \\ \langle G(a,b), y-a\rangle + \phi(y) - \phi(b) \geq 0, & \forall y \in X_2, \end{cases}$$

which is called system of nonlinear variational inequalities considered by Cho, Fang, Huang and Hwang [5]. Some special cases of problem (3.3) were studied by Kim and Kim [21], and Verma [24].

Case 3. *If $M_1(x) = \partial\delta_{K_1}(x)$ and $M_2(y) = \partial\delta_{K_2}(y)$, for all $x \in K_1$ and $y \in K_2$, where $K_1 \subset X_1$ and $K_2 \subset X_2$ are two nonempty, closed, and convex subsets, and δ_{K_1} and δ_{K_2} denote the indicator functions of K_1 and K_2, respectively. Then problem (3.2) reduces to the following system of variational inequalities: find $(a,b) \in K_1 \times K_2$ such that*

$$(3.4)\qquad \begin{cases} \langle F(a,b), x-a\rangle \geq 0, & \forall x \in K_1, \\ \langle G(a,b), y-a\rangle \geq 0, & \forall y \in K_2, \end{cases}$$

which is the problem in [20] with both F and G being single-valued.

Case 4. *If $X_1 = X_2 = X$, $K_1 = K_2 = K$, $F(X,y) = \rho T(y) + x - y$, and $G(x,y) = \gamma T(x) + y - x$, for all $x, y \in X$, where $T : K \to X$ is a nonlinear mapping, $\rho > 0$ and $\gamma > 0$ are two constants, then problem (3.4) reduces to the following system of variational inequalities: find $(a,b) \in K \times K$ such that*

$$(3.5)\qquad \begin{cases} \langle \rho T(b) + a - b, x-a\rangle \geq 0, & \forall x \in K, \\ \langle \gamma T(a) + b - a, x-b\rangle \geq 0, & \forall x \in K, \end{cases}$$

which is the system of nonlinear variational inequalities considered by Verma [25].

Case 5. *If A and B are both identity mappings, the problem (3.1) reduces to the following problem: $(a,b) \in X_1 \times X_2$ such that*

$$\begin{cases} 0 \in F(a,b) + M_1(a) \\ 0 \in G(a,b) + M_2(b) \end{cases} \tag{3.6}$$

which is the system of variational inclusions considered by Fang, Huang and Thompson [9].

4. Iterative Algorithm and Convergence

In this section, by using the resolvent operator method associated with (H, η)-monotone mappings, a new iterative algorithm for solving problem (3.1) is suggested. The convergence of the iterative sequence generated by the algorithm is proved.

Theorem 4.1. *For given $(a, b) \in X_1 \times X_2$, $u \in A(a)$, $v \in B(b)$, (a, b, u, v) is a solution of problem (3.1) if and only if (a, b, u, v) satisfies the relation*

$$\text{(4.1)} \qquad \begin{cases} a = R^{H_1,\eta_1}_{M_1,\rho_1}[H_1(a) - \rho_1 F(a, v)], \\ \\ b = R^{H_2,\eta_2}_{M_2,\rho_2}[H_2(b) - \rho_2 G(u, b)], \end{cases}$$

where $\rho_i > 0$ are two constants for $i = 1, 2$.

Proof. This directly follows from Definition 2.4. □

The relation (4.1) and Nadler [22] allows us to suggest the following iterative algorithm.

Algorithm 1.

Step 1. *Choose $(a_0, b_0) \in X_1 \times X_2$ and choose $u_0 \in A(a_0)$ and $v_0 \in B(b_0)$.*

Step 2. *Let*

$$\text{(4.2)} \qquad \begin{cases} a_{n+1} = (1-\lambda)a_n + \lambda R^{H_1,\eta_1}_{M_1,\rho_1}[H_1(a_n) - \rho_1 F(a_n, v_n)], \\ \\ b_{n+1} = (1-\lambda)b_n + \lambda R^{H_2,\eta_2}_{M_2,\rho_2}[H_2(b_n) - \rho_2 G(u_n, b_n)], \end{cases}$$

where $0 < \lambda \leq 1$ is a constant.

