Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 7, Issue 3, Article 115, 2006

ON A REVERSE OF A HARDY-HILBERT TYPE INEQUALITY

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Received 18 January, 2006; accepted 25 May, 2006
Communicated by L.-E. Persson

ABSTRACT. This paper deals with a reverse of the Hardy-Hilbert's type inequality with a best constant factor. The other reverse of the form is considered.

Key words and phrases: Hardy-Hilbert's inequality, Weight coefficient, Hölder's inequality.

2000 *Mathematics Subject Classification.* 26D15.

1. INTRODUCTION

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, $a_n, b_n \geq 0$, such that $0 < \sum_{n=0}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=0}^{\infty} b_n^q < \infty$, then we have the well known Hardy-Hilbert inequality (Hardy et al. [1]):

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left\{\sum_{n=0}^{\infty} a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=0}^{\infty} b_n^q\right\}^{\frac{1}{q}}, \tag{1.1}$$

where the constant factor $\pi/\sin(\pi/p)$ is the best possible. The equivalent form is (see Yang et al. [8]):

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\frac{a_m}{m+n+1}\right)^p < \left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^p \sum_{n=0}^{\infty} a_n^p, \tag{1.2}$$

where the constant factor $[\pi/\sin(\pi/p)]^p$ is still the best possible.

ISSN (electronic): 1443-5756

Research supported by Natural Science Foundation of Guangdong Institutions of Higher Learning, College and University (China, No. 0177).

016-06

Inequalities (1.1) and (1.2) are important in analysis and its applications (see Mitrinović, et al. [3]). In recent years, inequality (1.1) had been strengthened by Yang [5] as

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1} < \left\{\sum_{n=0}^{\infty}\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{\ln 2-\gamma}{(2n+1)^{1+\frac{1}{p}}}\right]a_n^p\right\}^{\frac{1}{p}} \times \left\{\sum_{n=0}^{\infty}\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{\ln 2-\gamma}{(2n+1)^{1+\frac{1}{q}}}\right]b_n^q\right\}^{\frac{1}{q}}, \tag{1.3}$$

where $\ln 2 - \gamma = 0.11593^+$ (γ is Euler's constant). Another strengthened version of (1.1) was given in [6]. By introducing a parameter λ, two extensions of (1.1) were proved in [8, 7] as:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+1)^\lambda} < k_\lambda(p)\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{1-\lambda}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{1-\lambda}b_n^q\right\}^{\frac{1}{q}}; \tag{1.4}$$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+1)^\lambda} < B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right) \times \left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{(p-1)(1-\lambda)}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{(q-1)(1-\lambda)}b_n^q\right\}^{\frac{1}{q}}, \tag{1.5}$$

where, the constant factors $k_\lambda(p) = B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)$ $(2-\min\{p,q\} < \lambda \le 2)$, and $B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)$ $(0 < \lambda \le \min\{p,q\})$ are the best possible ($B(u,v)$ is the β function). For $\lambda = 1$, both (1.4) and (1.5) reduce to (1.1). We call (1.4) and (1.5) Hardy-Hilbert type inequalities. Yang et al. [9] summarized the use of weight coefficients in research for Hardy-Hilbert type inequalities. But the problem on how to build the reverse of this type inequalities is unsolved.

The main objective of this paper is to deal with a reverse of inequality (1.4) for $\lambda = 2$. Another related reverse of the form is considered.

2. Some Lemmas

We need the formula of the β function $B(p,q)$ as follows (see [4]):

$$B(p,q) = B(q,p) = \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}}dt, \tag{2.1}$$

and the following inequality (see [5, 2]): If $f^4 \in C[0,\infty)$, $0 < \int_0^\infty f(t)dt < \infty$ and $(-1)^n f^{(n)}(x) > 0$, $f^{(n)}(\infty) = 0$ $(n = 0,1,2,3,4)$, then

$$\int_0^\infty f(t)dt + \frac{1}{2}f(0) < \sum_{k=0}^{\infty} f(k) < \int_0^\infty f(t)dt + \frac{1}{2}f(0) - \frac{1}{12}f'(0). \tag{2.2}$$

Lemma 2.1. *Define the weight function* $\omega(n)$ *as*

$$\omega(n) = \left(n+\frac{1}{2}\right)\sum_{k=0}^{\infty}\frac{1}{(k+n+1)^2}, \qquad n \in \mathbb{N}_0 (= \mathbb{N}\cup\{0\}), \tag{2.3}$$

then we have

$$1 - \frac{1}{4(n+1)^2} < \omega(n) < 1 - \frac{1}{6(n+1)(2n+1)} \qquad (n \in \mathbb{N}_0). \tag{2.4}$$

