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INEQUALITIES ON LINEAR FUNCTIONS AND CIRCULAR POWERS

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ABSTRACT. We prove some inequalities such as

$$F(x_1^{x_{\sigma(1)}}, \dots, x_n^{x_{\sigma(n)}}) \leq F(x_1^{x_1}, \dots, x_n^{x_n}),$$

where F is a linear function or a linear function in logarithms and σ is a permutation, which is a product of disjoint translation cycles. Stronger inequalities are proved for second-order recurrence sequences, generalizing those of Diaz.

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1. INTRODUCTION

Define the second-order recurrent sequence by

$$x_{n+1} = ax_n + bx_{n-1}, x_0 \geq 0, x_1 \geq 1, \tag{1.1}$$

with $a, b \geq 1$. If $a = b = 1$ and $x_0 = 0, x_1 = 1$ (or $x_0 = 2, x_1 = 1$), then x_n is the Fibonacci sequence, F_n (or Pell sequence, P_n). Inequalities on Fibonacci numbers were used recently by Bar-Noy *et.al* [1], to study a $9/8-$ approximation for a variant of the problem that models the Broadcast Disks application (model for efficient caching of web pages). In [2], J.L. Diaz proposed the following two inequalities:

(a) $F_n^{F_{n+1}} + F_{n+1}^{F_{n+2}} + F_{n+2}^{F_n} < F_n^{F_n} + F_{n+1}^{F_{n+1}} + F_{n+2}^{F_{n+2}}$,
(b) $F_n^{F_{n+1}} F_{n+1}^{F_{n+2}} F_{n+2}^{F_n} < F_n^{F_n} F_{n+1}^{F_{n+1}} F_{n+2}^{F_{n+2}}$.

In this note we show that the inequalities proposed by Diaz are not specific to the Fibonacci sequence, holding for any strictly increasing sequence. Moreover, we prove that stronger inequalities hold for any second-order recurrent sequence as in (1.1). Furthermore, we pose a problem for future research.

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2. THE RESULTS

We wondered if the inequalities $(a),(b)$ were dependent on the Fibonacci sequence or if they can be extended to binary recurrent sequences. From here on, we assume that all sequences have positive terms. Without too great a difficulty, we prove, for a binary sequence, that

Theorem 2.1. *For any positive integer* n,

$$x_n^{x_{n+1}}+x_{n+1}^{x_{n+2}}+x_{n+2}^{x_n}<x_n^{x_n}+x_{n+1}^{x_{n+1}}+x_{n+2}^{x_{n+2}}, \tag{2.1}$$
$$x_n^{x_{n+1}}x_{n+1}^{x_{n+2}}x_{n+2}^{x_n}<x_n^{x_n}x_{n+1}^{x_{n+1}}x_{n+2}^{x_{n+2}}. \tag{2.2}$$

Proof. We shall prove

$$x^y+y^{ax+by}+(ax+by)^x<x^x+y^y+(ax+by)^{ax+by}, \tag{2.3}$$

if $0<x<y$, which will imply our theorem. For easy writing, we denote $z=ax+by$. Then (2.3) is equivalent to

$$x^x\left(x^{y-x}-1\right)+y^y\left(y^{z-y}-1\right)<z^y\left(z^{z-y}-z^{-(y-x)}\right). \tag{2.4}$$

Now, $x^x+y^y<x^y+y^y<(x+y)^y\le z^y$, since $a,b\ge 1$. Moreover,

$$\begin{aligned}\left(x^{y-x}-1\right)+\left(y^{z-y}-1\right)&=x^{y-x}+y^{z-y}-2\\&<x^{z-y}+y^{z-y}-1\\&<(x+y)^{z-y}-1\\&<z^{z-y}-z^{-(y-x)}.\end{aligned}$$

Taking $A=x^x,B=y^y,C=x^{y-x}-1,D=y^{z-y}-1$ and using the inequality for positive numbers $AC+BD\le(A+B)(C+D)$, we obtain (2.4).

