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MONOTONE METHODS APPLIED TO SOME HIGHER ORDER BOUNDARY VALUE PROBLEMS

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ABSTRACT. We prove the existence of a solution for the nonlinear boundary value problem

$$u^{(2m+4)} = f\left(x, u, u'', \dots, u^{(2m+2)}\right), \qquad x \in [0,1],$$
$$u^{(2i)}(0) = 0 = u^{(2i)}(1), \qquad 0 \le i \le m+1,$$

where $f : [0,1] \times \mathbb{R}^{m+2} \to \mathbb{R}$ is continuous. The technique used here is a monotone method in the presence of upper and lower solutions. We introduce a new maximum principle which generalizes one due to Bai which in turn was an improvement of a maximum principle by Ma.

Key words and phrases: Differential Inequality, Monotone Methods, Upper and Lower Solutions, Maximum Principle.

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1. INTRODUCTION

In this paper, we are concerned with the existence of solutions of the higher order boundary value problem,

$$u^{(2m+4)} = f\left(x, u, u'', \dots, u^{(2m+2)}\right), \qquad x \in [0,1], \tag{1.1}$$

$$u^{(2i)}(0) = 0 = u^{(2i)}(1), \qquad 0 \le i \le m+1, \tag{1.2}$$

where $f : [0,1] \times \mathbb{R}^{m+2} \to \mathbb{R}$ is continuous, and m is a given nonnegative integer. Our results generalize those of Bai [2], whose own results were for $m = 0$ and involved an application of a new maximum principle for a fourth order two-parameter linear eigenvalue problem. The maximum principle was used in the presence of upper and lower solutions in developing a monotone method for obtaining solutions of the boundary value problem (1.1), (1.2).

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When $m = 0$, this boundary value problem arises from the study of static deflection of an elastic bending beam where u denotes the deflection of the beam and $f(x, u, u'')$ would represent the loading force that may depend on the deflection and the curvature of the beam; for example, see [1, 5, 9, 14, 15]. Some attention also has been given to (1.1), (1.2) in applications when $m \geq 1$, such as Meirovitch [13] who used higher even order boundary value problems in studying the open-loop control of a distributed structure, and Cabada [3] used upper and lower solutions methods to study higher order problems such as (1.1), (1.2).

The method of upper and lower solutions is thoroughly developed for second order equations, and several authors have used the method for fourth order problems (*i.e.*, when $m = 0$); see [1, 3, 4, 12, 18]. Kelly [10] and Klaasen [11] obtained early upper and lower solutions applications to higher order ordinary differential equations. Recently, Ehme, Eloe, and Henderson [6] employed truncations analogous to those of [10] and [11] and have extended the applications of upper and lower solutions to $2m$th order ordinary differential equations, where there was no dependency on odd order derivatives. Recently, Ehme, Eloe, and Henderson [7] generalized those results to any $2m$th order ordinary differential equation satisfying fully nonlinear boundary conditions using upper and lower solutions.

In their monotonicity method development, Ma, Zhang, and Fu [17] established results for the fourth order version of (1.1), (1.2) by requiring that $f(x, u, v)$ be nondecreasing in u and nonincreasing in v. Bai's [2] results were improvements of [17] in that Bai weakened the monotonicity constraints on f. This paper extends the methods and results of Bai. We obtain a maximum principle for a higher order operator in the context of this paper, and we develop a monotonicity method for appropriate higher order problems. The process yields extremal solutions of (1.1), (1.2).

2. A MAXIMUM PRINCIPLE

In this section, we obtain a maximum principle which generalizes the one given by Bai [2]. First, define

$$F = \Big\{ u \in C^{(2m+4)}[0,1] \;\Big|\; (-1)^i u^{(2m+2-2i)}(0) \leq 0 \text{ and } (-1)^i u^{(2m+2-2i)}(1) \leq 0 \text{ for } 0 \leq i \leq m+1 \Big\},$$

and then define the operator $\mathcal{L} : F \to C[0,1]$ by

$$\mathcal{L}u = u^{(2m+4)} - a u^{(2m+2)} + b u^{(2m)},$$

where $a, b \geq 0$, $a^2 - 4ab \geq 0$, and $u \in F$.

