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RATE OF CONVERGENCE FOR A GENERAL SEQUENCE OF DURRMEYER TYPE OPERATORS

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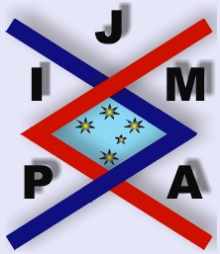

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Abstract

In the present paper we give the rate of convergence for the linear combinations of the generalized Durrmeyer type operators which includes the well known Szasz-Durrmeyer operators and Baskakov-Durrmeyer operators as special cases.

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Key words: Szasz-Durrmeyer operators, Baskakov-Durrmeyer operators, Rate of convergence, Order of approximation.

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1. Introduction

Durrmeyer [1] introduced the integral modification of Bernstein polynomials so as to approximate Lebesgue integrable functions on the interval $[0,1]$. We now consider the general family of Durrmeyer type operators, which is defined by

$$S_n(f(t),x) = (n-c)\sum_{v=0}^{\infty} p_{n,v}(x)\int_0^{\infty} p_{n,v}(t)\, f(t)\, dt, \quad x \in I \tag{1.1}$$

where $n \in \mathbb{N}$, $n > \max\{0,-c\}$ and $p_{n,v}(x) = (-1)^v \frac{x^v}{v!}\varphi_n^{(v)}(x)$. Also $\{\phi_n\}$ is a sequence of real functions having the following properties on $[0,a]$, where $a > 0$ and for all $n \in \mathbb{N}$, $v \in \mathbb{N} \cup \{0\}$, we have

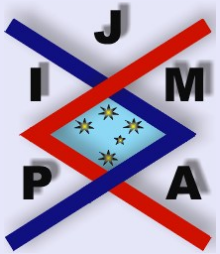
(I) $\phi_n \in C^{\infty}[0,a], \phi_n(0) = 1$.

(II) ϕ_n is complete monotonic.

(III) There exist $c \in \mathbb{N} : \phi_n^{(v+1)} = -n\phi_{n+c}^{(v)}, n > \max\{0,-c\}$.

Some special cases of the operators (1.1) are as follows:

1. If $c = 0$,$\phi_n(x) = e^{-nx}$, we get the Szasz-Durrmeyer operator.
2. If $c = 1, \phi_n(x) = (1+x)^{-n}$, we obtain the Baskakov-Durrmeyer operator.
3. If $c > 1, \phi_n(x) = (1+cx)^{\frac{-n}{c}}$, we obtain a general Baskakov-Durrmeyer operator.

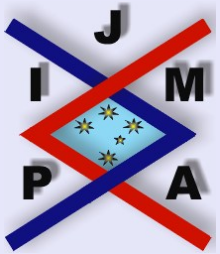


4. If $c = -1, \phi_n(x) = (1-x)^n$, we obtain the Bernstein-Durrmeyer operator.

Very recently Srivastava and Gupta [9] studied a similar type of operators and obtained the rate of convergence for functions of bounded variation. It is easily verified that the operators (1.1) are linear positive operators and these operators reproduce the constant ones, while the operators studied in [9] reproduce every linear functional for the case $c = 0$. Several researchers studied different approximation properties on the special cases of the operators (1.1), the pioneer work on Durrmeyer type operators is due to S. Guo [3], Vijay Gupta (see e. g. [4], [5]), R. P. Sinha et al. [8] and Wang and Guo [11], etc. It turns out that the order of approximation for such type of Durrmeyer operators is at best $O(n^{-1})$, how so ever smooth the function may be. In order to improve the order of approximation, we have to slacken the positivity condition of the operators, for this we consider the linear combinations of the operators (1.1). The technique of linear combinations is described as follows:

$$S_{n,k}(f,x) = \begin{vmatrix} 1 & d_0^{-1} & d_0^{-2} & ..... & d_0^{-k} \\ 1 & d_1^{-1} & d_1^{-2} & ..... & d_1^{-k} \\ ... & .. & .. & ..... & .. \\ ... & .. & .. & ..... & .. \\ 1 & d_k^{-1} & d_k^{-2} & ..... & d_k^{-k} \end{vmatrix}^{-1} \begin{vmatrix} S_{d_0 n}(f,x) & d_0^{-1} & d_0^{-2} & ..... & d_0^{-k} \\ S_{d_1 n}(f,x) & d_1^{-1} & d_1^{-2} & ..... & d_1^{-k} \\ ... & .. & .. & ..... & .. \\ ... & .. & .. & ..... & .. \\ S_{d_k n}(f,x) & d_k^{-1} & d_k^{-2} & ..... & d_k^{-k} \end{vmatrix}.$$

Such types of linear combinations were first considered by May [7] to improve the order of approximation of exponential type operators. In the alternative form



the above linear combinations can be defined as

$$S_{n,k}(f,x) = \sum_{j=0}^{k} c(j,k) S_{d_j n}(f,x),$$

where

$$C(j,k) = \prod_{\substack{i=0 \\ i\neq j}}^{k} \frac{d_j}{d_j - d_i}, k \neq 0 \qquad \text{and} \qquad C(0,0) = 1.$$

