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RATE OF CONVERGENCE FOR A GENERAL SEQUENCE OF DURRMEYER TYPE OPERATORS

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ABSTRACT. In the present paper we give the rate of convergence for the linear combinations of the generalized Durrmeyer type operators which includes the well known Szasz-Durrmeyer operators and Baskakov-Durrmeyer operators as special cases.

Key words and phrases: Szasz-Durrmeyer operators, Baskakov-Durrmeyer operators, Rate of convergence, Order of approximation.

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1. INTRODUCTION

Durrmeyer [1] introduced the integral modification of Bernstein polynomials so as to approximate Lebesgue integrable functions on the interval $[0,1]$. We now consider the general family of Durrmeyer type operators, which is defined by

$$S_n(f(t),x) = (n-c)\sum_{v=0}^{\infty} p_{n,v}(x)\int_0^{\infty} p_{n,v}(t)\, f(t)\, dt, \quad x \in I \tag{1.1}$$

where $n \in \mathbb{N}$, $n > \max\{0,-c\}$ and $p_{n,v}(x) = (-1)^v \frac{x^v}{v!}\varphi_n^{(v)}(x)$. Also $\{\phi_n\}$ is a sequence of real functions having the following properties on $[0,a]$, where $a > 0$ and for all $n \in \mathbb{N}$, $v \in \mathbb{N}\cup\{0\}$, we have

(I) $\phi_n \in C^{\infty}[0,a], \phi_n(0) = 1$.
(II) ϕ_n is complete monotonic.
(III) There exist $c \in \mathbb{N} : \phi_n^{(v+1)} = -n\phi_{n+c}^{(v)}, n > \max\{0,-c\}$.

Some special cases of the operators (1.1) are as follows:

(1) If $c = 0$,$\phi_n(x) = e^{-nx}$, we get the Szasz-Durrmeyer operator.

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(2) If $c = 1, \phi_n(x) = (1+x)^{-n}$, we obtain the Baskakov-Durrmeyer operator.
(3) If $c > 1, \phi_n(x) = (1+cx)^{\frac{-n}{c}}$, we obtain a general Baskakov-Durrmeyer operator.
(4) If $c = -1, \phi_n(x) = (1-x)^n$, we obtain the Bernstein-Durrmeyer operator.

Very recently Srivastava and Gupta [9] studied a similar type of operators and obtained the rate of convergence for functions of bounded variation. It is easily verified that the operators (1.1) are linear positive operators and these operators reproduce the constant ones, while the operators studied in [9] reproduce every linear functional for the case $c = 0$. Several researchers studied different approximation properties on the special cases of the operators (1.1), the pioneer work on Durrmeyer type operators is due to S. Guo [3], Vijay Gupta (see e. g. [4], [5]), R. P. Sinha et al. [8] and Wang and Guo [11], etc. It turns out that the order of approximation for such type of Durrmeyer operators is at best $O(n^{-1})$, how so ever smooth the function may be. In order to improve the order of approximation, we have to slacken the positivity condition of the operators, for this we consider the linear combinations of the operators (1.1). The technique of linear combinations is described as follows:

$$S_{n,k}(f,x) = \begin{vmatrix} 1 & d_0^{-1} & d_0^{-2} & \cdots\cdots & d_0^{-k} \\ 1 & d_1^{-1} & d_1^{-2} & \cdots\cdots & d_1^{-k} \\ \cdots & \cdot\cdot & \cdot\cdot & \cdots\cdots & \cdot\cdot \\ \cdots & \cdot\cdot & \cdot\cdot & \cdots\cdots & \cdot\cdot \\ 1 & d_k^{-1} & d_k^{-2} & \cdots\cdots & d_k^{-k} \end{vmatrix}^{-1} \begin{vmatrix} S_{d_0 n}(f,x) & d_0^{-1} & d_0^{-2} & \cdots\cdots & d_0^{-k} \\ S_{d_1 n}(f,x) & d_1^{-1} & d_1^{-2} & \cdots\cdots & d_1^{-k} \\ \cdots & \cdot\cdot & \cdot\cdot & \cdots\cdots & \cdot\cdot \\ \cdots & \cdot\cdot & \cdot\cdot & \cdots\cdots & \cdot\cdot \\ S_{d_k n}(f,x) & d_k^{-1} & d_k^{-2} & \cdots\cdots & d_k^{-k} \end{vmatrix}.$$

