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ON THE SECOND ORDER SLIP REYNOLDS EQUATION WITH MOLECULAR DYNAMICS: EXISTENCE AND UNIQUENESS

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ABSTRACT. In this paper we obtained existence and uniqueness results for the modified second order slip Reynolds equation modeling the performance of the slider head floating over a rotating disk inside a hard disk drive. The existence and the uniqueness are proved using the Ky-Fan's Lemma and some monotonicity techniques.

Key words and phrases: Reynolds equation, Ky-Fan's Lemma, Monotonicity techniques.

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1. INTRODUCTION

The advent of mini-fabrication and the ability to develop micro-machines for various applications have made micro-scale fluid dynamics increasingly important. In terms of application, microelectromechanical systems are devices having characteristic length of micrometer or even nanometer order. Microscale flows are found in micro-pumps and micro-turbines and in such applications, the flow cannot be considered as a continuum. This involves the selection of an appropriate model and boundary conditions. This deviation is measured by the Knudsen number (K_n) (the ratio of the molecular mean free path and the film thickness). Normally, flow can be classified into three categories [2]: $K_n \leq 10^{-3}$ the flow can be considered as a continuum; $K_n > 10$ the flow is considered to be a free molecular flow; $10^{-3} \leq K_n \leq 10$ the flow can neither be a continuum flow nor a free molecular one.

The conventional Navier-Stokes equations are based on a continuum assumption and it is no longer valid if the Kundsen number is beyond a certain limit [1]. A typical example is the case of the slider head floating over a rotating disk inside a hard disk drive (HDD).

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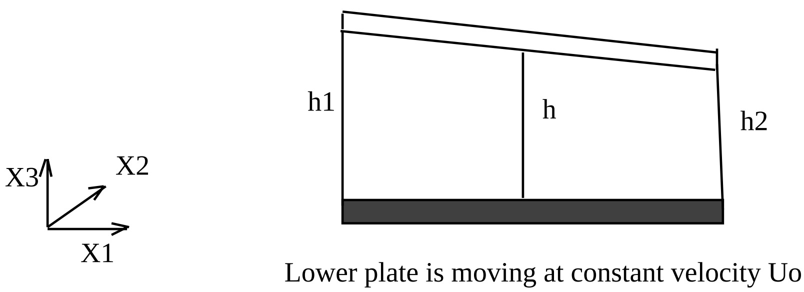

Figure 1.1: Slider-bearing flow geometry

This type of thin-film problem has been approximated by the famous Reynolds equation which is derived from the inertialess form of the Navier-Stokes equations combined with the continuity equation. Appropriate modifications such as slip boundary conditions are the realm of micro-fluid mechanics. Another approach is molecular-based models which are derived from kinetic theories.

1.1. **Reynolds Equation and Molecular Models.**

1.1.1. *Reynolds equation for thin film problems.* The well-known Reynolds equation in the continuum regime is [7]:

$$\frac{\partial}{\partial x_1}\left(\frac{\rho h^3}{\mu}\frac{\partial p}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\frac{\rho h^3}{\mu}\frac{\partial p}{\partial x_2}\right)=6\left(2\frac{\partial(\rho h)}{\partial t}+\frac{\partial(\rho U_0 h)}{\partial x_1}\right),$$

where h is the local gas bearing thickness, p the local pressure, ρ the local gas density, μ the viscosity and U_0 is the moving plate velocity.

In the slip regime the above equation needs modifications. Taking the Hsia's second order model, the boundary conditions are given as follows [9]:

$$\begin{aligned}
U_{x_1}(x_3=0)&=U_0+\frac{2-\tau}{\tau}\lambda\left.\frac{\partial U_{x_1}}{\partial x_3}\right|_{x_3=0}-\frac{\lambda^2}{2}\left.\frac{\partial^2 U_{x_1}}{\partial x_3^2}\right|_{x_3=0}+\cdots\\
U_{x_1}(x_3=h)&=-\frac{2-\tau}{\tau}\lambda\left.\frac{\partial U_{x_1}}{\partial x_3}\right|_{x_3=h}-\frac{\lambda^2}{2}\left.\frac{\partial^2 U_{x_1}}{\partial x_3^2}\right|_{x_3=h}+\cdots\\
U_{x_2}(x_3=0)&=\frac{2-\tau}{\tau}\lambda\left.\frac{\partial U_{x_2}}{\partial x_3}\right|_{x_3=0}-\frac{\lambda^2}{2}\left.\frac{\partial^2 U_{x_2}}{\partial x_3^2}\right|_{x_3=0}+\cdots\\
U_{x_2}(x_3=h)&=-\frac{2-\tau}{\tau}\lambda\left.\frac{\partial U_{x_2}}{\partial x_3}\right|_{x_3=h}-\frac{\lambda^2}{2}\left.\frac{\partial^2 U_{x_2}}{\partial x_3^2}\right|_{x_3=h}+\cdots
\end{aligned}$$