Step 3. *Choose* $u_{n+1} \in A(a_{n+1})$ *and* $v_{n+1} \in B(b_{n+1})$ *such that*

$$(4.3)\qquad \begin{cases} \|u_{n+1} - u_n\| \le (1 + (1+n)^{-1}) D_1(A(a_{n+1}), A(a_n)), \\ \\ \|v_{n+1} - v_n\| \le (1 + (1+n)^{-1}) D_2(B(b_{n+1}), B(b_n)), \end{cases}$$

where $D_i(\cdot,\cdot)$ *is the Hausdorff pseudo-metric on* 2^{X_i} *for* $i = 1, 2$.

Step 4. *If* $a_{n+1}, b_{n+1}, u_{n+1}$ *and* v_{n+1} *satisfy (4.2) to sufficient accuracy, stop; otherwise, set* $n := n + 1$ *and return to Step 2.*

Theorem 4.2. *Let* $\eta_i : X_i \times X_i \to X_i$ *be* τ_i*-Lipschitz continuous mappings,* $H_i : X_i \to X_i$ (r_i, η)*-strongly monotone and* β_i*-Lipschitz continuous mappings,* $M_i : X_i \to 2_i^X$ *be* (H_i, η_i)*-monotone mappings for* $i = 1, 2$*. Let* $A : X_1 \to C(X_1)$ *be* D_1*-*γ_1*-Lipschitz continuous and* $B : X_2 \to C(X_2)$ *be* D_2*-*γ_2*-Lipschitz continuous. Let* $F : X_1 \times X_2 \to X_1$ *be a nonlinear mapping such that for any given* $(a, b) \in X_1 \times X_2$*,* $F(\cdot, b)$ *is* μ_1*-strongly monotone with respect to* H_1 *and* α_1*-Lipschitz continuous and* $F(a, \cdot)$ *is* ζ_1*-Lipschitz continuous. Let* $G : X_1 \times X_2 \to X_2$ *be a nonlinear mapping such that for any given* $(x, y) \in X_1 \times X_2$*,* $G(x, \cdot)$ *is* μ_2*-strongly monotone with respect to* H_2 *and* α_2*-Lipschitz continuous and* $G(\cdot, y)$ *is* ζ_2*-Lipschitz continuous. If there exist constants* $\rho_i > 0$ *for* $i = 1, 2$ *such that*

$$(4.4)\qquad \begin{cases} \tau_1 r_2 \sqrt{\beta_1^2 - 2\rho_1\mu_1 + \rho_1^2\alpha_1^2} + \tau_2 r_1 \zeta_2 \gamma_1 < r_1 r_2, \\ \\ \tau_2 r_1 \sqrt{\beta_2^2 - 2\rho_2\mu_2 + \rho_2^2\alpha_2^2} + \tau_1 r_2 \zeta_1 \gamma_2 < r_1 r_2, \end{cases}$$

then problem (3.1) admits a solution (a, b, u, v) *and iterative sequences* $\{a_n\}$*,*

$\{b_n\}$, $\{u_n\}$ *and* $\{v_n\}$ *converge strongly to* a, b, u *and* v*, respectively, where* $\{a_n\}, \{b_n\}, \{u_n\}$ *and* $\{v_n\}$ *are the sequences generated by Algorithm 1.*

Proof. It follows from (4.2) and Lemma 2.2 that

$$\begin{aligned}
&\|a_{n+1} - a_n\| \\
&= \Big\|(1-\lambda)a_n + \lambda R^{H_1,\eta_1}_{M_1,\rho_1}(H_1(a_n) - \rho_1 F(a_n, v_n)) \\
&\qquad - \left[(1-\lambda)a_{n-1} + \lambda R^{H_1,\eta_1}_{M_1,\rho_1}(H_1(a_{n-1}) - \rho_1 F(a_{n-1}, v_{n-1}))\right]\Big\| \\
&\le (1-\lambda)\|a_n - a_{n-1}\| + \lambda \Big\|R^{H_1,\eta_1}_{M_1,\rho_1}(H_1(a_n) - \rho_1 F(a_n, v_n)) \\
&\qquad - R^{H_1,\eta_1}_{M_1,\rho_1}(H_1(a_{n-1}) - \rho_1 F(a_{n-1}, v_{n-1}))\Big\| \\
&\le (1-\lambda)\|a_n - a_{n-1}\| \\
&\qquad + \lambda \frac{\tau_1}{r_1}\|H_1(a_n) - H_1(a_{n-1}) - \rho_1[F(a_n, v_n) - F(a_{n-1}, v_{n-1})]\| \\
&\le (1-\lambda)\|a_n - a_{n-1}\| \\
&\qquad + \lambda \frac{\tau_1}{r_1}(\|H_1(a_n) - H_1(a_{n-1}) - \rho_1[F(a_n, v_n) - F(a_{n-1}, v_n)]\| \\
&\qquad + \|F(a_{n-1}, v_n) - F(a_{n-1}, v_{n-1})\|). \qquad (4.5)
\end{aligned}$$

