Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 3, Issue 1, Article 7, 2002

L'HOSPITAL TYPE RULES FOR MONOTONICITY: APPLICATIONS TO PROBABILITY INEQUALITIES FOR SUMS OF BOUNDED RANDOM VARIABLES

IOSIF PINELIS

DEPARTMENT OF MATHEMATICAL SCIENCES
MICHIGAN TECHNOLOGICAL UNIVERSITY
HOUGHTON, MI 49931, USA
ipinelis@mtu.edu

Received 29 January, 2001; accepted 6 September, 2001.
Communicated by C.E.M. Pearce

ABSTRACT. This paper continues a series of results begun by a l'Hospital type rule for monotonicity, which is used here to obtain refinements of the Eaton-Pinelis inequalities for sums of bounded independent random variables.

Key words and phrases: L'Hospital's Rule, Monotonicity, Probability inequalities, Sums of independent random variables, Student's statistic.

2000 *Mathematics Subject Classification.* Primary: 26A48, 26D10, 60E15; Secondary: 26D07, 62H15, 62F04, 62F35, 62G10, 62G15.

1. INTRODUCTION

In [8], the following criterion for monotonicity was given, which reminds one of the l'Hospital rule for computing limits.

Proposition 1.1. *Let $-\infty \leq a < b \leq \infty$. Let f and g be differentiable functions on an interval (a, b). Assume that either $g' > 0$ everywhere on (a, b) or $g' < 0$ on (a, b). Suppose that $f(a+) = g(a+) = 0$ or $f(b-) = g(b-) = 0$ and $\frac{f'}{g'}$ is increasing (decreasing) on (a, b). Then $\frac{f}{g}$ is increasing (respectively, decreasing) on (a, b). (Note that the conditions here imply that g is nonzero and does not change sign on (a, b).)*

Developments of this result and applications were given: in [8], applications to certain information inequalities; in [10], extensions to non-monotonic ratios of functions, with applications to certain probability inequalities arising in bioequivalence studies and to convexity problems; in [9], applications to monotonicity of the relative error of a Padé approximation for the complementary error function.

ISSN (electronic): 1443-5756

013-01

Here we shall consider further applications, to probability inequalities, concerning the Student t statistic.

Let $\eta_1, \dots, \eta_n$ be independent zero-mean random variables such that $\mathbb{P}(\,|\eta_i| \le 1) = 1$ for all i, and let $a_1, \dots, a_n$ be any real numbers such that $a_1^2 + \cdots + a_n^2 = 1$. Let ν stand for a standard normal random variable.

In [3] and [4], a multivariate version of the following inequality was given:

$$\mathbb{P}\left(|a_1\eta_1 + \cdots + a_n\eta_n| \ge u\right) < c \cdot \mathbb{P}\left(|\nu| \ge u\right) \quad \forall u \ge 0, \tag{1.1}$$

where

$$c := \frac{2e^3}{9} = 4.463\dots;$$

cf. Corollary 2.6 in [4] and the comment in the middle of page 359 therein concerning the Hunt inequality. For subsequent developments, see [5], [6], and [7].

Inequality (1.1) implies a conjecture made by Eaton [2]. In turn, (1.1) was obtained in [4] based on the inequality

$$\mathbb{P}\left(|a_1\eta_1 + \cdots + a_n\eta_n| \ge u\right) \le Q(u) \quad \forall u \ge 0, \tag{1.2}$$

where

$$Q(u) := \min\left[1, \frac{1}{u^2}, W(u)\right] \tag{1.3}$$

$$= \begin{cases} 1 & \text{if} \quad 0 \le u \le 1, \\ \dfrac{1}{u^2} & \text{if} \quad 1 \le u \le \mu_1, \\ W(u) & \text{if} \quad u \ge \mu_1, \end{cases} \tag{1.4}$$

$$\mu_1 := \frac{\mathbb{E}\,|\nu|^3}{\mathbb{E}\,|\nu|^2} = 2\sqrt{\frac{2}{\pi}} = 1.595\dots;$$

$$W(u) := \inf\left\{\frac{\mathbb{E}\,(|\nu| - t)_+^3}{(u-t)^3} : t \in (0, u)\right\};$$

cf. Lemma 3.5 in [4]. The bound $Q(u)$ possesses a certain optimality property; cf. (3.7) in [4] and the definition of $Q_r(u)$ therein. In [1], $Q(u)$ is denoted by $B_{\mathrm{EP}}(u)$, called the Eaton-Pinelis bound, and tabulated, along with other related bounds; various statistical applications are given therein.