$U_{x_1},\ U_{x_2}$: the velocity distributions.
τ: is the surface accommodation coefficient.
λ: is the mean free path, $\lambda=\frac{16}{5}\frac{\mu}{P}\sqrt{\frac{RT}{2\pi}}$ (where R is a gas constant, T is a local gas temperature and $P=\frac{p}{p_a}$ with p_a is the ambient temperature).
For these boundary conditions, the velocity distributions are obtained by solving the momentum

equation [9]:

$$U_{x_1} = \frac{1}{2\mu} \cdot \frac{\partial p}{\partial x_1} \left(x_3^2 - hx_3 - h\lambda - \lambda^2 \right) + U_0 \left(1 - \frac{\lambda + x_3}{h + 2\lambda} \right),$$
$$U_{x_2} = \frac{1}{2\mu} \cdot \frac{\partial p}{\partial x_2} \left(x_3^2 - hx_3 - h\lambda - \lambda^2 \right).$$

The second order modified Reynolds equation can hence be obtained by incorporating the expressions of U_{x_1} and U_{x_2} into the continuity equation and then integrating from $x_3 = 0$ to $x_3 = h$

$$\frac{\partial(\rho h)}{\partial t} + \frac{1}{2} \cdot \frac{\partial(\rho U_0 h)}{\partial x_1} = \frac{\partial}{\partial x_1} \left[\frac{1}{2\mu} \cdot \frac{\partial p}{\partial x_1} \rho \left(\frac{h^3}{6} + \lambda h^2 + \lambda^2 h \right) \right] + \frac{\partial}{\partial x_2} \left[\frac{1}{2\mu} \cdot \frac{\partial p}{\partial x_2} \rho \left(\frac{h^3}{6} + \lambda h^2 + \lambda^2 h \right) \right].$$

Normally, the non-dimensional second order slip Reynolds equation (in the stationary regime) is used which is given by [7]:

$$\nabla \left[\left(H^3 P + 6K_n H^2 + 6K_n^2 \frac{H}{P} \right) \nabla P \right] = \Lambda \cdot \nabla(PH), \tag{1.1}$$

Λ : is the bearing vector, $H = \frac{h}{h_2}$.

1.1.2. *The Molecular Models.* The mean free path is the average distance travelled by a molecule between collision and is defined as:

$$\lambda = \frac{\text{mean thermal speed}}{\text{collision frequency}}. \tag{1.2}$$

To obtain the mean free path, it is essential to calculate both the mean thermal speed and collision frequency, the terms in equation (1.2) depend on the molecular models used.

There exists three models: the (HS) Hard sphere model (equation (1.1)), the variable hard sphere model (VHS) [2] and the (VSS) variable soft sphere [10]. If we take the (HS) model as a reference, we can write a generalized mean free path λ' for the three cases (HS, VHS, VSS) where $\lambda' = \xi\lambda$ such that

- $\xi = 1$ for the (HS) model;
- $\xi = \frac{\Gamma\left(\frac{9}{2} - \varpi\right)}{6} \pi^{\frac{1}{2} - \varpi}$ for the (VHS) model;
- $\xi = \frac{\alpha\Gamma\left(\frac{9}{2} - \varpi\right)}{(\alpha+1)(\alpha+2)} \pi^{\frac{1}{2} - \varpi}$ for the (VSS) model,

where α, ϖ, Γ are determined by the type of gas and can be obtained from experimental data.

The non-dimensional modified Reynolds equation may be obtained as:

$$\nabla \left[\left(H^3 P + 6\xi K_n H^2 + 6\xi^2 K_n^2 \frac{H}{P} \right) \nabla P \right] = \Lambda \cdot \nabla(PH). \tag{1.3}$$

In [4] Chipot and Luskin studied an analogous equation without the $6\xi^2 K^2 \frac{H}{P}$ term, they proved existence and uniqueness by using a change of the unknown function which leads to a new problem in which the nonlinearity appears in the convection term.