If $f \in L_p[0,\infty)$, $1 \leq p < \infty$ and $0 < a_1 < a_3 < a_2 < b_2 < b_3 < b_1 < \infty$ and $I_i = [a_i, b_i]$, $i = 1,2,3$, the Steklov mean $f_{\eta,m}$ of m^{th} order corresponding to f is defined as

$$f_{\eta,m}(t) = \eta^{-m} \int_{\frac{-\eta}{2}}^{\frac{\eta}{2}} \cdots \int_{\frac{-\eta}{2}}^{\frac{\eta}{2}} \left(f(t) + (-1)^{m-1} \Delta^{m}_{\sum_{i=1}^{m} t_i} f(t) \right) \prod_{i=1}^{m} dt_i; \quad t \in I_1.$$

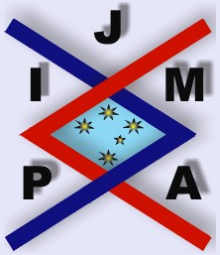
It can be verified [6, 10] that

(i) $f_{\eta,m}$ has derivative up to order m, $f^{(m-1)}_{\eta,m} \in AC(I_1)$ and $f^{(m)}_{\eta,m}$ exist almost every where and belongs to $L_p(I_1)$;

(ii) $\left\| f^{(r)}_{\eta,m} \right\|_{L_p(I_1)} \leq K_1 \eta^{-r} \omega_r(f,\eta,p,I_1), r = 1,2,3,\ldots,m;$

(iii) $\|f - f_{\eta,m}\|_{L_p(I_2)} \leq K_2 \omega_m(f,\eta,p,I_1);$

(iv) $\left\| f^{m}_{\eta,m} \right\|_{L_p(I_2)} \leq K_3 \eta^m \|f\|_{L_p(I_1)}$



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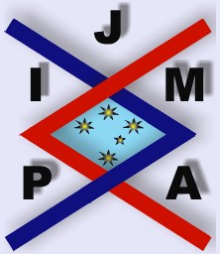
(v) $\left\| f_{\eta,m}^{(m)} \right\|_{L_p(I_1)} \leq K_4 \eta^{-r} \omega_r(f, \eta, p, I_1)$

where $K_i's$, $i = 1, 2, 3, 4$ are certain constants independent of f and η.

In the present paper we establish the rate of convergence for the combinations of the generalized Durrmeyer type operators in L_p-norm.



2. Auxiliary Results

To prove the rate of convergence we need the following lemmas:

Lemma 2.1. *Let $m \in \mathbb{N} \cup \{0\}$, also $\mu_{n,m}(x)$ is the m^{th} order central moment defined by*

$$\mu_{n,m} \equiv S_n((t-x)^m, x) = (n-c)\sum_{v=0}^{\infty} p_{n,v}(x) \int_0^{\infty} p_{n,v}(t)(t-x)^m dt,$$

then

(i) $\mu_{n,m}(x)$ *is a polynomial in* x *of degree* m.

(ii) $\mu_{n,m}(x)$ *is a rational function in* n *and for each* $0 \leq x < \infty \quad \mu_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$.

Remark 2.1. *Using Hölder's inequality, it can be easily verified that* $S_n(|t-x|^r, x) = O(n^{-r/2})$ *for each* $r > 0$ *and for every fixed* $0 \leq x < \infty$.

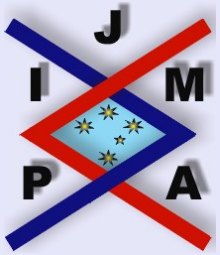
Lemma 2.2. *For sufficiently large* n *and* $q \in \mathbb{N}$, *there holds*

$$S_{n,k}\left((t-x)^q, x\right) = n^{-(k+1)}\{F(q,k,x) + o(1)\},$$

where $F(q,k,x)$ *are certain polynomials in* x *of degree* q *and* $0 \leq x < \infty$ *is arbitrary but fixed.*

Proof. For sufficiently large values of n we can write, from Lemma 2.1

$$S_n\left((t-x)^q, x\right) = \frac{Q_0(x)}{n^{[(q+1)/2]}} + \frac{Q_1(x)}{n^{[(q+1)/2]+1}} + \cdots + \frac{Q_{[q/2]}(x)}{n^q},$$



where $Q_i(x), i = 0, 1, 2, \ldots$ are certain polynomials in x of at most degree q.

Therefore $S_{n,k}\left((t-x)^q, x\right)$ is given by

$$\begin{vmatrix} 1 & d_0^{-1} & d_0^{-2} & ..... & d_0^{-k} \\ 1 & d_1^{-1} & d_1^{-2} & ..... & d_1^{-k} \\ ... & .. & .. & ..... & .. \\ ... & .. & .. & ..... & .. \\ 1 & d_k^{-1} & d_k^{-2} & ..... & d_k^{-k} \end{vmatrix}^{-1}$$

$$\times \begin{vmatrix} \frac{Q_0(x)}{(d_0 n)^{[(q+1)/2]}} + \frac{Q_1(x)}{(d_0 n)^{[(q+1)/2]+1}} + ... & d_0^{-1} & d_0^{-2} & ..... & d_0^{-k} \\ \frac{Q_0(x)}{(d_1 n)^{[(q+1)/2]}} + \frac{Q_1(x)}{(d_1 n)^{[(q+1)/2]+1}} + ... & d_1^{-1} & d_1^{-2} & ..... & d_1^{-k} \\ ..... & .. & .. & ..... & .. \\ ..... & .. & .. & ..... & .. \\ \frac{Q_0(x)}{(d_k n)^{[(q+1)/2]}} + \frac{Q_1(x)}{(d_k n)^{[(q+1)/2]+1}} + ... & d_k^{-1} & d_k^{-2} & ..... & d_k^{-k} \end{vmatrix}$$