Such types of linear combinations were first considered by May [7] to improve the order of approximation of exponential type operators. In the alternative form the above linear combinations can be defined as

$$S_{n,k}(f,x) = \sum_{j=0}^{k} c(j,k) S_{d_j n}(f,x),$$

where

$$C(j,k) = \prod_{\substack{i=0 \\ i \neq j}}^{k} \frac{d_j}{d_j - d_i}, k \neq 0 \qquad \text{and} \qquad C(0,0) = 1.$$

If $f \in L_p[0,\infty)$, $1 \leq p < \infty$ and $0 < a_1 < a_3 < a_2 < b_2 < b_3 < b_1 < \infty$ and $I_i = [a_i, b_i]$, $i = 1,2,3$, the Steklov mean $f_{\eta,m}$ of m^{th} order corresponding to f is defined as

$$f_{\eta,m}(t) = \eta^{-m} \int_{\frac{-\eta}{2}}^{\frac{\eta}{2}} \cdots \int_{\frac{-\eta}{2}}^{\frac{\eta}{2}} \left(f(t) + (-1)^{m-1} \Delta^m_{\sum_{i=1}^m t_i} f(t) \right) \prod_{i=1}^{m} dt_i; \quad t \in I_1.$$

It can be verified [6, 10] that

(i) $f_{\eta,m}$ has derivative up to order m, $f_{\eta,m}^{(m-1)} \in AC(I_1)$ and $f_{\eta,m}^{(m)}$ exist almost every where and belongs to $L_p(I_1)$;
(ii) $\left\| f_{\eta,m}^{(r)} \right\|_{L_p(I_1)} \leq K_1 \eta^{-r} \omega_r(f,\eta,p,I_1)$, $r = 1,2,3,\ldots,m$;
(iii) $\| f - f_{\eta,m} \|_{L_p(I_2)} \leq K_2 \omega_m(f,\eta,p,I_1)$;
(iv) $\| f_{\eta,m}^m \|_{L_p(I_2)} \leq K_3 \eta^m \|f\|_{L_p(I_1)}$
(v) $\left\| f_{\eta,m}^{(m)} \right\|_{L_p(I_1)} \leq K_4 \eta^{-r} \omega_r(f,\eta,p,I_1)$

where $K_i's$, $i = 1,2,3,4$ are certain constants independent of f and η.

In the present paper we establish the rate of convergence for the combinations of the generalized Durrmeyer type operators in L_p-norm.

2. AUXILIARY RESULTS

To prove the rate of convergence we need the following lemmas:

Lemma 2.1. *Let $m \in \mathbb{N} \cup \{0\}$, also $\mu_{n,m}(x)$ is the m^{th} order central moment defined by*

$$\mu_{n,m} \equiv S_n((t-x)^m, x) = (n-c)\sum_{v=0}^{\infty} p_{n,v}(x) \int_0^{\infty} p_{n,v}(t)(t-x)^m dt,$$

then

(i) $\mu_{n,m}(x)$ *is a polynomial in x of degree m.*
(ii) $\mu_{n,m}(x)$ *is a rational function in n and for each $0 \le x < \infty$* $\quad \mu_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$.

Remark 2.2. Using Hölder's inequality, it can be easily verified that $S_n(|t-x|^r, x) = O(n^{-r/2})$ for each $r > 0$ and for every fixed $0 \le x < \infty$.

Lemma 2.3. *For sufficiently large n and $q \in \mathbb{N}$, there holds*

$$S_{n,k}\left((t-x)^q, x\right) = n^{-(k+1)}\{F(q,k,x) + o(1)\},$$

where $F(q,k,x)$ are certain polynomials in x of degree q and $0 \le x < \infty$ is arbitrary but fixed.

Proof. For sufficiently large values of n we can write, from Lemma 2.1

$$S_n\left((t-x)^q, x\right) = \frac{Q_0(x)}{n^{[(q+1)/2]}} + \frac{Q_1(x)}{n^{[(q+1)/2]+1}} + \cdots + \frac{Q_{[q/2]}(x)}{n^q},$$

where $Q_i(x)$,$i = 0,1,2,\ldots$ are certain polynomials in x of at most degree q.