Let

$$\varphi(u) := \frac{1}{\sqrt{2\pi}}\, e^{-u^2/2}, \quad \Phi(u) := \int_{-\infty}^{u} \varphi(s)\, ds, \quad \text{and} \quad \overline{\Phi}(u) := 1 - \Phi(u)$$

denote, as usual, the density, distribution function, and tail function of the standard normal law.

It follows from [4] (cf. Lemma 3.6 therein) that the ratio

$$r(u) := \frac{Q(u)}{c \cdot \mathbb{P}\left(|\nu| \ge u\right)} = \frac{Q(u)}{c \cdot 2\overline{\Phi}(u)}, \quad u \ge 0, \tag{1.5}$$

of the upper bounds in (1.2) and (1.1) is less than 1 for all $u \ge 0$, so that (1.2) indeed implies (1.1). Moreover, it was shown in [4] that $r(u) \to 1$ as $u \to \infty$; cf. Proposition A.2 therein. Other methods of obtaining (1.1) are given in [5] and [6].

In Section 2 of this paper, we shall present monotonicity properties of the ratio r, from which it follows, once again, that

$$r < 1 \quad \text{on} \quad (0, \infty). \tag{1.6}$$

Combining the bounds (1.1) and (1.2) and taking (1.3) into account, one has the following improvement of the upper bound provided by (1.1):

$$\mathbb{P}\left(|a_1\eta_1 + \cdots + a_n\eta_n| \geq u\right) \leq V(u) := \min\left[1, \frac{1}{u^2}, c \cdot \mathbb{P}\left(|\nu| \geq u\right)\right] \quad \forall u \geq 0. \tag{1.7}$$

Monotonicity properties of the ratio

$$R := \frac{Q}{V} \tag{1.8}$$

of the upper bounds in (1.2) and (1.7) will be studied in Section 3.

Our approach is based on Proposition 1.1. Mainly, we follow here lines of [3].

2. Monotonocity Properties of the Ratio r given by (1.5)

Theorem 2.1.

1. *There is a unique solution to the equation* $2\overline{\Phi}(d) = d \cdot \varphi(d)$ *for* $d \in (1, \mu_1)$*; in fact,* $d = 1.190\ldots$.

2. *The ratio* r *is*

- **(a)** *increasing on* $[0, 1]$ *from* $r(0) = \frac{1}{c} = 0.224\ldots$ *to* $r(1) = \frac{1}{c \cdot 2\overline{\Phi}(1)} = 0.706\ldots$*;*
- **(b)** *decreasing on* $[1, d]$ *from* $r(1) = 0.706\ldots$ *to* $r(d) = \frac{\frac{1}{d^2}}{c \cdot 2\overline{\Phi}(d)} = 0.675\ldots$*;*
- **(c)** *increasing on* $[d, \infty)$ *from* $r(d) = 0.675\ldots$ *to* $r(\infty) = 1$.

Proof.

1. Consider the function

$$h(u) := 2\overline{\Phi}(u) - u\varphi(u).$$

One has $h(1) = 0.07\ldots > 0$, $h(\mu_1) = -0.06\ldots < 0$, and $h'(u) = (u^2 - 3)\varphi(u)$. Hence, $h'(u) < 0$ for $u \in [1, \mu_1]$, since $\mu_1 < \sqrt{3}$. This implies part 1 of the theorem.

2.

- **(a)** Part 2(a) of the theorem is immediate from (1.5) and (1.4).
- **(b)** For $u > 0$, one has

$$\frac{d}{du}\left(u^2\overline{\Phi}(u)\right) = uh(u),$$

where h is the function considered in the proof of part 1 of the theorem. Since $h > 0$ on $[1, d)$ and $r(u) = \frac{1}{2cu^2\overline{\Phi}(u)}$ for $u \in [1, \mu_1]$, part 2(b) now follows.