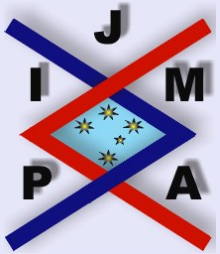
$$= n^{-(k+1)} \left\{F(q,k,x) + o(1)\right\}, \quad 0 \le x < \infty.$$

□

Lemma 2.3 ([4]). *Let $1 \le p < \infty$, $f \in L_p[a,b]$, $f^{(m)} \in AC[a,b]$ and $f^{(m+1)} \in L_p[a,b]$, then*

$$\left\|f^{(j)}\right\|_{L_p[a,b]} \le C \left\{\left\|f^{(m+1)}\right\|_{L_p[a,b]} + \|f\|_{L_p[a,b]}\right\},$$

$j = 1, 2, 3, \ldots, m$ and C is a constant depending on j, p, m, a, b.



Lemma 2.4. *Let $f \in L_p[0,\infty)$, $p > 1$. If $f^{(2k+1)} \in AC(I_1)$ and $f^{(2k+2)} \in L_p(I_1)$, then for all n sufficiently large*

$$\|S_{n,k}(f,\cdot) - f\|_{L_p(I_2)} \leq C_1 n^{-(k+1)} \left\{ \left\| f^{(2k+2)} \right\|_{L_p(I_2)} + \|f\|_{L_p[0,\infty)} \right\}. \tag{2.1}$$

Also if $f \in L_1[0,\infty)$, $f^{(2k+1)} \in L_1(I_1)$ with $f^{(2k)} \in AC(I_1)$ and $f^{(2k+1)} \in BV(I_1)$, then for all n sufficiently large

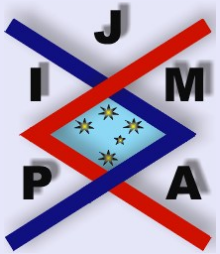
$$\begin{aligned} &\|S_{n,k}(f,\cdot) - f\|_{L_p(I_2)} \\ &\quad \leq C_2 n^{-(k+1)} \left\{ \left\| f^{(2k+1)} \right\|_{BV(I_1)} + \left\| f^{(2k+1)} \right\|_{L_1(I_2)} + \|f\|_{L_1[0,\infty)} \right\}. \end{aligned} \tag{2.2}$$

Proof. First let $p > 1$. By the hypothesis, for all $t \in [0,\infty)$ and $x \in I_2$, we have

$$\begin{aligned} S_{n,k}(f,x) - f(x) &= \sum_{i=1}^{2k+1} \frac{f^{(i)}(x)}{i!} S_{n,k}((t-x)^i, x) \\ &\quad + \frac{1}{(2k+1)!} S_{n,k}\left(\phi(t) \int_x^t (t-w)^{2k+1} f^{(2k+2)}(w)\,dw, x\right) \\ &\quad + S_{n,k}(F(t,x)(1-\phi(t)), x) \\ &= E_1 + E_2 + E_3, \quad \text{say,} \end{aligned}$$

where $\phi(t)$ denotes the characteristic function of I_1 and

$$F(t,x) = f(t) - \sum_{i=0}^{2k+1} \frac{(t-x)^i}{i!} f^{(i)}(x).$$



Applying Lemma 2.2 and Lemma 2.3, we have

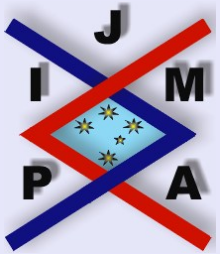
$$\|E_1\|_{L_p(I_2)} \leq C_3 n^{-(k+1)} \sum_{i=1}^{2k+1} \left\|f^{(i)}\right\|_{L_p(I_2)}$$
$$\leq C_4 n^{-(k+1)} \left\{\|f\|_{L_p(I_2)} + \left\|f^{(2k+2)}\right\|_{L_p(I_2)}\right\}.$$

Next we estimate E_2. Let H_f be the Hardy Littlewood maximal function [10] of $f^{(2k+2)}$ on I_1, using Hölder's inequality and Lemma 2.1, we have

$$|E_2| \leq \frac{1}{(2k+2)!} S_{n,k}\left(\phi(t)\left|\int_x^t |t-w|^{2k+1}\left|f^{(2k+2)}(w)\right|dw\right|, x\right)$$
$$\leq \frac{1}{(2k+2)!} S_{n,k}\left(\phi(t)\,|t-x|^{2k+1}\left|\int_x^t \left|f^{(2k+2)}(w)\right|dw\right|, x\right)$$
$$\leq \frac{1}{(2k+2)!} S_{n,k}\left(\phi(t)\,|t-x|^{2k+2}\,|H_f(t)|, x\right)$$
$$\leq \frac{1}{(2k+2)!}\left\{S_{n,k}\left(\phi(t)\,|t-x|^{q(2k+2)}, x\right)\right\}^{\frac{1}{q}}\left\{S_{n,k}\left(\phi(t)\,|H_f(t)|^p, x\right)\right\}^{\frac{1}{p}}$$
$$\leq C_5 n^{-(k+1)}\left(\sum_{j=0}^{k} C(j,k)\int_{a_1}^{b_1}(d_j n - c)\sum_{v=0}^{\infty} p_{d_j n,v}(x)p_{d_j n,v}(t)\,|H_f(t)|^p\,dt\right)^{\frac{1}{p}}.$$