Therefore $S_{n,k}\left((t-x)^q, x\right)$ is given by

$$\begin{vmatrix} 1 & d_0^{-1} & d_0^{-2} & \ldots\ldots & d_0^{-k} \\ 1 & d_1^{-1} & d_1^{-2} & \ldots\ldots & d_1^{-k} \\ \ldots & .. & .. & \ldots\ldots & .. \\ \ldots & .. & .. & \ldots\ldots & .. \\ 1 & d_k^{-1} & d_k^{-2} & \ldots\ldots & d_k^{-k} \end{vmatrix}^{-1} \begin{vmatrix} \frac{Q_0(x)}{(d_0 n)^{[(q+1)/2]}} + \frac{Q_1(x)}{(d_0 n)^{[(q+1)/2]+1}} + \ldots & d_0^{-1} & d_0^{-2} & \ldots\ldots & d_0^{-k} \\ \frac{Q_0(x)}{(d_1 n)^{[(q+1)/2]}} + \frac{Q_1(x)}{(d_1 n)^{[(q+1)/2]+1}} + \ldots & d_1^{-1} & d_1^{-2} & \ldots\ldots & d_1^{-k} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & .. & .. & \ldots\ldots & .. \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & .. & .. & \ldots\ldots & .. \\ \frac{Q_0(x)}{(d_k n)^{[(q+1)/2]}} + \frac{Q_1(x)}{(d_k n)^{[(q+1)/2]+1}} + \ldots & d_k^{-1} & d_k^{-2} & \ldots\ldots & d_k^{-k} \end{vmatrix}$$

$$= n^{-(k+1)}\left\{F(q,k,x) + o(1)\right\}, \quad 0 \le x < \infty.$$

□

Lemma 2.4 ([4]). *Let $1 \le p < \infty$, $f \in L_p[a,b]$, $f^{(m)} \in AC[a,b]$ and $f^{(m+1)} \in L_p[a,b]$, then*

$$\left\|f^{(j)}\right\|_{L_p[a,b]} \le C\left\{\left\|f^{(m+1)}\right\|_{L_p[a,b]} + \|f\|_{L_p[a,b]}\right\},$$

$j = 1,2,3,\ldots,m$ and C is a constant depending on j,p,m,a,b.

Lemma 2.5. *Let $f \in L_p[0,\infty)$, $p > 1$. If $f^{(2k+1)} \in AC(I_1)$ and $f^{(2k+2)} \in L_p(I_1)$, then for all n sufficiently large*

$$\|S_{n,k}(f,\cdot) - f\|_{L_p(I_2)} \le C_1 n^{-(k+1)}\left\{\left\|f^{(2k+2)}\right\|_{L_p(I_2)} + \|f\|_{L_p[0,\infty)}\right\}. \tag{2.1}$$

Also if $f \in L_1[0,\infty)$, $f^{(2k+1)} \in L_1(I_1)$ with $f^{(2k)} \in AC(I_1)$ and $f^{(2k+1)} \in BV(I_1)$, then for all n sufficiently large

$$\|S_{n,k}(f,\cdot) - f\|_{L_p(I_2)} \le C_2 n^{-(k+1)}\left\{\left\|f^{(2k+1)}\right\|_{BV(I_1)} + \left\|f^{(2k+1)}\right\|_{L_1(I_2)} + \|f\|_{L_1[0,\infty)}\right\}. \tag{2.2}$$

Proof. First let $p > 1$. By the hypothesis, for all $t \in [0, \infty)$ and $x \in I_2$, we have

$$\begin{aligned} S_{n,k}(f,x) - f(x) &= \sum_{i=1}^{2k+1} \frac{f^{(i)}(x)}{i!} S_{n,k}((t-x)^i, x) \\ &\qquad + \frac{1}{(2k+1)!} S_{n,k}(\phi(t) \int_x^t (t-w)^{2k+1} f^{(2k+2)}(w)dw, x) \\ &\qquad + S_{n,k}(F(t,x)(1-\phi(t)), x) \\ &= E_1 + E_2 + E_3, \quad \text{say,} \end{aligned}$$

where $\phi(t)$ denotes the characteristic function of I_1 and

$$F(t,x) = f(t) - \sum_{i=0}^{2k+1} \frac{(t-x)^i}{i!} f^{(i)}(x).$$

Applying Lemma 2.3 and Lemma 2.4, we have

$$\|E_1\|_{L_p(I_2)} \le C_3 n^{-(k+1)} \sum_{i=1}^{2k+1} \left\|f^{(i)}\right\|_{L_p(I_2)} \le C_4 n^{-(k+1)} \left\{ \|f\|_{L_p(I_2)} + \left\|f^{(2k+2)}\right\|_{L_p(I_2)} \right\}.$$