We will need the following result, which is a maximum principle that appears in Protter and Weinberger [16].

Lemma 2.1. *Suppose $u(x)$ satisfies*

$$u''(x) + g(x)u'(x) + h(x)u(x) \geq 0, \qquad x \in (a,b),$$

where $h(x) \leq 0$; g and h are bounded functions on any closed subset of (a,b); and there exists a $c \in (a,b)$ such that

$$M = u(c) = \max_{x \in (a,b)} u(x)$$

is a nonnegative maximum. Then $u(x) \equiv M$. Moreover, if $h(x) \not\equiv 0$, then $M = 0$.

Our next lemma extends maximum results from [2] and [17] in a manner useful for application to our (1.1), (1.2).

Lemma 2.2. *If $u \in F$ satisfies $\mathcal{L}u \geq 0$, then*

$$(-1)^i u^{(2m+2-2i)}(x) \leq 0, \qquad 1 \leq i \leq m+1. \tag{2.1}$$

Proof. Let $\mathcal{A}x = x''$. Then

$$\begin{aligned} \mathcal{L}u &= u^{(2m+4)} - au^{(2m+2)} + bu^{(2m)} \\ &= (\mathcal{A} - r_1)(\mathcal{A} - r_2)u^{(2m)} \\ &\geq 0 \end{aligned}$$

where

$$r_1, r_2 = \frac{a \pm \sqrt{a^2 - 4b}}{2} \geq 0.$$

Let

$$y = (\mathcal{A} - r_2)u^{(2m)} = u^{(2m+2)} - r_2 u^{(2m)}.$$

Then $(\mathcal{A} - r_1)y \geq 0$ and so $y'' \geq r_1 y$. On the other hand, $r_1, r_2 \geq 0$ and $u \in F$ imply

$$\begin{aligned} y(0) &= u^{(2m+2)}(0) - r_2 u^{(2m)}(0) \leq 0, \\ y(1) &= u^{(2m+2)}(1) - r_1 u^{(2m)}(1) \leq 0. \end{aligned}$$

By Lemma 2.1, we can conclude that $y(x) \leq 0$ for $x \in [0,1]$. Hence

$$u^{(2m+2)}(x) - r_2 u^{(2m)}(x) \leq 0, \qquad x \in [0,1].$$

Using this, Lemma 2.1, and the fact that

$$u^{(2m)}(0) \geq 0 \qquad \text{and} \qquad u^{(2m)}(1) \geq 0,$$

we get $u^{(2m)}(x) \geq 0$ for all $x \in [0,1]$. The boundary conditions (1.2) in turn imply (2.1). □

Lemma 2.3. [5] *Given $(a,b) \in \mathbb{R}^2$, the boundary value problem*

$$\begin{gathered} u^{(4)} - au'' + bu = 0, \\ u(0) = u''(0) = 0 = u(1) = u''(1), \end{gathered} \tag{2.2}$$

has a nontrivial solution if and only if

$$\frac{a}{(k\pi)^2} + \frac{b}{(k\pi)^4} + 1 = 0 \tag{2.3}$$

for some $k \in \mathbb{N}$.

In developing a monotonicity method relative to (1.1), (1.2), we will apply an extension of Lemma 2.3. This extension we can state as a corollary.

Corollary 2.4. *Given $(a,b) \in \mathbb{R}^2$, the boundary value problem*

$$\begin{gathered} u^{(2m+4)} - au^{(2m+2)} + bu^{(2m)} = 0, \\ u^{(2i)}(0) = 0 = u^{(2i)}(1), \qquad 0 \leq i \leq m+1, \end{gathered} \tag{2.4}$$

has a nontrivial solution if and only if (2.3) *holds for some $k \in \mathbb{N}$.*

Proof. Suppose u is a solution of (2.4). Let $v(x) = u^{(2m)}(x)$. Then

$$\begin{aligned} 0 &= u^{(2m+4)} - au^{(2m+2)} + bu^{(2m)} \\ &= \left(u^{(2m)}\right)^{(4)} - a\left(u^{(2m)}\right)'' + bu^{(2m)} \\ &= v^{(4)} - av'' + bv \end{aligned}$$

and

$$v(0) = 0 = v''(0)$$
$$v(1) = 0 = v''(1).$$

Hence $v(x)$ is a solution of (2.2) and so (2.3) holds. Each step is reversible and therefore the converse direction holds as well. $\square$

3. THE MONOTONE METHOD

In this section, we develop a monotone method which yields solutions of (1.1), (1.2).