Next applying Fubini's theorem, we have

$$\|E_2\|^p_{L_p(I_2)}$$
$$\leq C_6 n^{-p(k+1)}\sum_{j=0}^{k} C(j,k)\int_{a_2}^{b_2}\int_{a_1}^{b_1}(d_j n - c)\sum_{v=0}^{\infty} p_{d_j n,v}(x)p_{d_j n,v}(t)\,|H_f(t)|^p\,dtdx$$



$$\leq C_6 n^{-p(k+1)} \sum_{j=0}^{k} C(j,k) \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} (d_j n - c) \sum_{v=0}^{\infty} p_{d_j n,v}(x) p_{d_j n,v}(t) dx \right) |H_f(t)|^p dt$$

$$\leq C_7 n^{-p(k+1)} \|H_f\|^p_{L_p(I_1)} \leq C_8 n^{-p(k+1)} \left\| f^{(2k+2)} \right\|^p_{L_p(I_1)}.$$

Therefore

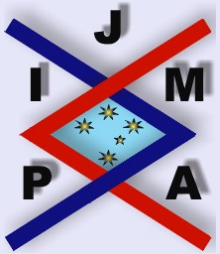
$$\|E_2\|_{L_p(I_2)} \leq C_8 n^{-(k+1)} \left\| f^{(2k+2)} \right\|_{L_p(I_1)}.$$

For $t \in [0,\infty)\backslash[a_1,b_1],\ x \in I_2$ there exists a $\delta > 0$ such that $|t-x| \geq \delta$. Thus

$$\begin{aligned} E_3 &\equiv S_{n,k}\left(F(t,x)(1-\phi(t)),x\right) \\ &\leq \delta^{-(2k+2)} \sum_{j=0}^{k} C(j,k) S_{d_j n}\left(|F(t,x)|\,(t-x)^{2k+2},x\right) \\ &\leq \delta^{-(2k+2)} \sum_{j=0}^{k} C(j,k) \Bigg[S_{d_j n}\left(|f(t)|\,(t-x)^{2k+2},x\right) \\ &\qquad + \sum_{i=0}^{2k+1} \frac{\left|f^{(i)}(x)\right|}{i!} S_{d_j n}\left(|t-x|^{2k+2+i},x\right) \Bigg] \\ &= E_{31} + E_{32}, \quad \text{say.} \end{aligned}$$

Applying Hölder's inequality and Lemma 2.1, we have

$$|E_{31}| \leq \delta^{-(2k+2)} \sum_{j=0}^{k} C(j,k) \left\{ S_{d_j n}(|f(t)|^p, x) \right\}^{\frac{1}{p}} \left\{ S_{d_j n}(|t-x|^{q(2k+2)}, x) \right\}^{\frac{1}{q}}$$



$$\leq C_9 \sum_{j=0}^{k} C(j,k) \left\{ S_{d_j n}(|f(t)|^p, x) \right\}^{\frac{1}{p}} \frac{1}{(d_j n)^{k+1}}.$$

Finally by Fubini's theorem, we obtain

$$\begin{aligned} \|E_{31}\|^p_{L_p(I_2)} &= \int_{a_2}^{b_2} |E_{31}|^p dx \\ &\leq C_9 \sum_{j=0}^{k} C(j,k)(d_j n)^{-p(k+1)} \\ &\qquad \times \int_{a_2}^{b_2} \int_0^{\infty} (d_j n - c) p_{d_j n,v}(x) p_{d_j n,v}(t) \, |f(t)|^p \, dt dx \\ &\leq C_{10} n^{-p(k+1)} \|f\|^p_{L_p[0,\infty)}. \end{aligned}$$

Again by Lemma 2.3, we have

$$\|E_{32}\|_{L_p(I_2)} \leq C_{11} n^{-(k+1)} \left\{ \|f\|_{L_p(I_2)} + \left\| f^{(2k+2)} \right\|_{L_p(I_2)} \right\}.$$

Thus

$$\|E_3\|_{L_p(I_2)} \leq C_{12} n^{-(k+1)} \left\{ \|f\|_{L_p[0,\infty)} + \left\| f^{(2k+2)} \right\|_{L_p(I_2)} \right\}.$$

Combining the estimates of E_1, E_2, E_3, we get (2.1).

Next suppose $p = 1$. By the hypothesis, for almost all $x \in I_2$ and for all

$t \in [0, \infty)$, we have

$$S_{n,k}(f,x) - f(x) = \sum_{i=1}^{2k+1} \frac{f^{(i)}(x)}{i!} S_{n,k}((t-x)^i, x)$$
$$+ \frac{1}{(2k+1)!} S_{n,k}(\phi(t) \int_x^t (t-w)^{2k+1} f^{(2k+1)}(w)dw, x)$$
$$+ S_{n,k}(F(t,x)(1-\phi(t)), x)$$
$$= M_1 + M_2 + M_3, \qquad \text{say,}$$

where$\phi(t)$ denotes the characteristic function of I_1 and

$$F(t,x) = f(t) - \sum_{i=0}^{2k+1} \frac{(t-x)^i}{i!} f^{(i)}(x),$$

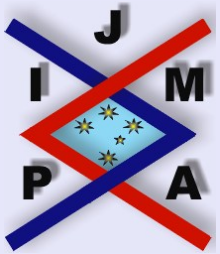
for almost all $x \in I_2$ and $t \in [0, \infty)$.