Proof. For fixed n, setting $f(t) = \frac{1}{(t+n+1)^2}$ $(t \in [0, \infty))$, we obtain

$$f(0) = \frac{1}{(n+1)^2}, \quad f'(0) = -\frac{2}{(n+1)^3}, \quad \text{and} \quad \int_0^\infty f(t)dt = \frac{1}{n+1}.$$

By (2.2), we find

$$\begin{aligned} \omega(n) &> \left[\int_0^\infty \frac{1}{(t+n+1)^2}dt + \frac{1}{2(n+1)^2}\right]\left(n+\frac{1}{2}\right) \\ &= \left[\frac{1}{(n+1)} + \frac{1}{2(n+1)^2}\right]\left[(n+1) - \frac{1}{2}\right] \\ &= 1 - \frac{1}{4(n+1)^2}. \end{aligned} \tag{2.5}$$

$$\begin{aligned} \omega(n) &< \left[\int_0^\infty \frac{1}{(t+n+1)^2}dt + \frac{1}{2(n+1)^2} + \frac{1}{6(n+1)^3}\right]\left(n+\frac{1}{2}\right) \\ &= \left[\frac{1}{n+1} + \frac{1}{2(n+1)^2} + \frac{1}{6(n+1)^3}\right]\left[(n+1) - \frac{1}{2}\right] \\ &= 1 - \left[\frac{1}{12(n+1)^2} + \frac{1}{12(n+1)^3}\right]. \end{aligned} \tag{2.6}$$

Since for $n \in \mathbb{N}_0$,

$$\begin{aligned} &\left[\frac{1}{12(n+1)^2} + \frac{1}{12(n+1)^3}\right]6(n+1)(2n+1) \\ &\qquad = \frac{2(n+1)-1}{2(n+1)} + \frac{2(n+1)-1}{2(n+1)^2} \\ &\qquad = 1 + \frac{1}{2(n+1)} - \frac{1}{2(n+1)^2} \geq 1, \end{aligned}$$

then we find

$$\frac{1}{12(n+1)^2} + \frac{1}{12(n+1)^3} \geq \frac{1}{6(n+1)(2n+1)},$$

and in view of (2.6), it follows that

$$\omega(n) < 1 - \frac{1}{6(n+1)(2n+1)}. \tag{2.7}$$

In virtue of (2.5) and (2.7), we have (2.4). The lemma is proved. □

Lemma 2.2. *For $0 < \varepsilon < p$, we have*

$$\begin{aligned} I &:= \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{1}{(m+n+1)^2}\left(m+\frac{1}{2}\right)^{-\frac{\varepsilon}{p}}\left(n+\frac{1}{2}\right)^{-\frac{\varepsilon}{q}} \\ &< (1+o(1))\sum_{n=0}^\infty \frac{1}{\left(n+\frac{1}{2}\right)^{1+\varepsilon}} \ (\varepsilon \to 0^+). \end{aligned} \tag{2.8}$$

Proof. For fixed n, setting $f(t) = \frac{\left(t+\frac{1}{2}\right)^{-\varepsilon/p}}{(t+n+1)^2}$ $(t \in (-\frac{1}{2}, \infty))$, then we have

$$f(0) = \frac{2^{\varepsilon/p}}{(n+1)^2}, \qquad f'(0) = -\frac{2^{1+\varepsilon/p}}{(n+1)^3} - \frac{\varepsilon 2^{1+\varepsilon/p}}{p(n+1)^2}.$$

Setting $u = \left(t+\frac{1}{2}\right)/\left(n+\frac{1}{2}\right)$ in the following integral, we obtain

$$\int_0^\infty f(t)dt < \int_{-1/2}^\infty f(t)dt = \frac{1}{\left(n+\frac{1}{2}\right)^{1+\frac{\varepsilon}{p}}}\int_0^\infty \frac{u^{-\frac{\varepsilon}{p}}}{(1+u)^2}du.$$