Next we estimate E_2. Let H_f be the Hardy Littlewood maximal function [10] of $f^{(2k+2)}$ on I_1, using Hölder's inequality and Lemma 2.1, we have

$$\begin{aligned} |E_2| &\le \frac{1}{(2k+2)!} S_{n,k}\left(\phi(t) \left| \int_x^t |t-w|^{2k+1} \left|f^{(2k+2)}(w)\right| dw \right|, x \right) \\ &\le \frac{1}{(2k+2)!} S_{n,k}\left(\phi(t) |t-x|^{2k+1} \left| \int_x^t \left|f^{(2k+2)}(w)\right| dw \right|, x \right) \\ &\le \frac{1}{(2k+2)!} S_{n,k}\left(\phi(t) |t-x|^{2k+2} |H_f(t)|, x \right) \\ &\le \frac{1}{(2k+2)!} \left\{ S_{n,k}\left(\phi(t) |t-x|^{q(2k+2)}, x \right) \right\}^{\frac{1}{q}} \left\{ S_{n,k}\left(\phi(t) |H_f(t)|^p, x \right) \right\}^{\frac{1}{p}} \\ &\le C_5 n^{-(k+1)} \left(\sum_{j=0}^{k} C(j,k) \int_{a_1}^{b_1} (d_j n - c) \sum_{v=0}^{\infty} p_{d_j n, v}(x) p_{d_j n, v}(t) |H_f(t)|^p \, dt \right)^{\frac{1}{p}}. \end{aligned}$$

Next applying Fubini's theorem, we have

$$\begin{aligned} &\|E_2\|_{L_p(I_2)}^p \\ &\le C_6 n^{-p(k+1)} \sum_{j=0}^{k} C(j,k) \int_{a_2}^{b_2} \int_{a_1}^{b_1} (d_j n - c) \sum_{v=0}^{\infty} p_{d_j n, v}(x) p_{d_j n, v}(t) |H_f(t)|^p \, dt dx \\ &\le C_6 n^{-p(k+1)} \sum_{j=0}^{k} C(j,k) \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} (d_j n - c) \sum_{v=0}^{\infty} p_{d_j n, v}(x) p_{d_j n, v}(t) dx \right) |H_f(t)|^p \, dt \\ &\le C_7 n^{-p(k+1)} \|H_f\|_{L_p(I_1)}^p \le C_8 n^{-p(k+1)} \left\|f^{(2k+2)}\right\|_{L_p(I_1)}^p. \end{aligned}$$

Therefore

$$\|E_2\|_{L_p(I_2)} \le C_8 n^{-(k+1)} \left\|f^{(2k+2)}\right\|_{L_p(I_1)}.$$

For $t \in [0,\infty)\backslash[a_1,b_1],\ x \in I_2$ there exists a $\delta > 0$ such that $|t-x| \geq \delta$. Thus

$$\begin{aligned}
E_3 &\equiv S_{n,k}\left(F(t,x)(1-\phi(t)),x)\right) \\
&\leq \delta^{-(2k+2)} \sum_{j=0}^{k} C(j,k) S_{d_jn}\left(|F(t,x)|\,(t-x)^{2k+2},x\right) \\
&\leq \delta^{-(2k+2)} \sum_{j=0}^{k} C(j,k) \Bigg[S_{d_jn}\left(|f(t)|\,(t-x)^{2k+2},x\right) \\
&\qquad + \sum_{i=0}^{2k+1} \frac{\left|f^{(i)}(x)\right|}{i!} S_{d_jn}\left(|t-x|^{2k+2+i},x\right)\Bigg] \\
&= E_{31} + E_{32}, \quad \text{say.}
\end{aligned}$$

Applying Hölder's inequality and Lemma 2.1, we have

$$\begin{aligned}
|E_{31}| &\leq \delta^{-(2k+2)} \sum_{j=0}^{k} C(j,k) \left\{S_{d_jn}(|f(t)|^p,x)\right\}^{\frac{1}{p}} \left\{S_{d_jn}(|t-x|^{q(2k+2)},x)\right\}^{\frac{1}{q}} \\
&\leq C_9 \sum_{j=0}^{k} C(j,k) \left\{S_{d_jn}(|f(t)|^p,x)\right\}^{\frac{1}{p}} \frac{1}{(d_jn)^{k+1}}.
\end{aligned}$$

Finally by Fubini's theorem, we obtain

$$\begin{aligned}
\|E_{31}\|^p_{L_p(I_2)} &= \int_{a_2}^{b_2} |E_{31}|^p dx \\
&\leq C_9 \sum_{j=0}^{k} C(j,k)(d_jn)^{-p(k+1)} \int_{a_2}^{b_2} \int_0^{\infty} (d_jn-c) p_{d_jn,v}(x) p_{d_jn,v}(t)\,|f(t)|^p\,dtdx \\
&\leq C_{10} n^{-p(k+1)} \|f\|^p_{L_p[0,\infty)}.
\end{aligned}$$

Again by Lemma 2.4, we have

$$\|E_{32}\|_{L_p(I_2)} \leq C_{11} n^{-(k+1)} \left\{\|f\|_{L_p(I_2)} + \left\|f^{(2k+2)}\right\|_{L_p(I_2)}\right\}.$$

Thus

$$\|E_3\|_{L_p(I_2)} \leq C_{12} n^{-(k+1)} \left\{\|f\|_{L_p[0,\infty)} + \left\|f^{(2k+2)}\right\|_{L_p(I_2)}\right\}.$$

Combining the estimates of E_1, E_2, E_3, we get (2.1).