The same proof technique does not work in our case due to the degenerate term $6\xi^2 K^2 \frac{H}{P}$, which motivated our intention to search in this sense.

In this work we will prove existence and uniqueness of weak solutions of equation (1.3) using a generalization of the Ky-Fan Lemma and preserving the idea of a new unknown function.

2. EXISTENCE AND UNIQUENESS OF SOLUTIONS

2.1. **Existence.** We consider the following problem ($\mathcal{P}$):

$$(\mathcal{P}) \qquad \begin{cases} \nabla\left[\left(H^3P + 6\xi K_n H^2 + 6\xi^2 K_n^2 \frac{H}{P}\right)\nabla P\right] = \Lambda.\nabla(PH), \quad x = (x_1, x_2) \in \Omega \\ P = \Psi \quad \text{in } \partial\Omega, \end{cases}$$

where Ω is a region of $\mathbb{R}^2$ with a smooth boundary $\partial\Omega$.

We assume that the functions $H : \Omega \to \mathbb{R}$ and $\Psi : \partial\Omega \to \mathbb{R}$ satisfy the following hypothesis:

$$(A_1) \qquad \begin{cases} H \in W^{1,\infty}(\Omega) \\ H \text{ is bounded in } W^{1,\infty}(\Omega) \text{ and } a \le H(x) \le b \text{ a.e in } \Omega \\ \text{with } a, b \text{ are two positives constants} \end{cases}$$

$$(A_2) \qquad \begin{cases} \Psi \text{ is the restriction to } \partial\Omega \text{ of a smooth function } \widetilde{\Psi} \text{ defined on } \Omega \\ \text{such that } \|\nabla\widetilde{\Psi}\|_{L^2(\Omega)} \le M \\ \text{with } M \text{ is a positive constant.} \end{cases}$$

We introduce the following set in order to give a variational formulation of ($\mathcal{P}$):

$$V := \left\{u \in H^1(\Omega) \cap L^\infty(\Omega) \;/\; \exists\alpha > 0 \text{ such that } u(x) \ge \alpha \text{ a.e in } \Omega\right\}.$$

For the following, we denote by $\|\cdot\|$ the norm in $L^2(\Omega)$.

Definition 2.1. We say that P is a weak solution of ($\mathcal{P}$) if $P - \widetilde{\Psi} \in H_0^1(\Omega)$, $P \in V$ and

$$\int_\Omega \left(H^3P + 6\xi K_n H^2 + 6\xi^2 K_n^2 \frac{H}{P}\right)\nabla P \cdot \nabla v \, dx = \int_\Omega PH\Lambda \cdot \nabla v \, dx \quad \forall v \in H_0^1(\Omega). \tag{2.1}$$

We prove the existence of a weak solution of ($\mathcal{P}$) by using a change of the unknown function. Let us write for $P > 0$,

$$\begin{aligned} \left(H^3P + 6\xi K_n H^2 + 6\xi^2 K_n^2 \frac{H}{P}\right)\nabla P &= H^3\nabla\left(\frac{P^2}{2} + 6\xi K_n \frac{P}{H} + 6\xi^2 K_n^2 \frac{\log(P)}{H^2}\right) \\ &\quad + 6\xi K_n PH\nabla H + 12\xi^2 K_n^2 \log(P)\nabla H. \end{aligned} \tag{2.2}$$

The new unknown function will be

$$u = \frac{P^2}{2} + 6\xi K_n \frac{P}{H} + 6\xi^2 K_n^2 \frac{\log(P)}{H^2}. \tag{2.3}$$