Definition 3.1. Let $\alpha \in C^{(2m+4)}[0,1]$. We say α is an *upper solution* of (1.1), (1.2) provided

$$\alpha^{(2m+4)}(x) \geq f(x, \alpha(x), \alpha''(x), \ldots, \alpha^{(2m+2)}(x)), \qquad x \in [0,1],$$
$$(-1)^i \alpha^{(2m+2-2i)}(0) \leq 0, \qquad 0 \leq i \leq m+1,$$
$$(-1)^i \alpha^{(2m+2-2i)}(1) \leq 0, \qquad 0 \leq i \leq m+1.$$

Definition 3.2. Let $\beta \in C^{(2m+4)}[0,1]$. We say β is a *lower solution* of (1.1), (1.2) provided

$$\beta^{(2m+4)}(x) \leq f(x, \beta(x), \beta''(x), \ldots, \beta^{(2m+2)}(x)), \qquad x \in [0,1],$$
$$(-1)^i \beta^{(2m+2-2i)}(0) \leq 0, \qquad 0 \leq i \leq m+1,$$
$$(-1)^i \beta^{(2m+2-2i)}(1) \leq 0, \qquad 0 \leq i \leq m+1.$$

Definition 3.3. A function $v \in C^{(2m)}[0,1]$ is in the *order interval* $[\beta, \alpha]$ if, for each $0 \leq i \leq m$,

$$(-1)^i \beta^{(2m-2i)}(x) \leq (-1)^i v^{(2m-2i)}(x) \leq (-1)^i \alpha^{(2m-2i)}(x), \qquad x \in [0,1].$$

For $a, b \geq 0$ and $f : [0,1] \times \mathbb{R}^{m+2} \to \mathbb{R}$, define

$$f^*(x, u_0, u_1, \ldots, u_{m+1}) = f(x, u_0, u_1, \ldots, u_{m+1}) + bu_m - au_{m+1}.$$

Then (1.1) is equivalent to

$$\mathcal{L}u = u^{(2m+4)} - au^{(2m+2)} + bu^{(2m)} = f^*(x, u, u'', \ldots, u^{(2m+2)}). \tag{3.1}$$

Therefore, if α is an upper solution of (1.1), (1.2), then α is an upper solution for (3.1), (1.2). The same is true for the lower solution, β.

Our main goal now is to obtain solutions of (3.1), (1.2).

Theorem 3.1. *Let α and β be upper and lower solutions, respectively, for (1.1), (1.2) which satisfy*

$$\beta^{(2m)}(x) \leq \alpha^{(2m)}(x) \quad \textit{and} \quad \beta^{(2m+2)}(x) + r(\alpha - \beta)^{(2m)}(x) \geq \alpha^{(2m+2)}(x),$$

for $x \in [0,1]$ and where $f : [0,1] \times \mathbb{R}^{m+2} \to \mathbb{R}$ is continuous. Let $a, b \geq 0$, $a^2 - 4b \geq 0$, and

$$r_1, r_2 = \frac{a \pm \sqrt{a^2 - 4b}}{2}.$$

Suppose

$$f(x, u_0, u_1, \ldots, s, u_{m+1}) - f(x, u_0, u_1, \ldots, t, u_{m+1}) \geq -b\,(s-t)\,,$$

for

$$\beta^{(2m)}(x) \leq t \leq s \leq \alpha^{(2m)}(x),$$

where $u_0, u_1, \ldots, u_{m-1}, u_{m+1} \in \mathbb{R}$ and $x \in [0,1]$. Suppose also that

$$f(x, u_0, u_1, \ldots, u_m, \rho) - f(x, u_0, u_1, \ldots, u_m, \sigma) \leq a\,(\rho - \sigma)\,,$$