Next suppose $p=1$. By the hypothesis, for almost all $x \in I_2$ and for all $t \in [0,\infty)$, we have

$$\begin{aligned}
S_{n,k}(f,x) - f(x) &= \sum_{i=1}^{2k+1} \frac{f^{(i)}(x)}{i!} S_{n,k}((t-x)^i,x) \\
&\qquad + \frac{1}{(2k+1)!} S_{n,k}(\phi(t) \int_x^t (t-w)^{2k+1} f^{(2k+1)}(w)dw, x) \\
&\qquad + S_{n,k}(F(t,x)(1-\phi(t)),x) \\
&= M_1 + M_2 + M_3, \qquad \text{say,}
\end{aligned}$$

where$\phi(t)$ denotes the characteristic function of I_1 and

$$F(t,x) = f(t) - \sum_{i=0}^{2k+1} \frac{(t-x)^i}{i!} f^{(i)}(x),$$

for almost all $x \in I_2$ and $t \in [0, \infty)$.

Applying Lemma 2.3 and Lemma 2.4, we have

$$\|M_1\|_{L_1(I_2)} \leq C_{13} n^{-(k+1)} \left\{ \|f\|_{L_1(I_2)} + \left\| f^{(2k+1)} \right\|_{L_1(I_2)} \right\}.$$

Next, we have

$$\begin{aligned} &\|M_2\|_{L_1(I_2)} \\ &\quad \leq \frac{1}{(2k+1)!} \sum_{j=0}^{k} |C(j,k)| \int_{a_2}^{b_2} \int_{a_1}^{b_1} (d_j n - c) \sum_{v=0}^{\infty} p_{d_j n, v}(x) p_{d_j n, v}(t) \left| t - x \right|^{2k+1} \\ &\qquad \times \left| \int_x^t \left| df^{(2k+1)}(w) \right| \right| dt dx. \end{aligned}$$

For each $d_j n$ there exists a non negative integer $r = r(d_j n)$ satisfying

$$r(d_j n)^{-1/2} < \max(b_1 - a_2, b_2 - a_1) \leq (r+1)(d_j n)^{-1/2}.$$

Thus

$$\begin{aligned} \|M_2\|_{L_1(I_2)} &\leq \frac{1}{(2k+1)!} \sum_{j=0}^{k} |C(j,k)| \sum_{l=0}^{r} \int_{a_2}^{b_2} \left\{ \int_{x+(l)(d_j n)^{-1/2}}^{x+(l+1)(d_j n)^{-1/2}} \phi(t)(d_j n - c) \right. \\ &\cdot \left. \sum_{v=0}^{\infty} p_{d_j n, v}(x) p_{d_j n, v}(t) \left| t - x \right|^{2k+1} \right\} \left(\int_x^{x+(l+1)(d_j n)^{-1/2}} \phi(w) \left| df^{(2k+1)}(w) \right| \right) dt dx \\ &+ \frac{1}{(2k+1)!} \sum_{j=0}^{k} |C(j,k)| \sum_{l=0}^{r} \int_{a_2}^{b_2} \left\{ \int_{x+(l+1)(d_j n)^{-1/2}}^{x-(l)(d_j n)^{-1/2}} \phi(t)(d_j n - c) \right. \\ &\times \left. \sum_{v=0}^{\infty} p_{d_j n, v}(x) p_{d_j n, v}(t) \left| t - x \right|^{2k+1} \right\} \cdot \left(\int_{x-(l+1)(d_j n)^{-1/2}}^{x} \phi(w) \left| df^{(2k+1)}(w) \right| \right) dt dx. \end{aligned}$$