Hence by (2.2) and (2.1), we find

$$\begin{aligned} I &= \sum_{n=0}^\infty \left(n+\frac{1}{2}\right)^{-\frac{\varepsilon}{q}}\left[\sum_{m=0}^\infty \frac{1}{(m+n+1)^2}\left(m+\frac{1}{2}\right)^{-\frac{\varepsilon}{p}}\right] \\ &< \sum_{n=0}^\infty \left(n+\frac{1}{2}\right)^{-\frac{\varepsilon}{q}}\left[\frac{1}{\left(n+\frac{1}{2}\right)^{1+\frac{\varepsilon}{p}}}\int_0^\infty \frac{u^{-\frac{\varepsilon}{p}}}{(1+u)^2}du \right. \\ &\quad \left. + \frac{2^{\varepsilon/p}}{2(n+1)^2} + \frac{2^{1+\varepsilon/p}}{12(n+1)^3} + \frac{\varepsilon 2^{1+\varepsilon/p}}{12p(n+1)^2}\right] \\ &= \sum_{n=0}^\infty \frac{1}{\left(n+\frac{1}{2}\right)^{1+\varepsilon}} B\left(1-\frac{\varepsilon}{p}, 1+\frac{\varepsilon}{p}\right) + O(1) \quad (\varepsilon \to 0^+). \end{aligned}$$

Since $B\left(1-\frac{\varepsilon}{p}, 1+\frac{\varepsilon}{p}\right) \to B(1,1) = 1\,(\varepsilon \to 0^+)$, then (2.8) is valid. The lemma is proved. □

3. MAIN RESULTS

Theorem 3.1. *If* $0 < p < 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $a_n, b_n \geq 0$, *such that* $0 < \sum_{n=0}^\infty \frac{a_n^p}{2n+1} < \infty$ *and* $0 < \sum_{n=0}^\infty \frac{b_n^q}{2n+1} < \infty$, *then*

$$\sum_{n=0}^\infty \sum_{m=0}^\infty \frac{a_m b_n}{(m+n+1)^2} > 2\left\{\sum_{n=0}^\infty \left[1-\frac{1}{4(n+1)^2}\right]\frac{a_n^p}{2n+1}\right\}^{\frac{1}{p}} \times \left\{\sum_{n=0}^\infty \left[1-\frac{1}{6(n+1)(2n+1)}\right]\frac{b_n^q}{2n+1}\right\}^{\frac{1}{q}}, \tag{3.1}$$

where the constant factor 2 is the best possible. In particular, one has

$$\sum_{n=0}^\infty \sum_{m=0}^\infty \frac{a_m b_n}{(m+n+1)^2} > 2\left\{\sum_{n=0}^\infty \left[1-\frac{1}{4(n+1)^2}\right]\frac{a_n^p}{2n+1}\right\}^{\frac{1}{p}}\left\{\sum_{n=0}^\infty \frac{b_n^q}{2n+1}\right\}^{\frac{1}{q}}. \tag{3.2}$$

Proof. By the reverse of Hölder's inequality and (2.3), one has

$$\begin{aligned} &\sum_{n=0}^\infty \sum_{m=0}^\infty \frac{a_m b_n}{(m+n+1)^2} \\ &= \sum_{n=0}^\infty \sum_{m=0}^\infty \left[\frac{a_m}{(m+n+1)^{\frac{2}{p}}}\right]\left[\frac{b_n}{(m+n+1)^{\frac{2}{q}}}\right] \\ &\geq \left\{\sum_{n=0}^\infty \sum_{m=0}^\infty \frac{a_m^p}{(m+n+1)^2}\right\}^{\frac{1}{p}}\left\{\sum_{n=0}^\infty \sum_{m=0}^\infty \frac{b_n^q}{(m+n+1)^2}\right\}^{\frac{1}{q}} \end{aligned}$$

$$= \left\{ \sum_{m=0}^{\infty} \left[\sum_{n=0}^{\infty} \frac{m+\frac{1}{2}}{(m+n+1)^2} \right] \frac{2a_m^p}{2m+1} \right\}^{\frac{1}{p}} \left\{ \sum_{n=0}^{\infty} \left[\sum_{m=0}^{\infty} \frac{n+\frac{1}{2}}{(m+n+1)^2} \right] \frac{2b_n^q}{2n+1} \right\}^{\frac{1}{q}}$$
$$= 2 \left\{ \sum_{m=0}^{\infty} \omega(m) \frac{a_m^p}{2m+1} \right\}^{\frac{1}{p}} \left\{ \sum_{n=0}^{\infty} \omega(n) \frac{b_n^q}{2n+1} \right\}^{\frac{1}{q}}.$$

Since $0 < p < 1$ and $q < 0$, by (2.4), it follows that (3.1) is valid.