- **(c)** Since $h < 0$ on $(d, \mu_1]$, it also follows from above that r is increasing on $[d, \mu_1]$. It remains to show that r is increasing on $[\mu_1, \infty)$. This is the main part of the proof, and it requires some notation and facts from [4]. Let

$$
\begin{aligned}
C &:= \frac{1}{\int_0^\infty e^{-s^2/2} ds}, \\
\gamma(u) &:= \int_u^\infty (s-u)^3 e^{-s^2/2} ds, \\
\gamma^{(j)}(u) &:= \frac{d^j \gamma(u)}{du^j} \quad \left(\gamma^{(0)} := \gamma\right), \\
\mu(t) &:= t - \frac{3\gamma(t)}{\gamma'(t)}, \\
F(t,u) &:= C \frac{\gamma(t)}{(u-t)^3}, \quad t < u;
\end{aligned} \tag{2.1}
$$

cf. notation on pages 361–363 in [4], in which we presently take $r = 1$.
Then $\forall j \in \{0,1,2,3,4,5\}$

$$(-1)^j \gamma^{(j)} > 0 \quad \text{on} \quad (0,\infty), \tag{2.2}$$

$$(-1)^j \gamma^{(j)}(u) = 6u^{j-4} e^{-u^2/2} (1 + o(1)) \quad \text{as} \quad u \to \infty, \tag{2.3}$$

$$\gamma^{(4)}(u) = 6e^{-u^2/2} \quad \text{and} \quad \gamma^{(5)}(u) = -6ue^{-u^2/2}; \tag{2.4}$$

cf. Lemma 3.3 in [4]. Moreover, it was shown in [4] (see page 363 therein) that on $[0,\infty)$

$$\mu' > 0, \tag{2.5}$$

so that the formula

$$t \leftrightarrow u = \mu(t)$$

defines an increasing correspondence between $t \geq 0$ and $u \geq \mu(0) = \mu_1$, so that the inverse map

$$\mu^{-1} : [\mu_1, \infty) \to [0, \infty)$$

is correctly defined and is a bijection. Finally, one has (cf. (3.11) in [4] and (1.4) and (2.1) above)

$$\forall u \geq \mu_1 \quad Q(u) = W(u) = F(t,u) = -\frac{C}{27} \frac{\gamma'(t)^3}{\gamma(t)^2}; \tag{2.6}$$

here and in the rest of this proof, t stands for $\mu^{-1}(u)$ and, equivalently, u for $\mu(t)$. Now equation (2.6) implies

$$Q'(u) = \frac{\dfrac{dQ(\mu(t))}{dt}}{\dfrac{d\mu(t)}{dt}} = -\frac{C}{27} \frac{\gamma'(t)^4}{\gamma(t)^3}. \tag{2.7}$$

for $u \geq \mu_1$; here we used the formula

$$\mu'(t) = \frac{3\gamma(t)\gamma''(t) - 2\gamma'(t)^2}{\gamma'(t)^2}. \tag{2.8}$$

Next,

$$\begin{aligned}\gamma'(t)\mu(t) &= t\gamma'(t) - 3\gamma(t) \\ &= -3\int_t^\infty \left[t(s-t)^2 + (s-t)^3\right]\, e^{-s^2/2}\, ds \\ &= -3\int_t^\infty (s-t)^2\, se^{-s^2/2}\, ds \\ &= -6\int_t^\infty (s-t)\, e^{-s^2/2}\, ds \\ &= -\gamma''(t);\end{aligned}$$

for the fourth of the five equalities here, integration by parts was used. Hence, on $[0,\infty)$,

$$\mu = -\frac{\gamma''}{\gamma'}, \tag{2.9}$$

whence

$$\mu' = \frac{\gamma''^2 - \gamma'\gamma'''}{\gamma'^2};$$

this and (2.5) yield

$$\gamma''^2 - \gamma'\gamma''' > 0. \tag{2.10}$$

Let (cf. (1.5) and use (2.7))

$$\rho(u) := \frac{Q'(u)}{c\cdot 2\overline{\Phi}'(u)} = \frac{C}{54c}\frac{\gamma'(t)^4}{\gamma(t)^3\varphi(\mu(t))}. \tag{2.11}$$