Suppose $\phi_{x,c,s}(w)$ denotes the characteristic function of the interval $\left[x - c(d_j n)^{-1/2}, x + s(d_j n)^{-1/2} \right]$, where c, s are nonnegative integers. Then we have

$$\begin{aligned} \|M_2\|_{L_1(I_2)} \leq \frac{1}{(2k+1)!} &\sum_{j=0}^{k} |C(j,k)| \sum_{l=1}^{r} \int_{a_2}^{b_2} \left\{ \int_{x+(l)(d_j n)^{-1/2}}^{x+(l+1)(d_j n)^{-1/2}} \phi(t)(d_j n)^2 (d_j n - c) \right. \\ &\times \left. \sum_{v=0}^{\infty} p_{d_j n, v}(x) p_{d_j n, v}(t) l^{-4} \left| t - x \right|^{2k+5} \right\} \\ &\times \left(\int_x^{x+(l+1)(d_j n)^{-1/2}} \phi(w) \phi_{x,0,l+1}(w) \left| df^{(2k+1)}(w) \right| \right) dt dx \\ &+ \frac{1}{(2k+1)!} \sum_{j=0}^{k} |C(j,k)| \sum_{l=1}^{r} \int_{a_2}^{b_2} \left\{ \int_{x+(l+1)(d_j n)^{-1/2}}^{x-(l)(d_j n)^{-1/2}} \phi(t)(d_j n)^2 (d_j n - c). \right. \\ &\times \left. \sum_{v=0}^{\infty} b_{d_j n, v}(x) p_{d_j n, v}(t) l^{-4} \left| t - x \right|^{2k+5} \right\} \end{aligned}$$

$$
\begin{aligned}
&\times\left(\int_{x-(l+1)(d_jn)^{-1/2}}^{x} \phi(w)\phi_{x,l+1,0}(w)\left|df^{(2k+1)}(w)\right|\right) dtdx\\
&\times \frac{1}{(2k+1)!}\sum_{j=0}^{k}|C(j,k)|\int_{a_2}^{b_2}\left\{\int_{(d_jn)^{-1/2}}^{a_1+(d_jn)^{-1/2}} \phi(t)(d_jn-c)\right.\\
&\left.\times\sum_{v=0}^{\infty}p_{d_jn,v}(x)p_{d_jn,v}(t)\left|t-x\right|^{2k+1}\right\}\\
&\times\left(\int_{x-(d_jn)^{-1/2}}^{x+(d_jn)^{-1/2}} \phi(w)\phi_{x,1,1}(w)\left|df^{(2k+1)}(w)\right|\right) dtdx\\
\leq{}& \frac{1}{(2k+1)!}\sum_{j=0}^{k}|C(j,k)|\sum_{l=1}^{r}l^{-4}\int_{a_2}^{b_2}\left\{\int_{x+(l)(d_jn)^{-1/2}}^{x+(l+1)(d_jn)^{-1/2}} \phi(t)(d_jn-c)\right.\\
&\left.\times\sum_{v=0}^{\infty}p_{d_jn,v}(x)p_{d_jn,v}(t)\left|t-x\right|^{2k+5}\right\}\\
&\times\left(\int_{a_1}^{b_1} \phi_{x,0,l+1}(w)\left|df^{(2k+1)}(w)\right|\right) dtdx\\
&+\frac{1}{(2k+1)!}\sum_{j=0}^{k}|C(j,k)|\sum_{l=1}^{r}l^{-4}\int_{a_2}^{b_2}\left\{\int_{x+(l+1)(d_jn)^{-1/2}}^{x-(l)(d_jn)^{-1/2}} \phi(t)(d_jn-c)\right.\\
&\left.\times\sum_{v=0}^{\infty}p_{d_jn,v}(x)p_{d_jn,v}(t)\left|t-x\right|^{2k+5}\right\}\\
&\times\left(\int_{a_1}^{b_1} \phi_{x,l+1,0}(w)\left|df^{(2k+1)}(w)\right|\right) dtdx\\
&+\frac{1}{(2k+1)!}\sum_{j=0}^{k}|C(j,k)|\int_{a_2}^{b_2}\left\{\int_{(d_jn)^{-1/2}}^{a_1+(d_jn)^{-1/2}} \phi(t)(d_jn-c)\right.\\
&\left.\times\sum_{v=0}^{\infty}p_{d_jn,v}(x)p_{d_jn,v}(t)\left|t-x\right|^{2k+1}\right\}\\
&\times\left(\int_{a_1}^{b_1} \phi(w)\phi_{x,1,1}(w)\left|df^{(2k+1)}(w)\right|\right) dtdx.
\end{aligned}
$$