Applying Lemma 2.2 and Lemma 2.3, we have

$$\|M_1\|_{L_1(I_2)} \leq C_{13} n^{-(k+1)} \left\{ \|f\|_{L_1(I_2)} + \left\|f^{(2k+1)}\right\|_{L_1(I_2)} \right\}.$$

Next, we have

$$\|M_2\|_{L_1(I_2)}$$
$$\leq \frac{1}{(2k+1)!} \sum_{j=0}^{k} |C(j,k)| \int_{a_2}^{b_2} \int_{a_1}^{b_1} (d_j n - c) \sum_{v=0}^{\infty} p_{d_j n,v}(x) p_{d_j n,v}(t) \, |t-x|^{2k+1}$$
$$\times \left| \int_x^t \left| df^{(2k+1)}(w) \right| \right| dtdx.$$



For each d_jn there exists a non negative integer $r = r(d_jn)$ satisfying

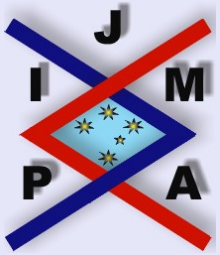
$$r(d_jn)^{-1/2} < \max(b_1 - a_2, b_2 - a_1) \leq (r+1)(d_jn)^{-1/2}.$$

Thus

$$\|M_2\|_{L_1(I_2)} \leq \frac{1}{(2k+1)!} \sum_{j=0}^{k} |C(j,k)| \sum_{l=0}^{r} \int_{a_2}^{b_2} \left\{ \int_{x+(l)(d_jn)^{-1/2}}^{x+(l+1)(d_jn)^{-1/2}} \phi(t)(d_jn - c) \right.$$
$$\left. \cdot \sum_{v=0}^{\infty} p_{d_jn,v}(x) p_{d_jn,v}(t) \, |t-x|^{2k+1} \right\} \left(\int_{x}^{x+(l+1)(d_jn)^{-1/2}} \phi(w) \left| df^{(2k+1)}(w) \right| \right) dtdx$$
$$+ \frac{1}{(2k+1)!} \sum_{j=0}^{k} |C(j,k)| \sum_{l=0}^{r} \int_{a_2}^{b_2} \left\{ \int_{x+(l+1)(d_jn)^{-1/2}}^{x-(l)(d_jn)^{-1/2}} \phi(t)(d_jn - c) \right.$$
$$\left. \times \sum_{v=0}^{\infty} p_{d_jn,v}(x) p_{d_jn,v}(t) \, |t-x|^{2k+1} \right\} \cdot \left(\int_{x-(l+1)(d_jn)^{-1/2}}^{x} \phi(w) \left| df^{(2k+1)}(w) \right| \right) dtdx.$$

Suppose $\phi_{x,c,s}(w)$ denotes the characteristic function of the interval $\left[x - c(d_jn)^{-1/2}, x + s(d_jn)^{-1/2}\right]$, where c, s are nonnegative integers. Then we have

$$\|M_2\|_{L_1(I_2)}$$
$$\leq \frac{1}{(2k+1)!} \sum_{j=0}^{k} |C(j,k)| \sum_{l=1}^{r} \int_{a_2}^{b_2} \left\{ \int_{x+(l)(d_jn)^{-1/2}}^{x+(l+1)(d_jn)^{-1/2}} \phi(t)(d_jn)^2 (d_jn - c) \right.$$
$$\left. \times \sum_{v=0}^{\infty} p_{d_jn,v}(x) p_{d_jn,v}(t) l^{-4} \, |t-x|^{2k+5} \right\}$$



$$\times \left(\int_x^{x+(l+1)(d_jn)^{-1/2}} \phi(w)\phi_{x,0,l+1}(w) \left| df^{(2k+1)}(w) \right| \right) dtdx$$

$$+ \frac{1}{(2k+1)!} \sum_{j=0}^{k} |C(j,k)| \sum_{l=1}^{r} \int_{a_2}^{b_2} \left\{ \int_{x+(l+1)(d_jn)^{-1/2}}^{x-(l)(d_jn)^{-1/2}} \phi(t)(d_jn)^2(d_jn-c). \right.$$

$$\left. \times \sum_{v=0}^{\infty} b_{d_jn,v}(x) p_{d_jn,v}(t) l^{-4} |t-x|^{2k+5} \right\}$$

$$\times \left(\int_{x-(l+1)(d_jn)^{-1/2}}^{x} \phi(w)\phi_{x,l+1,0}(w) \left| df^{(2k+1)}(w) \right| \right) dtdx$$