Using (2.11) and then (2.9) and (2.8), one has

$$\frac{d\ln\rho(u)}{dt} = \frac{d}{dt}\left(4\ln|\gamma'(t)| - 3\ln\gamma(t) + \frac{\mu(t)^2}{2}\right) = -\frac{3D(t)^2\gamma''(t)^2}{\gamma(t)\gamma'(t)^3} \tag{2.12}$$

for all $t > 0$, where

$$D := \frac{\gamma'^2}{\gamma''} - \gamma.$$

Further, on $(0,\infty)$,

$$D' = \frac{\gamma'}{\gamma''^2}\left(\gamma''^2 - \gamma'\gamma'''\right) < 0, \tag{2.13}$$

in view of (2.2) and (2.10). On the other hand, it follows from (2.3) that $D(t)\to 0$ as $t\to\infty$. Hence, (2.13) implies that on $(0,\infty)$

$$D > 0. \tag{2.14}$$

Now (2.12), (2.14), and (2.2) imply that ρ is increasing on (μ_1,∞). Also, it follows from (2.6) and (2.3) that $Q(u)\to 0$ as $u\to\infty$; it is obvious that $c\cdot 2\overline{\Phi}(u)\to 0$ as $u\to\infty$. It remains to refer to (1.5), (2.11), Proposition 1.1, and also (for $r(\infty)=1$) to Proposition A.2 [4].

□

3. MONOTONOCITY PROPERTIES OF THE RATIO R GIVEN BY (1.8)

Theorem 3.1.

1. *There is a unique solution to the equation*

$$\frac{1}{z^2} = c \cdot \mathbb{P}\left(|\nu| \geq z\right) \tag{3.1}$$

for $z > \mu_1$*; in fact,* $z = 1.834\ldots$.

2.

$$V(u) = \begin{cases} 1 & \text{if} \quad 0 \leq u \leq 1, \\ \dfrac{1}{u^2} & \text{if} \quad 1 \leq u \leq z, \\ c \cdot \mathbb{P}\left(|\nu| \geq u\right) & \text{if} \quad u \geq z. \end{cases} \tag{3.2}$$

3. **(a)** $R = 1$ *on* $[0, \mu_1]$ *;*
   (b) R *is decreasing on* $[\mu_1, z]$ *from* $R(\mu_1) = 1$ *to* $R(z) = 0.820\ldots$*;*
   (c) R *is increasing on* $[z, \infty)$ *from* $R(z) = 0.820\ldots$ *to* $R(\infty) = 1[= r(\infty)]$.

Thus, the upper bound V is quite close to the optimal Eaton-Pinelis bound $Q = B_{\text{EP}}$ given by (1.3), exceeding it by a factor of at most $\dfrac{1}{R(z)} = 1.218\ldots$. In addition, V is asymptotic (at ∞) to and as universal as Q. On the other hand, V is much more transparent and tractable than Q.

Proof of Theorem 3.1.

1. Consider the function

$$\lambda(u) := \frac{c\mathbb{P}\left(|\nu| \geq u\right)}{\dfrac{1}{u^2}} = 2cu^2\bar{\Phi}(u). \tag{3.3}$$

Then

$$\lambda'(u) = 2cuh(u),$$

where h is the same as in the beginning of the proof of Theorem 2.1 on page 3, with $h'(u) = (u^2 - 3)\varphi(u)$, so that $\sqrt{3}$ is the only root of the equation $h'(u) = 0$. Since $h(\mu_1) = -0.06\ldots < 0$, $h(\sqrt{3}) = -0.07\ldots < 0$, and $h(\infty) = 0$, it follows that $h < 0$ on $[\mu_1, \infty)$, and then so is λ'. Hence, λ is decreasing on $[\mu_1, \infty)$ from $\lambda(\mu_1) = 1.2\ldots$ to $\lambda(\infty) = 0$. Now part 1 of the theorem follows.

2. It also follows from the above that $\lambda \geq 1$ on $[\mu_1, z]$ and $\lambda \leq 1$ on $[z, \infty)$. In addition, by (3.3), (1.5), and (1.4), one has $\lambda = \dfrac{1}{r}$ on $[1, \mu_1]$, whence $\lambda > 1$ on $[1, \mu_1]$ by (1.6). Thus, $\lambda \geq 1$ on $[1, z]$ and $\lambda \leq 1$ on $[z, \infty)$; in particular, $c\mathbb{P}\left(|\nu| \geq 1\right) = \lambda(1) \geq 1$. Now part 2 of the theorem follows.