for

$$\alpha^{(2m+2)}(x) - r(\alpha - \beta)^{(2m)}(x) \leq \sigma \leq \rho + r(\alpha - \beta)^{(2m)}(x)$$

with

$$\rho \leq \beta^{(2m+2)}(x) + r(\alpha-\beta)^{(2m)}(x),$$

where $u_0, u_1, \ldots, u_m \in \mathbb{R}$ *and* $x \in [0,1]$. *Then there exist sequences* $\{\alpha_n\}_{n=0}^{\infty}$ *and* $\{\beta_n\}_{n=0}^{\infty}$ *in* $C^{(2m+4)}$ *such that*

$$\alpha_0 = \alpha \quad \textit{and} \quad \beta_0 = \beta,$$

which converge in $C^{(2m+4)}$ *to extremal solutions of* (1.1), (1.2) *in the order interval* $[\beta, \alpha]$. *Furthermore, if* m *is even, these sequences satisfy the montonicity conditions*

$$\begin{aligned}
&\left\{\alpha_n^{(2i)}\right\}_{n=0}^{\infty} \textit{ is nonincreasing for } i \textit{ even},\\
&\left\{\alpha_n^{(2i)}\right\}_{n=0}^{\infty} \textit{ is nondecreasing for } i \textit{ odd},\\
&\left\{\beta_n^{(2i)}\right\}_{n=0}^{\infty} \textit{ is nondecreasing for } i \textit{ even},\\
&\left\{\beta_n^{(2i)}\right\}_{n=0}^{\infty} \textit{ is nonincreasing for } i \textit{ odd}.
\end{aligned}$$

If m *is odd, the sequences satisfy the montonicity conditions*

$$\begin{aligned}
&\left\{\alpha_n^{(2i)}\right\}_{n=0}^{\infty} \textit{ is nondecreasing for } i \textit{ even},\\
&\left\{\alpha_n^{(2i)}\right\}_{n=0}^{\infty} \textit{ is nonincreasing for } i \textit{ odd},\\
&\left\{\beta_n^{(2i)}\right\}_{n=0}^{\infty} \textit{ is nonincreasing for } i \textit{ even},\\
&\left\{\beta_n^{(2i)}\right\}_{n=0}^{\infty} \textit{ is nondecreasing for } i \textit{ odd}.
\end{aligned}$$

Proof. Consider the associated problem

$$u^{(2m+4)}(x) - au^{(2m+2)}(x) + bu^{(2m)}(x) = f\left(x, \varphi, \varphi'', \ldots, \varphi^{(2m+2)}\right), \tag{3.2}$$

satisfying (1.2), where $\varphi \in C^{(2m+2)}[0,1]$. Since $a, b \geq 0$, (a,b) is not an eigenvalue pair of (2.2). By Lemma 2.3 and the Fredholm Alternative [8], the problem (3.2), (1.2) has a unique solution, u. Based on this, we can define the operator

$$\mathcal{T} : C^{(2m+2)}[0,1] \to C^{(2m+4)}[0,1]$$

by $\mathcal{T}\varphi = u$. Next, let

$$\begin{aligned}
C = \Big\{\varphi \in C^{(2m+2)}[0,1] \,\Big|\, &(-1)^i\alpha^{(2i)} \leq (-1)^i\varphi^{(2i)} \leq (-1)^i\beta^{(2i)},\ 0 \leq i \leq m,\\
&\text{and } \alpha^{(2m+2)} - r(\alpha-\beta)^{(2m)} \leq \varphi^{(2m+2)} \leq \beta^{(2m+2)} + r(\alpha-\beta)^{(2m)}\Big\}.
\end{aligned}$$