Applying Lemma 2.1 and using Fubini's theorem we get

$$
\begin{aligned}
&\|M_2\|_{L_1(I_2)}\\
&\leq C_{14}\sum_{j=0}^{k}|C(j,k)|(d_jn)^{-(2k+1)/2}\sum_{l=1}^{r}l^{-4}\left\{\int_{a_2}^{b_2}\int_{a_1}^{b_1}\phi_{x,0,l+1}(w)\left|df^{(2k+1)}(w)\right|dx\right\}\\
&\quad+C_{14}\sum_{j=0}^{k}|C(j,k)|(d_jn)^{-(2k+1)/2}\sum_{l=1}^{r}l^{-4}\int_{a_2}^{b_2}\left\{\int_{a_1}^{b_1}\phi_{x,l+1,0}(w)\left|df^{(2k+1)}(w)\right|dx\right.\\
&\quad\left.+\int_{a_1}^{b_1}\phi_{x,1,1}(w)\left|df^{(2k+1)}(w)\right|dx\right\}
\end{aligned}
$$

$$\le C_{14}\sum_{j=0}^{k}|C(j,k)|(d_jn)^{-(2k+1)/2}\sum_{l=1}^{r}l^{-4}\left\{\int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\phi_{x,0,l+1}(w)dx\right)\left|df^{(2k+1)}(w)\right|\right\}$$

$$+C_{14}\sum_{j=0}^{k}|C(j,k)|(d_jn)^{-(2k+1)/2}\sum_{l=1}^{r}l^{-4}\int_{a_1}^{b_1}\left\{\int_{a_2}^{b_2}\phi_{x,l+1,0}(w)\left|df^{(2k+1)}(w)\right|dx\right.$$

$$\left.+\int_{a_2}^{b_2}\phi_{x,1,1}(w)\left|df^{(2k+1)}(w)\right|dx\right\}$$

$$\le C_{15}\sum_{j=0}^{k}|C(j,k)|(d_jn)^{-(2k+1)/2}\sum_{l=1}^{r}l^{-4}\left\{\int_{a_1}^{b_1}\left(\int_{w-(l+1)(d_jn)^{-1/2}}^{w}dx\right)\left|df^{(2k+1)}(w)\right|\right\}$$

$$+C_{15}\sum_{j=0}^{k}|C(j,k)|(d_jn)^{-(2k+1)/2}\sum_{l=1}^{r}l^{-4}\int_{a_1}^{b_1}\left\{\left(\int_{w}^{w+(l+1)(d_jn)^{-1/2}}dx\right)\left|df^{(2k+1)}(w)\right|\right.$$

$$\left.+\int_{a_1}^{b_1}\left(\int_{w-(d_jn)^{-1/2}}^{w+(d_jn)^{-1/2}}dx\right)\left|df^{(2k+1)}(w)\right|\right\}$$

$$\le C_{16}n^{-(k+1)}\left\|f^{(2k+1)}\right\|_{B.V.(I_1)}.$$

Finally we estimate M_3. It is sufficient to choose the expression without the linear combinations. For all $t\in[0,\infty)\backslash[a_1,b_1]$ and all $x\in I_2$, we choose a $\delta>0$ such that $|t-x|\ge\delta$. Thus

$$\begin{aligned}
&\|S_n(F(t,x)(1-\phi(t)),x)\|_{L_1(I_2)}\\
&\le\int_{a_2}^{b_2}\int_0^{\infty}(n-c)\sum_{v=0}^{\infty}p_{n,v}(x)p_{n,v}(t)\left|f(t)\right|(1-\phi(t))dtdx\\
&\le\sum_{i=0}^{2k+1}\frac{1}{i!}\int_{a_2}^{b_2}\int_0^{\infty}(n-c)\sum_{v=0}^{\infty}p_{n,v}(x)p_{n,v}(t)\left|f^{(i)}(t)\right|.\left|t-x\right|^{i}(1-\phi(t))dtdx\\
&=M_4+M_5,\qquad\text{say.}
\end{aligned}$$

For sufficiently large t there exist positive constants C_{17},C_{18} such that $\frac{(t-x)^{2k+2}}{t^{2k+2}+1}>C_{17}$ for all $t\ge C_{18}$, $t\in I_2$. Applying Fubini's theorem and Lemma 2.1, we obtain

$$\begin{aligned}
M_4&=\left(\int_0^{C_{18}}\int_{a_2}^{b_2}+\int_{C_{18}}^{\infty}\int_{a_2}^{b_2}\right)\sum_{v=0}^{\infty}(n-c)p_{n,v}(x)p_{n,v}(t)\left|f(t)\right|(1-\phi(t))dtdx\\
&\le C_{19}n^{-(k+1)}\left(\int_0^{C_{20}}|f(t)|\,dt\right)+\frac{1}{C_{17}}\int_{C_{20}}^{\infty}\int_{a_2}^{b_2}\sum_{v=0}^{\infty}p_{n,v}(x)p_{n,v}(t)\frac{(t-x)^{2k+2}}{(t^{2k+2}+1)}\left|f(t)\right|dxdt\\
&\le C_{20}n^{-(k+1)}\left\{\left(\int_0^{C_{20}}|f(t)|\,dt\right)+\left(\int_{C_{20}}^{\infty}|f(t)|\,dt\right)\right\}\le C_{21}n^{-(k+1)}\|f\|_{L_1[0,\infty)}.
\end{aligned}$$