We consider the function $g :]0, +\infty[\to \mathbb{R}$

$$g(t) = \frac{t^2}{2} + 6\xi K_n t + 6\xi^2 K_n^2 \log(t).$$

It is easy to see that g is an increasing and bijective function. We have from the above equality

$$P = \frac{1}{H}\kappa(x, u), \tag{2.4}$$

with

$$\kappa(x, u) = g^{-1}\left(H^2 u + 6\xi^2 K_n^2 \log H\right). \tag{2.5}$$

Our initial problem ($\mathcal{P}$) becomes in u

$$(\mathcal{P}_u)\qquad \begin{cases} \nabla\cdot(H^3u) = \nabla\cdot\left[(\Lambda-6\xi K_n\nabla H)\,\kappa(x,u)-12\xi^2K_n^2\log\kappa(x,u)\nabla H\right] \\ \qquad\qquad +\nabla\cdot\left[12\xi K_n\log H\nabla H\right] \quad \text{in } \Omega \\ u=\Psi_u=\dfrac{\Psi^2}{2}+6\xi K_n\dfrac{\Psi}{H}+6\xi^2K_n^2\dfrac{\log(\Psi)}{H^2} \quad \text{in } \partial\Omega. \end{cases}$$

We set

$$\widetilde{\Psi}_u=\frac{\widetilde{\Psi}^2}{2}+6\xi K_n\frac{\widetilde{\Psi}}{H}+6\xi^2K_n^2\frac{\log(\widetilde{\Psi})}{H^2},$$

while keeping (due to (A_2)) the fact that $\|\nabla\widetilde{\Psi}_u\|\le M_1$ (with M_1 is a positive constant).

Definition 2.2. We say that u is a weak solution of ($\mathcal{P}_u$) if $u-\widetilde{\Psi}_u\in H_0^1(\Omega)$ and

$$\begin{aligned}\int_\Omega H^3\nabla u.\nabla v\,dx &= \int_\Omega (\Lambda-6\xi K_n\nabla H)\,\kappa(x,u)\nabla v\,dx \\ &\quad -\int_\Omega 12\xi^2K_n^2\log\kappa(x,u)\nabla H\nabla v\,dx \\ &\quad +\int_\Omega 12\xi K_n\log H\nabla H\nabla v\,dx \quad \forall v\in H_0^1(\Omega).\end{aligned}\tag{2.6}$$

The equivalence between ($\mathcal{P}$) and ($\mathcal{P}_u$) is given by the following result.

Lemma 2.1. *u is a weak solution of ($\mathcal{P}_u$) if and only if P, given by (2.4), is a weak solution of ($\mathcal{P}$).*

Proof. It is clear from (2.2) that the two variational formulas are equivalent. And from (2.3) it is obvious that if $P\in V$ then $u\in H^1(\Omega)$. It remains to show that if u is a solution of ($\mathcal{P}_u$) then $P\in V$. From (2.4) we have that $P\in H^1(\Omega)$ since $(g^{-1})'$ is bounded. On the other hand, we have classically $u\in L^\infty(\Omega)$. From (2.4) we deduce that P belongs to $L^\infty(\Omega)$ with P bounded away from 0, and the proof is ended. □

Proposition 2.2. *Under hypotheses (A_1) and (A_2), if we have*

$$\frac{a^3}{C_pb^2\left(\frac{\|\Lambda\|_e}{6\xi K_n}+3\|\nabla H\|\right)}>1\tag{2.7}$$

(where C_p is the constant of Poincaré [3], *$\|\Lambda\|_e$ is the Euclidean norm of Λ), then, for all solution z_1 of the following inequality*

$$\begin{aligned}&\int_\Omega H^3\nabla\left(z_1+\widetilde{\Psi}_u\right)\cdot\nabla z_1dx \\ &\qquad\le\int_\Omega(\Lambda-6\xi K_n\nabla H)\,\kappa(x,z_1+\widetilde{\Psi}_u)\nabla z_1dx \\ &\qquad\quad -\int_\Omega 12\xi^2K_n^2\log\kappa(x,z_1+\widetilde{\Psi}_u)\nabla H\nabla z_1dx+\int_\Omega 12\xi K_n\log H\nabla H\nabla z_1dx,\end{aligned}$$

we have

$$\|\nabla z_1\|\le C.\tag{2.8}$$

Proof. We have

$$\begin{aligned}
&\int_\Omega H^3\nabla\left(z_1+\widetilde{\Psi}_u\right)\cdot\nabla z_1 dx\\
&\qquad\le \int_\Omega \left(\Lambda-6\xi K_n\nabla H\right)\kappa(x,z_1+\widetilde{\Psi}_u)\nabla z_1 dx\\
&\qquad\qquad-\int_\Omega 12\xi^2K_n^2\log\kappa(x,z_1+\widetilde{\Psi}_u)\nabla H\nabla z_1 dx+\int_\Omega 12\xi K_n\log H\nabla H\nabla z_1 dx,
\end{aligned}$$