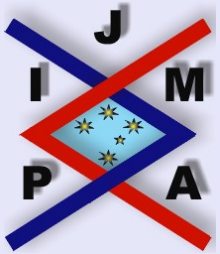
$$\times \frac{1}{(2k+1)!} \sum_{j=0}^{k} |C(j,k)| \int_{a_2}^{b_2} \left\{ \int_{(d_jn)^{-1/2}}^{a_1+(d_jn)^{-1/2}} \phi(t)(d_jn-c) \right.$$

$$\left. \times \sum_{v=0}^{\infty} p_{d_jn,v}(x) p_{d_jn,v}(t) |t-x|^{2k+1} \right\}$$

$$\times \left(\int_{x-(d_jn)^{-1/2}}^{x+(d_jn)^{-1/2}} \phi(w)\phi_{x,1,1}(w) \left| df^{(2k+1)}(w) \right| \right) dtdx$$

$$\leq \frac{1}{(2k+1)!} \sum_{j=0}^{k} |C(j,k)| \sum_{l=1}^{r} l^{-4} \int_{a_2}^{b_2} \left\{ \int_{x+(l)(d_jn)^{-1/2}}^{x+(l+1)(d_jn)^{-1/2}} \phi(t)(d_jn-c) \right.$$

$$\left. \times \sum_{v=0}^{\infty} p_{d_jn,v}(x) p_{d_jn,v}(t) |t-x|^{2k+5} \right\}$$



Rate Of Convergence For A General Sequence Of Durrmeyer Type Operators

Niraj Kumar

$$\times \left(\int_{a_1}^{b_1} \phi_{x,0,l+1}(w) \left| df^{(2k+1)}(w) \right| \right) dtdx$$

$$+ \frac{1}{(2k+1)!} \sum_{j=0}^{k} |C(j,k)| \sum_{l=1}^{r} l^{-4} \int_{a_2}^{b_2} \left\{ \int_{x+(l+1)(d_j n)^{-1/2}}^{x-(l)(d_j n)^{-1/2}} \phi(t)(d_j n - c) \right.$$

$$\left. \times \sum_{v=0}^{\infty} p_{d_j n,v}(x) p_{d_j n,v}(t) \left| t - x \right|^{2k+5} \right\}$$

$$\times \left(\int_{a_1}^{b_1} \phi_{x,l+1,0}(w) \left| df^{(2k+1)}(w) \right| \right) dtdx$$

$$+ \frac{1}{(2k+1)!} \sum_{j=0}^{k} |C(j,k)| \int_{a_2}^{b_2} \left\{ \int_{(d_j n)^{-1/2}}^{a_1+(d_j n)^{-1/2}} \phi(t)(d_j n - c) \right.$$

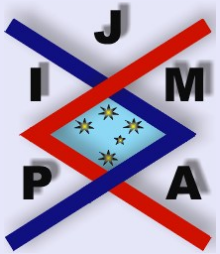
$$\left. \times \sum_{v=0}^{\infty} p_{d_j n,v}(x) p_{d_j n,v}(t) \left| t - x \right|^{2k+1} \right\}$$

$$\times \left(\int_{a_1}^{b_1} \phi(w) \phi_{x,1,1}(w) \left| df^{(2k+1)}(w) \right| \right) dtdx.$$

Applying Lemma 2.1 and using Fubini's theorem we get

$$\|M_2\|_{L_1(I_2)}$$

$$\leq C_{14} \sum_{j=0}^{k} |C(j,k)| (d_j n)^{-\frac{2k+1}{2}} \sum_{l=1}^{r} l^{-4} \left\{ \int_{a_2}^{b_2} \int_{a_1}^{b_1} \phi_{x,0,l+1}(w) \left| df^{(2k+1)}(w) \right| dx \right\}$$



$$+ C_{14} \sum_{j=0}^{k} |C(j,k)| (d_j n)^{-\frac{2k+1}{2}} \sum_{l=1}^{r} l^{-4} \int_{a_2}^{b_2} \left\{ \int_{a_1}^{b_1} \phi_{x,l+1,0}(w) \left| df^{(2k+1)}(w) \right| dx \right.$$

$$\left. + \int_{a_1}^{b_1} \phi_{x,1,1}(w) \left| df^{(2k+1)}(w) \right| dx \right\}$$

$$\leq C_{14} \sum_{j=0}^{k} |C(j,k)| (d_j n)^{-\frac{2k+1}{2}} \sum_{l=1}^{r} l^{-4} \left\{ \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} \phi_{x,0,l+1}(w) dx \right) \left| df^{(2k+1)}(w) \right| \right\}$$

$$+ C_{14} \sum_{j=0}^{k} |C(j,k)| (d_j n)^{-\frac{2k+1}{2}} \sum_{l=1}^{r} l^{-4} \int_{a_1}^{b_1} \left\{ \int_{a_2}^{b_2} \phi_{x,l+1,0}(w) \left| df^{(2k+1)}(w) \right| dx \right.$$

$$\left. + \int_{a_2}^{b_2} \phi_{x,1,1}(w) \left| df^{(2k+1)}(w) \right| dx \right\}$$