For $0 < \varepsilon < p$, setting $\tilde{a}_n = \left(n + \frac{1}{2}\right)^{-\varepsilon/p}$, $\tilde{b}_n = \left(n + \frac{1}{2}\right)^{-\varepsilon/q}$, $n \in \mathbb{N}_0$, we find

$$\begin{aligned}
(3.3)\qquad & \left\{ \sum_{n=0}^{\infty} \left[1 - \frac{1}{4(n+1)^2} \right] \frac{\tilde{a}_n^p}{2n+1} \right\}^{\frac{1}{p}} \left\{ \sum_{n=0}^{\infty} \frac{\tilde{b}_n^q}{2n+1} \right\}^{\frac{1}{q}} \\
&= \left\{ \sum_{n=0}^{\infty} \left[1 - \frac{1}{4(n+1)^2} \right] \frac{1}{2\left(n+\frac{1}{2}\right)^{1+\varepsilon}} \right\}^{\frac{1}{p}} \left\{ \sum_{n=0}^{\infty} \frac{1}{2\left(n+\frac{1}{2}\right)^{1+\varepsilon}} \right\}^{\frac{1}{q}} \\
&> \frac{1}{2} \left\{ \sum_{n=0}^{\infty} \frac{1}{\left(n+\frac{1}{2}\right)^{1+\varepsilon}} - \sum_{n=0}^{\infty} \frac{1}{2(n+1)^2(2n+1)} \right\}^{\frac{1}{p}} \left\{ \sum_{n=0}^{\infty} \frac{1}{\left(n+\frac{1}{2}\right)^{1+\varepsilon}} \right\}^{\frac{1}{q}} \\
&= \frac{1}{2} \{1 - \tilde{o}(1)\}^{\frac{1}{p}} \sum_{n=0}^{\infty} \frac{1}{\left(n+\frac{1}{2}\right)^{1+\varepsilon}} \ (\varepsilon \to 0^+).
\end{aligned}$$

If the constant factor 2 in (3.1) is not the best possible, then there exists a real number k with $k > 2$, such that (3.1) is still valid if one replaces 2 by k. In particular, one has

$$(3.4)\qquad \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\tilde{a}_m \tilde{b}_n}{(m+n+1)^2} > k \left\{ \sum_{n=0}^{\infty} \left[1 - \frac{1}{4(n+1)^2} \right] \frac{\tilde{a}_n^p}{2n+1} \right\}^{\frac{1}{p}} \left\{ \sum_{n=0}^{\infty} \frac{\tilde{b}_n^q}{2n+1} \right\}^{\frac{1}{q}}.$$

Hence by (2.8) and (3.3), it follows that

$$(1 + o(1)) \sum_{n=0}^{\infty} \frac{1}{\left(n+\frac{1}{2}\right)^{1+\varepsilon}} > \frac{k}{2} \{1 - \tilde{o}(1)\}^{\frac{1}{p}} \sum_{n=0}^{\infty} \frac{1}{\left(n+\frac{1}{2}\right)^{1+\varepsilon}},$$

and then $2 \geq k \, (\varepsilon \to 0^+)$. This contradicts the fact that $k > 2$. Hence the constant factor 2 in (3.1) is the best possible. The theorem is proved. □

Theorem 3.2. *If* $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n \geq 0$, *such that* $0 < \sum_{n=0}^{\infty} \frac{a_n^p}{2n+1} < \infty$, *then*

$$(3.5)\qquad \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right)^{p-1} \left[\sum_{m=0}^{\infty} \frac{a_m}{(m+n+1)^2} \right]^p > 2 \sum_{n=0}^{\infty} \left[1 - \frac{1}{4(n+1)^2} \right] \frac{a_n^p}{2n+1},$$

where the constant factor 2 *is the best possible.*

Proof. By the reverse of Hölder's inequality, (2.3) and (2.4), one has $\omega(n) < 1$ and

$$\left[\sum_{m=0}^{\infty} \frac{a_m}{(m+n+1)^2}\right]^p = \left\{\sum_{m=0}^{\infty} \left[\frac{a_m}{(m+n+1)^{\frac{2}{p}}}\right]\left[\frac{1}{(m+n+1)^{\frac{2}{q}}}\right]\right\}^p \tag{3.6}$$

$$\geq \left\{\sum_{m=0}^{\infty} \frac{a_m^p}{(m+n+1)^2}\right\}\left\{\sum_{m=0}^{\infty} \frac{1}{(m+n+1)^2}\right\}^{p-1}$$

$$= \left\{\sum_{m=0}^{\infty} \frac{a_m^p}{(m+n+1)^2}\right\}\left\{\omega(n)\left(n+\frac{1}{2}\right)^{-1}\right\}^{p-1}$$

$$> \left(n+\frac{1}{2}\right)^{1-p} \sum_{m=0}^{\infty} \frac{a_m^p}{(m+n+1)^2}.$$