Similarly, we can prove that

$$\begin{aligned}
\|b_{n+1} - b_n\| &\le (1-\lambda)\|b_n - b_{n-1}\| \\
&\quad + \lambda \frac{\tau_2}{r_2}(\|H_2(b_n) - H_2(b_{n-1}) - \rho_2[G(u_n, b_n) - G(u_n, b_{n-1})]\| \\
&\qquad + \|G(u_n, b_{n-1}) - G(u_{n-1}, b_{n-1})\|). \qquad (4.6)
\end{aligned}$$

Since H_i are β_i-Lipschitz continuous for $i = 1, 2$, $F(\cdot, b)$ is μ_1-strongly monotone with respect to H_1 and α_1-Lipschitz continuous, $G(x, \cdot)$ is μ_2-strongly monotone with respect to H_2 and α_2-Lipschitz continuous, we obtain

$$\begin{aligned} &\|H_1(a_n) - H_1(a_{n-1}) - \rho_1[F(a_n, v_n) - F(a_{n-1}, v_n)]\|^2 \\ &= \|H_1(a_n) - H_1(a_{n-1})\|^2 \\ &\qquad - 2\rho_1\langle F(a_n, v_n) - F(a_{n-1}, v_n), H_1(a_n) - H_1(a_{n-1})\rangle \\ &\qquad\qquad + \rho_1^2\|F(a_n, v_n) - F(a_{n-1}, v_n)\|^2 \\ &\leq (\beta_1^2 - 2\rho_1\mu_1 + \rho_1^2\alpha_1^2)\|a_n - a_{n-1}\|^2 \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} &\|H_2(b_n) - H_2(b_{n-1}) - \rho_2[G(u_n, b_n) - G(u_n, b_{n-1})]\|^2 \\ &= \|H_2(b_n) - H_2(b_{n-1})\|^2 \\ &\qquad - 2\rho_2\langle G(u_n, b_n) - G(u_n, b_{n-1}), H_2(b_n) - H_2(b_{n-1})\rangle \\ &\qquad\qquad + \rho_2^2\|G(u_n, b_n) - G(u_n, b_{n-1})\|^2 \\ &\leq (\beta_2^2 - 2\rho_2\mu_2 + \rho_2^2\alpha_2^2)\|b_n - b_{n-1}\|^2. \end{aligned} \tag{4.8}$$

Further, from the assumptions, we have

$$\begin{aligned} \|F(a_{n-1}, v_n) - F(a_{n-1}, v_{n-1})\| &\leq \zeta_1\|v_n - v_{n-1}\| \\ &\leq \zeta_1\gamma_2(1 + n^{-1})\|b_n - b_{n-1}\|, \end{aligned} \tag{4.9}$$

$$\begin{aligned} \|G(u_n, b_{n-1}) - G(u_{n-1}, b_{n-1})\| &\leq \zeta_2\|u_n - u_{n-1}\| \\ &\leq \zeta_2\gamma_1(1 + n^{-1})\|a_n - a_{n-1}\|. \end{aligned} \tag{4.10}$$

It follows from (4.5) – (4.10) that

$$\left\{\begin{aligned} &\|a_{n+1}-a_n\| \\ &\quad\le\left(1-\lambda+\lambda\tfrac{\tau_1}{r_1}\sqrt{\beta_1^2-2\rho_1\mu_1+\rho_1^2\alpha_1^2}\right)\|a_n-a_{n-1}\| \\ &\qquad+\lambda\tfrac{\tau_1}{r_1}\zeta_1\gamma_2(1+n^{-1})\|b_n-b_{n-1}\|, \\ &\|b_{n+1}-b_n\| \\ &\quad\le\left(1-\lambda+\lambda\tfrac{\tau_2}{r_2}\sqrt{\beta_2^2-2\rho_2\mu_2+\rho_2^2\alpha_2^2}\right)\|b_n-b_{n-1}\| \\ &\qquad+\lambda\tfrac{\tau_2}{r_2}\zeta_2\gamma_1(1+n^{-1})\|a_n-a_{n-1}\|. \end{aligned}\right. \tag{4.11}$$