The inequality (2.2) is implied by

$$x^yy^zz^x<x^xy^yz^z\iff x^{y-x}y^{z-y}<z^{z-x}. \tag{2.5}$$

But $z^{z-x}=z^{(z-y)+(y-x)}\ge(x+y)^{z-y}(x+y)^{(y-x)}>y^{z-y}x^{y-x}$. The theorem is proved. $\square$

Remark 2.2. We preferred to give this proof since it can be seen that the two inequalities are far from being tight. We remark that inequality (2.2) can be also shown by using Theorem 2.7.

With a little effort, while not attempting to have the best bound, we can improve it, and also prove that the gaps are approaching infinity.

Theorem 2.3. *We have*

$$x_n^{x_{n+1}}+x_{n+1}^{x_{n+2}}+x_{n+2}^{x_n}<x_n^{x_n}+x_{n+1}^{x_{n+1}}+x_{n+2}^{x_{n+2}}-{x_{n+2}}^{x_{n+1}-x_n}$$
$$x_n^{x_{n+1}}x_{n+1}^{x_{n+2}}x_{n+2}^{x_n}<x_n^{x_n}x_{n+1}^{x_{n+1}}x_{n+2}^{x_{n+2}}-3x_n^{x_n}x_{n+1}^{x_{n+1}}x_{n+2}^{x_n}.$$

In particular,

$$\lim_{n\to\infty}\left[\left(x_n^{x_n}+x_{n+1}^{x_{n+1}}+x_{n+2}^{x_{n+2}}\right)-\left(x_n^{x_{n+1}}+x_{n+1}^{x_{n+2}}+x_{n+2}^{x_n}\right)\right]=\infty$$
$$\lim_{n\to\infty}\left[x_n^{x_n}x_{n+1}^{x_{n+1}}x_{n+2}^{x_{n+2}}-x_n^{x_{n+1}}x_{n+1}^{x_{n+2}}x_{n+2}^{x_n}\right]=\infty.$$

In fact, the inequalities (2.1), (2.2) are not dependent on binary sequences, at all. A much more general statement is true. Take σ a permutation, which is a product of disjoint *cyclic* (translations by a fixed number, $c(i)=i+t$) permutations.

Theorem 2.4. *Let* $n\ge 2$ *and* $1\le x_1<x_2<\ldots<x_n$ *a strictly increasing sequence. Then*

$$\sum_{i=1}^n x_i^{x_{\sigma(i)}}\le\sum_{i=1}^n x_i^{x_i}, \tag{2.6}$$

and

$$\prod_{i=1}^{n} x_i^{x_{\sigma(i)}} \le \prod_{i=1}^{n} x_i^{x_i}, \tag{2.7}$$

with strict inequality if σ is not the identity.

Proof. If σ is the identity permutation, the equality is obvious. Now, assume that $\sigma(i) = i + t$. We take the case of $t = 1$ (the others are similar). We prove (2.6) by induction on n. If $n = 2$, we need

$$x_1^{x_2} + x_2^{x_1} < x_1^{x_1} + x_2^{x_2},$$

which is equivalent to

$$x_1^{x_1}\left(x_1^{x_2-x-1} - 1\right) < x_2^{x_1}\left(x_2^{x_2-x-1} - 1\right).$$

The last inequality is certainly valid, since $x_1^{x_1} < x_2^{x_1}$ and $x_1^{x_2-x_1} - 1 < x_2^{x_2-x_1} - 1$.