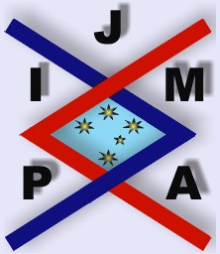
$$\leq C_{15} \sum_{j=0}^{k} |C(j,k)| (d_j n)^{-\frac{2k+1}{2}} \sum_{l=1}^{r} l^{-4} \left\{ \int_{a_1}^{b_1} \left(\int_{w-(l+1)(d_j n)^{-\frac{1}{2}}}^{w} dx \right) \left| df^{(2k+1)}(w) \right| \right\}$$

$$+ C_{15} \sum_{j=0}^{k} |C(j,k)| (d_j n)^{-\frac{2k+1}{2}} \sum_{l=1}^{r} l^{-4} \int_{a_1}^{b_1} \left\{ \left(\int_{w}^{w+(l+1)(d_j n)^{-\frac{1}{2}}} dx \right) \left| df^{(2k+1)}(w) \right| \right.$$

$$\left. + \int_{a_1}^{b_1} \left(\int_{w-(d_j n)^{-\frac{1}{2}}}^{w+(d_j n)^{-\frac{1}{2}}} dx \right) \left| df^{(2k+1)}(w) \right| \right\}$$

$$\leq C_{16} n^{-(k+1)} \left\| f^{(2k+1)} \right\|_{B.V.(I_1)}.$$

Finally we estimate M_3. It is sufficient to choose the expression without the

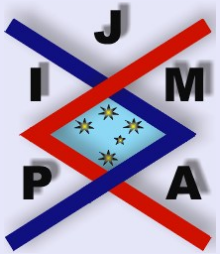


linear combinations. For all $t \in [0,\infty)\backslash[a_1,b_1]$ and all $x \in I_2$, we choose a $\delta > 0$ such that $|t-x| \geq \delta$. Thus

$$\begin{aligned}
&\|S_n(F(t,x)(1-\phi(t)),x)\|_{L_1(I_2)} \\
&\leq \int_{a_2}^{b_2}\int_0^{\infty}(n-c)\sum_{v=0}^{\infty}p_{n,v}(x)p_{n,v}(t)\,|f(t)|\,(1-\phi(t))dtdx \\
&\leq \sum_{i=0}^{2k+1}\frac{1}{i!}\int_{a_2}^{b_2}\int_0^{\infty}(n-c)\sum_{v=0}^{\infty}p_{n,v}(x)p_{n,v}(t)\left|f^{(i)}(t)\right|.|t-x|^i\,(1-\phi(t))dtdx \\
&= M_4 + M_5, \qquad \text{say.}
\end{aligned}$$

For sufficiently large t there exist positive constants C_{17}, C_{18} such that $\frac{(t-x)^{2k+2}}{t^{2k+2}+1} > C_{17}$ for all $t \geq C_{18}$, $t \in I_2$. Applying Fubini's theorem and Lemma 2.1, we obtain

$$\begin{aligned}
M_4 &= \left(\int_0^{C_{18}}\int_{a_2}^{b_2} + \int_{C_{18}}^{\infty}\int_{a_2}^{b_2}\right)\sum_{v=0}^{\infty}(n-c)p_{n,v}(x)p_{n,v}(t)\,|f(t)|\,(1-\phi(t))dtdx \\
&\leq C_{19}n^{-(k+1)}\left(\int_0^{C_{20}}|f(t)|\,dt\right) \\
&\qquad + \frac{1}{C_{17}}\int_{C_{20}}^{\infty}\int_{a_2}^{b_2}\sum_{v=0}^{\infty}p_{n,v}(x)p_{n,v}(t)\frac{(t-x)^{2k+2}}{(t^{2k+2}+1)}\,|f(t)|\,dxdt \\
&\leq C_{20}n^{-(k+1)}\left\{\left(\int_0^{C_{20}}|f(t)|\,dt\right) + \left(\int_{C_{20}}^{\infty}|f(t)|\,dt\right)\right\} \\
&\leq C_{21}n^{-(k+1)}\,\|f\|_{L_1[0,\infty)}\,.
\end{aligned}$$



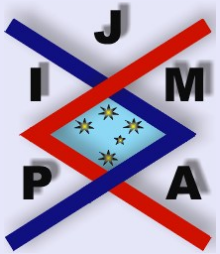
Finally by the Lemma 2.3, we have

$$M_5 \leq C_{22} n^{-(k+1)} \left\{ \|f\|_{L_1(I_2)} + \left\| f^{(2k+1)} \right\|_{L_1(I_2)} \right\}.$$

Combining M_4 and M_5, we obtain

$$M_3 \leq C_{23} n^{-(k+1)} \left\{ \|f\|_{L_1[0,\infty)} + \left\| f^{(2k+1)} \right\|_{L_1(I_2)} \right\}.$$

This completes the proof of (2.2) of the lemma. □



3. Rate Of Convergence

Theorem 3.1. *Let $f \in L_p[0,\infty), p \geq 1$. Then for n sufficiently large*

$$\|S_{n,k}(f,\cdot) - f\|_{L_p(I_2)} \leq C_{24}\left\{\omega_{2k+2}(f, n^{-1/2}, p, I_1) + n^{-(k+1)}\|f\|_{L_p[0,\infty)}\right\},$$

where C_{24} is a constant independent of f and n.