3. **(a)** Part 3(a) of the theorem is immediate from (1.4), (3.2), and the inequality $z > \mu_1$.
   (b) Of all the parts of the theorem, part 3(b) is the most difficult to prove. In view of (3.2), the inequalities $z > \mu_1 > 1$, (2.6), and (2.9), one has

$$R(u) = u^2 Q(u) = -\frac{C}{27} \frac{\gamma'(t)\gamma''(t)^2}{\gamma(t)^2} \quad \forall u \in [\mu_1, z]; \tag{3.4}$$

here and to the rest of this proof, t again stands for $\mu^{-1}(u)$ and, equivalently, u for $\mu(t)$. It follows that for all $u \in [\mu_1, z]$ or, equivalently, for all $t \in [0, \mu^{-1}(z)]$,

$$\frac{d}{dt} \ln R(u) = L(t) := \frac{\gamma''(t)}{\gamma'(t)} + 2\frac{\gamma'''(t)}{\gamma''(t)} - 2\frac{\gamma'(t)}{\gamma(t)}. \tag{3.5}$$

Comparing (2.1) and (2.9), one has for all $t > 0$

$$\frac{\gamma''(t)}{\gamma'(t)} = 3\frac{\gamma(t)}{\gamma'(t)} - t = -\left(t + \frac{3}{\kappa(t)}\right), \tag{3.6}$$

where

$$\kappa(t) := -\frac{\gamma'(t)}{\gamma(t)}; \tag{3.7}$$

similarly,

$$\frac{\gamma'''(t)}{\gamma''(t)} = 2\frac{\gamma'(t)}{\gamma''(t)} - t = \frac{2}{\dfrac{\gamma''(t)}{\gamma'(t)}} - t; \tag{3.8}$$

this and (3.6) yield

$$\frac{\gamma'''(t)}{\gamma''(t)} = -\frac{(t^2+2)\ \kappa(t) + 3t}{t\,\kappa(t) + 3}. \tag{3.9}$$

Now (3.5), (3.6), and (3.9) lead to

$$L(t) = -\frac{N(t, \kappa(t))}{\kappa(t)\ (t\kappa(t) + 3)}, \tag{3.10}$$

where

$$N(t,k) := -2t\,k^3 + \left(3t^2 - 2\right)\ k^2 + 12t\,k + 9.$$

Next, for $t > 0$,

$$-\frac{1}{6t}\frac{\partial N}{\partial k} = k^2 - \left(t - \frac{2}{3t}\right)\ k - 2,$$

which is a monic quadratic polynomial in k, the product of whose roots is -2, negative, so that one has $k_1(t) < 0 < k_2(t)$, where $k_1(t)$ and $k_2(t)$ are the two roots. It follows that $\dfrac{\partial N}{\partial k} > 0$ on $(0, k_2(t))$ and $\dfrac{\partial N}{\partial k} < 0$ on $(k_2(t), \infty)$.
Hence, $N(t,k)$ is increasing in $k \in (0, k_2(t))$ and decreasing in $k \in (k_2(t), \infty)$. On the other hand, it follows from (3.7) and (2.2) that

$$\kappa(t) > 0 \quad \forall t > 0. \tag{3.11}$$

Therefore,

$$(\kappa(t) < \kappa^*(t) \quad \forall t > 0) \implies (\ N(t, \kappa(t)) > \min\left(N(t,0), N(t, \kappa^*(t))\right) \quad \forall t > 0\ ); \tag{3.12}$$

at this point, κ^* may be any function which majorizes κ on $(0, \infty)$.
Let us now show the function $\kappa^*(t) := t + 2$ is such a majorant of $\kappa(t)$. Toward this end, introduce

$$\gamma^{(-1)}(t) := -\frac{1}{4}\int_t^\infty (s-t)^4\, e^{-s^2/2}\, ds,$$

so that

$$\left(\gamma^{(-1)}\right)' = \gamma.$$

Similarly to (3.6) and (3.8),

$$\kappa(t) = -\frac{\gamma'(t)}{\gamma(t)} = -4\frac{\gamma^{(-1)}(t)}{\gamma(t)} + t. \tag{3.13}$$