Assuming the statement holds true for n, we prove it for $n + 1$. We need

$$\sum_{i=1}^{n+1} x_i^{x_{i+1}} < \sum_{i=1}^{n+1} x_i^{x_i}, \tag{2.8}$$

where $x_{n+2} := x_1$. We re-write (2.8) as

$$\left(x_1^{x_2} + x_2^{x_3} + \cdots + x_n^{x_1}\right) + x_n^{x_{n+1}} + x_{n+1}^{x_1} - x_n^{x_1} < x_1^{x_1} + x_2^{x_2} + \cdots + x_n^{x_n} + x_{n+1}^{x_{n+1}},$$

and using induction, it suffices to prove that

$$x_n^{x_{n+1}} + x_{n+1}^{x_1} - x_n^{x_1} < x_{n+1}^{x_{n+1}}.$$

The previous inequality is equivalent to

$$x_n^{x_1}\left(x_n^{x_{n+1}-x_1} - 1\right) < x_{n+1}^{x_1}\left(x_{n+1}^{x_{n+1}-x_1} - 1\right),$$

which is obviously true, since $x_n < x_{n+1}$.

The inequality (2.7) (when $\sigma(i) = i + t$) can be proved by induction, as well. If $n = 2$, then

$$x_1^{x_2} x_2^{x_1} < x_1^{x_1} x_2^{x_2} \iff x_1^{x_2-x_1} < x_2^{x_2-x_1},$$

which is true since $x_1 < x_2$. Assuming the inequality holds true for n, we prove it for $n + 1$. We need

$$x_1^{x_2} \cdots x_n^{x_{n+1}} x_n^{x_1} = x_1^{x_2} \cdots x_{n-1}^{x_n} x_n^{x_1} x_n^{x_{n+1}-x_1} x_{n+1}^{x_1} < x_1^{x_1} \cdots x_{n+1}^{x_{n+1}}.$$

Using the induction step, it suffices to prove

$$x_n^{x_{n+1}-x_1} x_{n+1}^{x_1} < x_{n+1}^{x_{n+1}} \iff x_n^{x_{n+1}-x_1} < x_{n+1}^{x_{n+1}-x_1},$$

which is valid since $x_n < x_{n+1}$.

Now, take the general permutation $\sigma \ne$identity, which is a product of disjoint cyclic permutations. Thus, σ can be written as a product of disjoint cycles as $\sigma = C_1 \times C_2 \times \cdots \times C_m$. Recall that σ was taken such that all of its cycles C_k are translations by a fixed number, say t_k. Take C_k a cycle of length e_k and choose an index in C_k, say i_k. Since σ is not the identity, then there is an index k such that $e_k \ne 1$. We write the inequalities (2.6) and (2.7) as

$$\sum_{k=1}^{m} \sum_{j=0}^{e_k-1} x_{\sigma^j(i_k)}^{x_{\sigma^{j+1}(i_k)}} < \sum_{k=1}^{m} \sum_{j=0}^{e_k-1} x_{\sigma^j(i_k)}^{x_{\sigma^j(i_k)}},$$

$$\prod_{k=1}^{m} \prod_{j=0}^{e_k-1} x_{\sigma^j(i_k)}^{x_{\sigma^{j+1}(i_k)}} < \prod_{k=1}^{m} \prod_{j=0}^{e_k-1} x_{\sigma^j(i_k)}^{x_{\sigma^j(i_k)}},$$

Therefore, it suffices to prove that, for any k, with $e_k \neq 1$, we have

$$\sum_{j=0}^{e_k-1} x_{\sigma^j(i_k)}^{x_{\sigma^{j+1}(i_k)}} < \sum_{j=0}^{e_k-1} x_{\sigma^j(i_k)}^{x_{\sigma^j(i_k)}},$$
$$\prod_{j=0}^{e_k-1} x_{\sigma^j(i_k)}^{x_{\sigma^{j+1}(i_k)}} < \prod_{j=0}^{e_k-1} x_{\sigma^j(i_k)}^{x_{\sigma^j(i_k)}},$$

that is,

$$\sum_{j=0}^{e_k-1} x_{\sigma^j(i_k)}^{x_{\sigma^j(i_k)+t_k}} < \sum_{j=0}^{e_k-1} x_{\sigma^j(i_k)}^{x_{\sigma^j(i_k)}},$$
$$\prod_{j=0}^{e_k-1} x_{\sigma^j(i_k)}^{x_{\sigma^j(i_k)+t_k}} < \prod_{j=0}^{e_k-1} x_{\sigma^j(i_k)}^{x_{\sigma^j(i_k)}}.$$