Hence we find

$$\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{p-1}\left[\sum_{m=0}^{\infty} \frac{a_m}{(m+n+1)^2}\right]^p > \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_m^p}{(m+n+1)^2} \tag{3.7}$$

$$= \sum_{m=0}^{\infty}\left[\sum_{n=0}^{\infty} \frac{m+\frac{1}{2}}{(m+n+1)^2}\right]\frac{2a_m^p}{2m+1}$$

$$= 2\sum_{m=0}^{\infty} \omega(m)\frac{a_m^p}{2m+1}.$$

By (2.4), we have (3.5).

Setting $b_n \geq 0$ and $0 < \sum_{n=0}^{\infty} \frac{b_n^q}{2n+1} < \infty$, by the reverse of Hölder's inequality, one has

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+1)^2} \tag{3.8}$$

$$= \sum_{n=0}^{\infty}\left[\left(n+\frac{1}{2}\right)^{\frac{1}{q}}\sum_{m=0}^{\infty} \frac{a_m}{(m+n+1)^2}\right]\left[\left(n+\frac{1}{2}\right)^{-\frac{1}{q}} b_n\right]$$

$$\geq \left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{p-1}\left[\sum_{m=0}^{\infty} \frac{a_m}{(m+n+1)^2}\right]^p\right\}^{\frac{1}{p}}\left\{\sum_{n=0}^{\infty} \frac{2b_n^q}{2n+1}\right\}^{\frac{1}{q}}.$$

If the constant factor 2 in (3.5) is not the best possible, then by (3.6), we can get a contradiction that the constant factor 2 in (3.1) is not the best possible. The theorem is proved. □

Remark 3.3. If $a_n,\, b_n$ satisfy the conditions of (1.4) for $\lambda = 2,\, r > 1,\, \frac{1}{r} + \frac{1}{s} = 1$, and (3.2) for $0 < p < 1,\, \frac{1}{p} + \frac{1}{q} = 1$, then one can get the following two-sided inequality:

$$0 < \left\{\frac{3}{4}\sum_{n=0}^{\infty} \frac{a_n^p}{2n+1}\right\}^{\frac{1}{p}}\left\{\sum_{n=0}^{\infty} \frac{b_n^q}{2n+1}\right\}^{\frac{1}{q}} \tag{3.9}$$

$$< \frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+1)^2}$$

$$< \left\{\sum_{n=0}^{\infty} \frac{a_n^r}{2n+1}\right\}^{\frac{1}{r}}\left\{\sum_{n=0}^{\infty} \frac{b_n^s}{2n+1}\right\}^{\frac{1}{s}} < \infty.$$

REFERENCES

[1] G.H. HARDY, J.E. LITTLEWOOD AND G. POLYA, *Inequalities*, Cambridge Univ. Press, London, 1952.

[2] JICHANG KUANG, On new generalizations of Hilbert's inequality and their applications, *J. Math. Anal. Appl.*, **245** (2000), 248–265.

[3] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Inequalities Involving Functions and their Integrals and Derivatives*, Kluwer Academic, Boston, 1991.

[4] ZHUXI WANG AND DUNRIN GUO, *An Introduction to Special Functions*, Science Press, Beijing, 1979.

[5] BICHENG YANG, On a strengthened version of the more accurate Hardy-Hilbert's inequality, *Acta Math. Sinica*, **42** (1999), 1103–1110.

[6] BICHENG YANG, On a strengthened Hardy-Hilbert's inequality, *J. Ineq. Pure. Appl. Math.*, **1**(2) (2000), Art. 22. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=116`]

[7] BICHENG YANG, On a new extension of Hardy-Hilbert's inequality and its applications, *Internat. J. Pure Appl. Math.*, **5** (2003), 57–66.

[8] BICHENG YANG AND L. DEBNATH, On a new generalization of Hardy-Hilbert's inequality and its applications, *J. Math. Anal. Appl.*, **233** (1999), 484–497.

[9] BICHENG YANG AND Th.M. RASSIAS, On the way of weight coefficient and research for the Hilbert-type inequalities, *Math. Inequal. Appl.*, **6** (2003), 625–658.