C is a nonempty, closed, bounded subset of $C^{(2m+2)}[0,1]$. For $\psi \in C$, set $\omega = \mathcal{T}\psi$. Then, for $x \in [0,1]$,

$$\begin{aligned}
\mathcal{L}(\alpha-\omega)(x) &= (\alpha-\omega)^{(2m+4)}(x) - a(\alpha-\omega)^{(2m+2)}(x) + b(\alpha-\omega)^{(2m)}(x)\\
&\geq f^*\left(x, \alpha(x), \ldots, \alpha^{(2m+2)}(x)\right) - f^*\left(x, \psi(x), \ldots, \psi^{(2m+2)}(x)\right)\\
&= f\left(x, \alpha(x), \ldots, \alpha^{(2m+2)}(x)\right) - f\left(x, \psi(x), \ldots, \psi^{(2m+2)}(x)\right)\\
&\qquad - a(\alpha-\psi)^{(2m+2)}(x) + b(\alpha-\psi)^{(2m)}(x)\\
&\geq 0,
\end{aligned}$$

and by the definition of α,

$$\begin{aligned}
(-1)^i(\alpha-\omega)^{(2m+2-2i)}(0) &\leq 0, \qquad 0 \leq i \leq m+1,\\
(-1)^i(\alpha-\omega)^{(2m+2-2i)}(1) &\leq 0, \qquad 0 \leq i \leq m+1.
\end{aligned}$$

Employing Lemma 2.2, we have

$$(-1)^i(\alpha-\omega)^{(2m+2-2i)}(x) \leq 0, \qquad 1 \leq i \leq m+1,\ x \in [0,1].$$

By a similar argument, we see that

$$(-1)^i(\omega-\beta)^{(2m+2-2i)}(x) \le 0, \qquad 1 \le i \le m+1,\ x \in [0,1].$$

Hence

$$(-1)^i\alpha^{(2m+2-2i)} \le (-1)^i\omega^{(2m+2-2i)} \le (-1)^i\beta^{(2m+2-2i)}, \qquad 1 \le i \le m+1.$$

Note

$$(\alpha-\omega)^{(2m+2)}(x) - r(\alpha-\omega)^{(2m)}(x) \le 0, \qquad x \in [0,1],$$

or

$$\omega^{(2m+2)}(x) + r(\alpha-\omega)^{(2m)}(x) \ge \alpha^{(2m+2)}(x), \qquad x \in [0,1]. \tag{3.3}$$

Using (3.3) we have

$$\begin{aligned} \omega^{(2m+2)}(x) + r(\alpha-\beta)^{(2m)}(x) &\ge \omega^{(2m+2)}(x) + r(\alpha-\omega)^{(2m)}(x) \\ &\ge \alpha^{(2m+2)}(x) \end{aligned}$$

or

$$\alpha^{(2m+2)}(x) - r(\alpha-\beta)^{(2m)}(x) \le \omega^{(2m+2)}(x), \qquad x \in [0,1].$$

By a similar argument, we can conclude

$$\omega^{(2m+2)}(x) \le \beta^{(2m+2)}(x) + r(\alpha-\beta)^{(2m)}(x), \qquad x \in [0,1].$$

Therefore, $\mathcal{T} : C \to C$.

Next, let $u_1 = \mathcal{T}\varphi_1$ and $u_2 = \mathcal{T}\varphi_2$ where $\varphi_1, \varphi_2 \in C$ with

$$\begin{gathered} (-1)^i\varphi_2^{(2i)} \le (-1)^i\varphi_1^{(2i)}, \qquad 0 \le i \le m, \\ \varphi_1^{(2m+2)} + r(\alpha-\beta)^{(2m)} \ge \varphi_2^{(2m+2)}. \end{gathered}$$

We claim that the analogous inequalities hold in terms of u_1, u_2. That is,

$$\begin{gathered} (-1)^i u_2^{(2i)} \le (-1)^i u_1^{(2i)}, \qquad 0 \le i \le m, \\ u_1^{(2m+2)} + r(\alpha-\beta)^{(2m)} \ge u_2^{(2m+2)}. \end{gathered} \tag{3.4}$$