Finally by the Lemma 2.4, we have

$$M_5\le C_{22}n^{-(k+1)}\left\{\|f\|_{L_1(I_2)}+\left\|f^{(2k+1)}\right\|_{L_1(I_2)}\right\}.$$

Combining M_4 and M_5, we obtain

$$M_3\le C_{23}n^{-(k+1)}\left\{\|f\|_{L_1[0,\infty)}+\left\|f^{(2k+1)}\right\|_{L_1(I_2)}\right\}.$$

This completes the proof of (2.2) of the lemma. □

3. RATE OF CONVERGENCE

Theorem 3.1. *Let $f \in L_p[0,\infty), p \geq 1$. Then for n sufficiently large*

$$\|S_{n,k}(f,\cdot) - f\|_{L_p(I_2)} \leq C_{24}\left\{\omega_{2k+2}(f, n^{-1/2}, p, I_1) + n^{-(k+1)}\|f\|_{L_p[0,\infty)}\right\},$$

where C_{24} is a constant independent of f and n.

Proof. We can write

$$\begin{aligned}\|S_{n,k}(f,*) - f\|_{L_p(I_2)} &\leq \|S_{n,k}(f - f_{\eta,2k+2},*)\|_{L_p(I_2)} \\ &\quad + \|S_{n,k}(f_{\eta,2k+2},*) - f_{\eta,2k+2})\|_{L_p(I_2)} + \|(f_{\eta,2k+2} - f)\|_{L_p(I_2)} \\ &= E_1 + E_2 + E_3, \qquad \text{say.}\end{aligned}$$

It is well known that

$$\left\|f^{(2k+1)}_{\eta,2k+2}\right\|_{B.V.(I_3)} = \left\|f^{(2k+2)}_{\eta,2k+2}\right\|_{L_1(I_3)}.$$

Therefore from Lemma 2.5 $(p > 1)$ and $(p = 1)$ we have

$$\begin{aligned}E_2 &\leq C_{25}n^{-(k+1)}\left(\left\|f^{(2k+2)}_{\eta,2k+2}\right\|_{L_p(I_3)} + \|f_{\eta,2k+2}\|_{L_p[0,\infty)}\right) \\ &\leq C_{26}n^{-(k+1)}\left(n^{-(k+2)}\omega_{2k+2}(f,\eta,p,I_1) + \|f\|_{L_p[0,\infty)}\right),\end{aligned}$$

which follows from the properties of Steklov means.

Let $\phi(t)$ be the characteristic function of I_3, we have

$$\begin{aligned}S_n((f - f_{\eta,2k+2})(t),x) &= S_n(\phi(t)(f - f_{\eta,2k+2})(t),x) + S_n((1-\phi(t))(f - f_{\eta,2k+2})(t),x) \\ &= E_4 + E_5, \qquad \text{say.}\end{aligned}$$

By Hölder's inequality

$$\int_{a_2}^{b_2} |E_4|^p dx \leq \int_{a_2}^{b_2}\int_{a_1}^{b_1}(n-c)\sum_{v=0}^{\infty} p_{n,v}(x)p_{n,v}(t)\,|(f - f_{\eta,2k+2})(t)|^p\,dtdx.$$

Applying Fubini's theorem, we have

$$\int_{a_2}^{b_2} |E_4|^p dx \leq \|f - f_{\eta,2k+2}\|_{Lp(I_2)}.$$

Similarly, for all $p \geq 1$

$$\|E_5\|_{Lp(I_2)} \leq C_{27}\eta^{-(k+1)}\|f - f_{\eta,2k+2}\|_{Lp(0,\infty)}.$$

Consequently, via the property of Steklov means, we find that

$$\|S_n\left(f - f\eta, 2k+2, \cdot\right)\|_{Lp(I2)} \leq C_{28}\left\{\omega_{2k+2}\left(f,\eta,p,I_1\right) + \eta^{-(k+1)}\|f\|_{Lp[0,\infty)}\right\}.$$

Hence

$$E_1 \leq C_{29}\left\{\omega_{2k+2}\left(f,\eta,p,I_1\right) + \eta^{-(k+1)}\|f\|_{Lp[0,\infty)}\right\}.$$

Thus, with $\eta = n^{-1/2}$, the result follows.

This completes the proof of the theorem. □

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