Again with $\gamma^{(0)} := \gamma$, one has for $t > 0$

$$\frac{\left(-\gamma^{(j-1)}\right)'}{\left(\gamma^{(j)}\right)'} = \frac{-\gamma^{(j)}}{\gamma^{(j+1)}} \quad \forall j \in \{0, 1, \ldots\},$$

and, in view of (2.4), $\dfrac{-\gamma^{(4)}(t)}{\gamma^{(5)}(t)} = \dfrac{1}{t}$ is decreasing in $t > 0$. In addition, (2.3) implies that $\gamma^{(j)}(t) \to 0$ as $t \to \infty$, for every $j \in \{-1, 0, 1, \ldots\}$. Using now Proposition 1.1 repeatedly, 5 times, one sees that $\dfrac{-\gamma^{(-1)}}{\gamma}$ is decreasing on $(0, \infty)$, whence $\forall t > 0$

$$\frac{-\gamma^{(-1)}(t)}{\gamma(t)} < \frac{-\gamma^{(-1)}(0)}{\gamma(0)} = \frac{3\sqrt{2\pi}}{16} < \frac{1}{2}.$$

This and (3.13) imply that

$$\kappa(t) < t + 2 \quad \forall t > 0.$$

Hence, in view of (3.12),

$$N(t, \kappa(t)) > \min\left(N(t, 0), N(t, t+2)\right) \quad \forall t > 0.$$

But $N(t, 0) = 9 > 0$ and $N(t, t+2) = \left(t^2 - 1\right)^2 \geq 0$ for all t. Therefore, $N(t, \kappa(t)) > 0 \quad \forall t > 0$. Recalling now (3.5), (3.10) and (3.11), one concludes that R is decreasing on $[\mu_1, z]$. To compute $R(z)$, use (3.4). Now part 3(b) of the theorem is proved.

(c) In view of (1.5) and (3.2), one has $R = r$ on $[z, \infty)$. Part 3(c) of the theorem now follows from part 2(c) of Theorem 2.1 and inequalities $d < \mu_1 < z$.

□

REFERENCES

[1] J.-M. DUFOUR And M. HALLIN, Improved Eaton bounds for linear combinations of bounded random variables, with statistical applications, *JASA*, **88** (1993), 1026–1033.

[2] M. EATON, A probability inequality for linear combinations of bounded random variables, *Ann. Stat.*, **2** (1974), 609–613.

[3] I. PINELIS, Extremal probabilistic problems and Hotelling's T^2 test under symmetry condition, *Preprint* (1991).

[4] I. PINELIS, Extremal probabilistic problems and Hotelling's T^2 test under a symmetry condition. *Ann. Stat.*, **22** (1994), 357–368.

[5] I. PINELIS, Optimal tail comparison based on comparison of moments. *High dimensional probability* (Oberwolfach, 1996), 297–314, *Progr. Probab.*, **43**, Birkhäuser, Basel, 1998.

[6] I. PINELIS, Fractional sums and integrals of r-concave tails and applications to comparison probability inequalities. *Advances in stochastic inequalities* (Atlanta, GA, 1997), 149–168, *Contemp. Math.*, **234**, Amer. Math. Soc., Providence, RI, 1999.

[7] I. PINELIS, On exact maximal Khinchine inequalities. *High dimensional probability II* (University of Washington, 1999), 49–63, *Progr. Probab.*, **47**, Birkhäuser, Boston, 2000.

[8] I. PINELIS, L'Hospital type rules for monotonicity, with applications, *J. Ineq. Pure & Appl. Math.*, **3**(1) (2002), Article 5. (`http://jipam.vu.edu.au/v3n1/010_01.html`).

[9] I. PINELIS, Monotonicity properties of the relative error of a Padé approximation for Mills' ratio, *J. Ineq. Pure & Appl. Math.*, **3**(2) (2002), Article 20. (`http://jipam.vu.edu.au/v3n2/012_01.html`).

[10] I. PINELIS, L'Hospital type rules for oscillation, with applications, *J. Ineq. Pure & Appl. Math.*, **2**(3) (2001), Article 33. (`http://jipam.vu.edu.au/v2n3/011_01.html`).