Proof. We can write

$$\begin{aligned}
&\|S_{n,k}(f,*) - f\|_{L_p(I_2)} \\
&\quad \leq \|S_{n,k}(f - f_{\eta,2k+2},*)\|_{L_p(I_2)} \\
&\qquad + \|S_{n,k}(f_{\eta,2k+2},*) - f_{\eta,2k+2})\|_{L_p(I_2)} + \|(f_{\eta,2k+2} - f)\|_{L_p(I_2)} \\
&\quad = E_1 + E_2 + E_3, \qquad \text{say.}
\end{aligned}$$

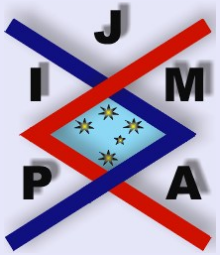
It is well known that

$$\left\|f^{(2k+1)}_{\eta,2k+2}\right\|_{B.V.(I_3)} = \left\|f^{(2k+2)}_{\eta,2k+2}\right\|_{L_1(I_3)}.$$

Therefore from Lemma 2.4 $(p > 1)$ and $(p = 1)$ we have

$$\begin{aligned}
E_2 &\leq C_{25}n^{-(k+1)}\left(\left\|f^{(2k+2)}_{\eta,2k+2}\right\|_{L_p(I_3)} + \|f_{\eta,2k+2}\|_{L_p[0,\infty)}\right) \\
&\leq C_{26}n^{-(k+1)}\left(n^{-(k+2)}\omega_{2k+2}(f,\eta,p,I_1) + \|f\|_{L_p[0,\infty)}\right),
\end{aligned}$$

which follows from the properties of Steklov means.



Let $\phi(t)$ be the characteristic function of I_3, we have

$$\begin{aligned} &S_n((f - f_{\eta,2k+2})(t), x) \\ &\quad = S_n(\phi(t)(f - f_{\eta,2k+2})(t), x) + S_n((1 - \phi(t))(f - f_{\eta,2k+2})(t), x) \\ &\quad = E_4 + E_5, \qquad \text{say.} \end{aligned}$$

By Hölder's inequality

$$\int_{a_2}^{b_2} |E_4|^p dx \leq \int_{a_2}^{b_2} \int_{a_1}^{b_1} (n-c) \sum_{v=0}^{\infty} p_{n,v}(x) p_{n,v}(t) \left|(f - f_{\eta,2k+2})(t)\right|^p dt dx.$$

Applying Fubini's theorem, we have

$$\int_{a_2}^{b_2} |E_4|^p dx \leq \left\| f - f_{\eta,2k+2} \right\|_{Lp(I_2)}.$$

Similarly, for all $p \geq 1$

$$\left\| E_5 \right\|_{Lp(I_2)} \leq C_{27} \eta^{-(k+1)} \left\| f - f_{\eta,2k+2} \right\|_{Lp(0,\infty)}.$$

Consequently, via the property of Steklov means, we find that

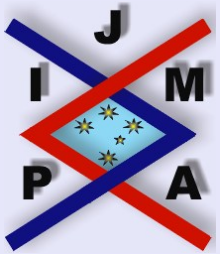
$$\left\| S_n\left(f - f\eta, 2k+2, \cdot\right) \right\|_{Lp(I2)} \leq C_{28} \left\{ \omega_{2k+2}\left(f, \eta, p, I_1\right) + \eta^{-(k+1)} \left\| f \right\|_{Lp[0,\infty)} \right\}.$$

Hence

$$E_1 \leq C_{29} \left\{ \omega_{2k+2}\left(f, \eta, p, I_1\right) + \eta^{-(k+1)} \left\| f \right\|_{Lp[0,\infty)} \right\}.$$

Thus, with $\eta = n^{-1/2}$, the result follows.

This completes the proof of the theorem. □



References

[1] J.L. DURRMEYER, Une Formula d' inversion de la Transformee de Laplace: Application a la Theorie des Moments, *These de 3c cycle, Faculte des Science de l' University de Paris, 1967.*

[2] S. GOLDBERG AND V. MEIR, Minimum moduli of ordinary differential operators, *Proc. London Math. Soc.*, **23** (3) (1971), 1–15.

[3] S. GUO, On the rate of convergence of Durrmeyer operator for functions of bounded variation, *J. Approx. Theory,* **51**(1987), 183–192.

[4] V. GUPTA, Simultaneous approximation by Szasz-Durrmeyers operators, *The Math. Student,* **64** (1-4) (1995), 27–36.

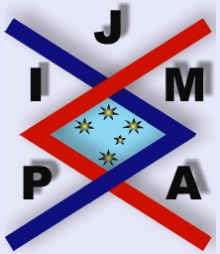
[5] V. GUPTA, A note on modified Baskakov operators, *Approx.Theory and its Appl.*, **10**(3) (1994), 74–78.

[6] E. HEWITT AND K. STROMBERG, *Real and Abstract Analysis*, McGraw Hill, New York (1956).

[7] C.P. MAY, Saturation and inverse theorem for combination of a class of exponential type operators, *Canad. J. Math.*, **28** (1976), 1224–1250.

[8] R.P. SINHA, P.N. AGRAWAL AND V. GUPTA, On simultaneous approximation by modified Baskakov operators, *Bull Soc. Math Belg. Ser. B.*, **44** (1991), 177–188.

[9] H.M. SRIVASTAVA AND V. GUPTA, A certain family of summation integral type operators, *Math and Comput. Modelling,* **37** (2003), 1307–1315.



[10] A.F. TIMAN, *Theory of Approximation of Functions of a Real Variable*, Hindustan Publ. Corp., Delhi 1966.

[11] Y. WANG AND S. GUO, Rate of approximation of function of bounded variation by modified Lupas operators, *Bull. Austral. Math. Soc.*, **44** (1991), 177–188.