To verify the claim, note first that

$$\begin{aligned} \mathcal{L}(u_2-u_1)(x) &= f\left(x, \varphi_2, \varphi_2'', \ldots, \varphi_2^{(2m+2)}\right) - f\left(x, \varphi_1, \varphi_1'', \ldots, \varphi_1^{(2m+2)}\right) \\ &\ge 0 \end{aligned}$$

and

$$(u_2-u_1)^{(2i)}(0) = 0 = (u_2-u_1)^{(2i)}(1), \qquad 0 \le i \le m.$$

By Lemma 2.2, we have

$$(-1)^i(u_2-u_1)^{(2m+2-2i)}(x) \le 0, \qquad 1 \le i \le m+1,\ x \in [0,1],$$

or

$$(-1)^i u_2^{(2m+2-2i)}(x) \le (-1)^i u_1^{(2m+2-2i)}(x), \qquad 1 \le i \le m+1,\ x \in [0,1].$$

By the same reasoning used to show $\mathcal{T} : C \to C$, we deduce

$$u_1^{(2m+2)} + r(\alpha-\beta)^{(2m)} \ge u_2^{(2m+2)}.$$

Therefore (3.4) holds.

Finally, we construct our sequences. Define

$$\begin{aligned} \alpha_0 &= \alpha, \qquad \alpha_n = \mathcal{T}\alpha_{n-1}, \qquad n \ge 1, \\ \beta_0 &= \beta, \qquad \beta_n = \mathcal{T}\beta_{n-1}, \qquad n \ge 1. \end{aligned}$$

Then $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty} \subset C^{(2m+4)}$. But, in particular, from the earlier portion of the proof, $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty} \subset C$ and

$$(-1)^i \alpha_0^{(2m+2-2i)} \le (-1)^i \beta_0^{(2m+2-2i)}, \qquad 1 \le i \le m+1,$$
$$\alpha_0^{(2m+2)} \le \beta_0^{(2m+2)} + r(\alpha_0 - \beta_0)^{(2m)}.$$

We can argue as before that

$$\begin{aligned} (-1)^i \alpha_0^{(2m+2-2i)} &\ge (-1)^i \alpha_1^{(2m+2-2i)} \\ &\ge \cdots \ge (-1)^i \beta_1^{(2m+2-2i)} \ge (-1)^i \beta_0^{(2m+2-2i)}, \qquad 1 \le i \le m+1, \end{aligned} \tag{3.5}$$

and

$$\begin{gathered} \beta^{(2m+2)} = \beta_0^{(2m+2)}, \qquad \alpha^{(2m+2)} = \alpha_0^{(2m+2)}, \\ \alpha_0^{(2m+2)} - r(\alpha_0 - \beta_0)^{(2m)} \le \alpha_n^{(2m+2)}, \\ \beta_n^{(2m+2)} \le \beta_0^{(2m+2)} + r(\alpha_0 - \beta_0)^{(2m)}. \end{gathered} \tag{3.6}$$

From the definition of $\mathcal{T}$,

$$\begin{aligned} \mathcal{L}\alpha_n(x) &= \alpha_n^{(2m+4)}(x) - a\alpha_n^{(2m+2)}(x) + b\alpha_n^{(2m)}(x) \\ &= f^*\left(x, \alpha_{n-1}(x), \ldots, \alpha_{n-1}^{(2m+2)}(x)\right) \end{aligned}$$

and

$$\alpha_n^{(2i)}(0) = 0 = \alpha_n^{(2i)}(1), \qquad 0 \le i \le m+1.$$

This in turn yields

$$\begin{aligned} \alpha_n^{(2m+4)}(x) &= f^*\left(x, \alpha_{n-1}(x), \ldots, \alpha_{n-1}^{(2m+2)}(x)\right) + a\alpha_n^{(2m+2)}(x) - b\alpha_n^{(2m)}(x) \\ &\le f^*\left(x, \alpha_{n-1}(x), \ldots, \alpha_{n-1}^{(2m+2)}(x)\right) \\ &\qquad + a\left[\beta^{(2m+2)} + r(\alpha - \beta)^{(2m)}\right](x) - b\beta^{(2m)}(x) \end{aligned} \tag{3.7}$$

and

$$\alpha_n^{(2i)}(0) = 0 = \alpha_n^{(2i)}(1), \qquad 0 \le i \le m+1. \tag{3.8}$$