Now (4.11) implies that

$$\begin{aligned} &\|a_{n+1}-a_n\|+\|b_{n+1}-b_n\| \\ &\le\left(1-\lambda+\lambda\frac{\tau_1}{r_1}\sqrt{\beta_1^2-2\rho_1\mu_1+\rho_1^2\alpha_1^2}\right. \\ &\qquad\left.+\lambda\frac{\tau_2}{r_2}\zeta_2\gamma_1(1+n^{-1})\right)\|a_n-a_{n-1}\| \\ &\quad+\left(1-\lambda+\lambda\frac{\tau_2}{r_2}\sqrt{\beta_2^2-2\rho_2\mu_2+\rho_2^2\alpha_2^2}\right. \\ &\qquad\left.+\lambda\frac{\tau_1}{r_1}\zeta_1\gamma_2(1+n^{-1})\right)\|b_n-b_{n-1}\| \\ &\le(1-\lambda+\lambda\theta_n)(\|a_n-a_{n-1}\|+\|b_n-b_{n-1}\|), \end{aligned} \tag{4.12}$$

where

$$\theta_n = \max\left\{ \frac{\tau_1}{r_1}\sqrt{\beta_1^2 - 2\rho_1\mu_1 + \rho_1^2\alpha_1^2} + \frac{\tau_2}{r_2}\zeta_2\gamma_1(1+n^{-1}) \right., \\ \left. \frac{\tau_2}{r_2}\sqrt{\beta_2^2 - 2\rho_2\mu_2 + \rho_2^2\alpha_2^2} + \frac{\tau_1}{r_1}\zeta_1\gamma_2(1+n^{-1}) \right\}.$$

Letting

$$\theta = \max\left\{ \frac{\tau_1}{r_1}\sqrt{\beta_1^2 - 2\rho_1\mu_1 + \rho_1^2\alpha_1^2} + \frac{\tau_2}{r_2}\zeta_2\gamma_1 \right., \\ \left. \frac{\tau_2}{r_2}\sqrt{\beta_2^2 - 2\rho_2\mu_2 + \rho_2^2\alpha_2^2} + \frac{\tau_1}{r_1}\zeta_1\gamma_2 \right\},$$

we have that $\theta_n \to \theta$ as $n \to \infty$. It follows from condition (4.4) that $0 < \theta < 1$. Therefore, by (4.12) and $0 < \lambda \leq 1$, $\{a_n\}$ and $\{b_n\}$ are both Cauchy sequences and so there exist $a \in X_1$ and $b \in X_2$ such that $a_n \to a$ and $b_n \to b$ as $n \to \infty$.

Now we prove that $u_n \to u \in A(u)$ and $v_n \to v \in B(b)$ as $n \to \infty$. In fact, it follows from (4.9) and (4.10) that $\{u_n\}$ and $\{v_n\}$ are also Cauchy sequences. Therefore, there exist $u \in X_1$ and $v \in X_2$ such that $u_n \to u$ and $v_n \to v$ as $n \to \infty$. Further,

$$\begin{aligned} d(u, A(u)) &= \inf\{\|u - t\| : t \in A(a)\} \\ &\leq \|u - u_n\| + d(u_n, A(a)) \\ &\leq \|u - u_n\| + D_1(A(a_n), A(a)) \\ &\leq \|u - u_n\| + \zeta_1\|a_n - a\| \to 0. \end{aligned}$$

Hence, since $A(a)$ is closed, we have $u \in A(a)$. Similarly, we can prove that $v \in B(b)$.

By continuity, a, b, u and v satisfy the following relation

$$\begin{cases} a = R_{M_1,\rho_1}^{H_1,\eta_1}[H_1(a) - \rho_1 F(a,v)], \\ b = R_{M_2,\rho_2}^{H_2,\eta_2}[H_2(b) - \rho_2 G(u,b)]. \end{cases}$$

By Theorem 4.1, we know that (a, b, u, v) is a solution of problem (3.1). This completes the proof. □

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