For k fixed, the above inequalities are just applications of the previous step (of $\sigma(i) = i + t$), by taking a sequence y_l to be $x_{\sigma^j(i_k)}$ in increasing order (we could take from the beginning i_k to be the minimum index in each cycle C_k). $\square$

We can slightly extend the previous result (for a similar permutation σ) in the following (we omit the proof).

Theorem 2.5. *For any increasing sequence $0 < x_1 < \cdots < x_n$, we have*

$$\sum_{i=1}^{n} a_i x_i^{x_{\sigma(i)}} \leq \sum_{i=1}^{n} a_i x_i^{x_i}, \text{ and}$$
$$\prod_{i=1}^{n} a_i x_i^{x_{\sigma(i)}} \leq \prod_{i=1}^{n} a_i x_i^{x_i}, \tag{2.9}$$

where $a_i \geq 0$.

A parallel result involving logarithms is also true (σ is a permutation as before).

Theorem 2.6. *For any finite increasing sequence, $0 < x_1 < x_2 < \cdots < x_n$, and any positive real numbers a_i, we have*

$$\sum_{i=1}^{n} a_i x_{\sigma(i)} \log(x_i) \leq \sum_{i=1}^{n} a_i x_i \log(x_i), \text{ and}$$
$$\prod_{i=1}^{n} a_i x_{\sigma(i)} \log(x_i) = \prod_{i=1}^{n} a_i x_i \log(x_i).$$

The second identity is easily true since every a_i, x_i and $\log x_i$ occurs in both sides. We omit the proof of the first inequality, since it can be deduced easily (as the referee observed) from the known fact (see [3, p. 261])

Theorem 2.7. *Given two increasing sequences $u_1 \leq u_2 \leq \cdots \leq u_n$ and $w_1 \leq w_2 \leq \cdots \leq w_n$, then*

$$\sum_{i=1}^{n} u_i w_{n+1-i} \leq \sum_{i=1}^{n} u_{\tau(i)} w_{\sigma(i)} \leq \sum_{i=1}^{n} u_i w_i,$$

for any permutations σ, τ.

3. FURTHER COMMENTS

We believe that other inequalities of the type occurring in our theorems can also be constructed. Let $F : \mathbb{R}^n \to \mathbb{R}$ be a function, with the properties

$$\text{(3.1)} \qquad \textit{If } x_i \leq y_i, i = 1, \dots, n, \textit{ then } F(x_1, \dots, x_n) \leq F(y_1, \dots, y_n), \\ \textit{with strict inequality if there is an index } i \textit{ such that } x_i < y_i.$$

and

$$\text{(3.2)} \qquad \textit{For } 0 < x_1 < x_2 < \cdots < x_n, \textit{ then, } F(x_1^{x_2}, x_2^{x_3}, \dots, x_n^{x_1}) \leq F(x_1^{x_1}, x_2^{x_2}, \dots, x_n^{x_n}).$$

As examples, we have the linear polynomial $F(x_1, \dots, x_n) = \sum_{i=1}^n a_i x_i$, the linear form in logarithms $F(x_1, \dots, x_n) = \sum_{i=1}^n a_i \log(x_i)$, and the corresponding products.

We ask for more examples of functions satisfying (3.1) and (3.2), which cannot be derived trivially from the previous examples (by raising each variable to the same power, for instance). Is it true that any symmetric polynomial satisfies (3.1) and (3.2)? In addition to more examples, it might be worth investigating the general form of polynomial functions that satisfy these properties.

This looks like a mathematical version of the philosophy saying:
Going one step at the time it is far better than jumping too fast and then at the end falling to the bottom.

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