Analogously,

$$\begin{aligned} \beta_n^{(2m+4)}(x) &\le f^*\left(x, \beta_{n-1}(x), \ldots, \beta_{n-1}^{(2m+2)}(x)\right) \\ &\qquad + a\left[\beta^{(2m+2)} + r(\alpha - \beta)^{(2m)}\right](x) - b\beta^{(2m)}(x), \\ \beta_n^{(2i)}(0) &= 0 = \beta_n^{(2i)}(1), \qquad 0 \le i \le m+1. \end{aligned}$$

By (3.5)–(3.7), there exists a constant $M_{\alpha,\beta} > 0$ (independent of n and x) such that

$$\left|\alpha_n^{(2m+4)}(x)\right| \le M_{\alpha,\beta} \qquad \text{for all } x \in [0,1]. \tag{3.9}$$

By (3.8), for each $n \in \mathbb{N}$, there exists a $t_n \in (0,1)$ such that $\alpha_n^{(2m+3)}(t_n) = 0$. Using this and (3.9), we obtain

$$\left|\alpha_n^{(2m+3)}(x)\right| = \left|\alpha_n^{(2m+3)}(t_n) + \int_{t_n}^{x} \alpha_n^{(2m+4)}(s)\, ds\right| \le M_{\alpha,\beta}. \tag{3.10}$$

Combining (3.6) and (3.8) and arguing as above, we know that there is a constant $N_{\alpha,\beta} > 0$ (independent of n and x) such that

$$\left|\alpha_n^{(i)}(x)\right| \le N_{\alpha,\beta}, \qquad 1 \le i \le 2m+2,\ x \in [0,1]. \tag{3.11}$$

By (3.5), (3.10), and (3.11), we have $\{\alpha_n\}_{n=0}^{\infty}$ is bounded in $C^{(2m+4)}$-norm. Similarly, $\{\beta_n\}_{n=0}^{\infty}$ is bounded in $C^{(2m+4)}$-norm as well.

Appropriate equicontinuity conditions are satisfied as well, and then by standard convergence theorems as well as the monotonicity of $\left\{\alpha_n^{(2i)}\right\}_{n=0}^{\infty}$ and $\left\{\beta_n^{(2i)}\right\}_{n=0}^{\infty}$, $0 \le i \le m$, it follows that $\{\alpha_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ converge to the extremal solutions of (3.1), (1.2) and hence to the extremal solutions of (1.1), (1.2). □

4. EXAMPLES

We conclude the paper with two examples which illustrate the usefulness of Theorem 3.1 above.

Example 4.1. Consider the boundary value problem

$$\begin{aligned} u^{(6)} &= -u''(x) - \frac{1}{\pi^2}u^{(4)} + \sin \pi x, \qquad x \in [0,1], \\ u(0) &= u''(0) = u^{(4)}(0) = 0, \\ u(1) &= u''(1) = u^{(4)}(1) = 0. \end{aligned} \tag{4.1}$$

One can easily verify that the conditions of Theorem 3.1 are satisfied if we take $\alpha(x) = -\frac{1}{\pi^2}\sin \pi x$ as an upper solution and $\beta(x) \equiv 0$ as a lower solution of (4.1). We then conclude that there exists a solution, $u(x)$, of (4.1) such that $-\frac{1}{\pi^2}\sin \pi x \le u(x) \le 0$ for $x \in [0,1]$.

Example 4.2. Consider the boundary value problem

$$\begin{aligned} u^{(6)} &= -u''(x) + \frac{1}{\pi^4}\left(u^{(4)}\right)^2 + \sin \pi x, \qquad x \in [0,1], \\ u(0) &= u''(0) = u^{(4)}(0) = 0, \\ u(1) &= u''(1) = u^{(4)}(1) = 0. \end{aligned} \tag{4.2}$$

Again, the hypotheses of Theorem 3.1 hold for the upper solution $\alpha(x) = -\frac{1}{\pi}\cos \pi x$ and the lower solution $\beta(x) \equiv 0$. Hence, there exists a solution, $u(x)$, of (4.2) satisfying $-\frac{1}{\pi}\cos \pi x \le u(x) \le 0$ for $x \in [0,1]$.

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