then

$$\begin{aligned}
\int_\Omega H^3(\nabla z_1)^2 \le &\int_\Omega \left(\Lambda-6\xi K_n\nabla H\right)\kappa(x,z_1+\widetilde{\Psi}_u)\nabla z_1\\
&-\int_\Omega 12\xi^2K_n^2\log\kappa(x,z_1+\widetilde{\Psi}_u)\nabla H\nabla z_1\\
&+\int_\Omega 12\xi K_n\log H\nabla H\nabla z_1-\int_\Omega H^3\nabla z_1\nabla\widetilde{\Psi}_u.
\end{aligned}$$

Due to the fact that, for all $s\in\mathbb{R}$,

$$0\le\frac{dg^{-1}}{ds}(s)=\frac{g^{-1}(s)}{(g^{-1})^2(s)+6\xi K_n g^{-1}(s)+6\xi^2K_n^2}\le\frac{1}{6\xi K_n}, \tag{2.9}$$
$$0\le\frac{d}{ds}\log(g^{-1}(s))=\frac{1}{(g^{-1})^2(s)+6\xi K_n g^{-1}(s)+6\xi^2K_n^2}\le\frac{1}{6\xi^2K_n^2},$$

it follows that

$$\begin{aligned}
&a^3\left\|\nabla z_1\right\|^2\\
&\le\left\|(\Lambda-6\xi K_n\nabla H)\right\|_\infty\frac{1}{6\xi K_n}\left[\left\|H^2\left(z_1+\widetilde{\Psi}_u\right)+6\xi^2K_n^2\log H-1\right\|+|\Omega|^{1/2}g^{-1}(1)\right]\left\|\nabla z_1\right\|\\
&\quad+2\left\|\nabla H\right\|_\infty\left[\left\|H^2\left(z_1+\widetilde{\Psi}_u\right)+6\xi^2K_n^2\log H-1\right\|+|\Omega|^{1/2}\log\left(g^{-1}(1)\right)\right]\left\|\nabla z_1\right\|\\
&\quad+12\xi K_n\left\|\log H\nabla H\right\|\left\|\nabla z_1\right\|+b^3\left\|\nabla\widetilde{\Psi}_u\right\|\left\|\nabla z_1\right\|,
\end{aligned}$$

i.e.

$$\begin{aligned}
&\left(a^3-\left\|(\Lambda-6\xi K_n\nabla H)\right\|_\infty\frac{1}{6\xi K_n}C_pb^2-2\left\|\nabla H\right\|_\infty C_pb^2\right)\left\|\nabla z_1\right\|\\
&\le\left\|(\Lambda-6\xi K_n\nabla H)\right\|_\infty\frac{1}{6\xi K_n}\left[\left\|H^2\widetilde{\Psi}_u+6\xi^2K_n^2\log H-1\right\|+|\Omega|^{1/2}g^{-1}(1)\right]\\
&\quad+2\left\|\nabla H\right\|_\infty\left[\left\|H^2\widetilde{\Psi}_u+6\xi^2K_n^2\log H-1\right\|+|\Omega|^{1/2}\log\left(g^{-1}(1)\right)\right]\\
&\quad+12\xi K_n\left\|\log H\nabla H\right\|+b^3\left\|\nabla\widetilde{\Psi}_u\right\|,
\end{aligned}$$

where $|\Omega|$ is the measure of Ω.

However, if

$$\frac{a^3}{C_pb^2\left(\frac{\|\Lambda\|_e}{6\xi K_n}+3\left\|\nabla H\right\|_\infty\right)}>1,$$

hence

$$\left(a^3-\left\|(\Lambda-6\xi K_n\nabla H)\right\|_\infty\frac{1}{6\xi K_n}C_pb^2-2\left\|\nabla H\right\|_\infty C_pb^2\right)>0,$$

then $\|\nabla z_1\| \leq C$, where

$$C = \frac{cte}{a^3 - \|(\Lambda - 6\xi K_n \nabla H)\|_\infty \frac{1}{6\xi K_n} C_p b^2 - 2\|\nabla H\|_\infty C_p b^2}$$

such that

$$\begin{aligned} cte = \|(\Lambda - 6\xi K_n \nabla H)\|_\infty \frac{1}{6\xi K_n} \left[\left\| H^2 \widetilde{\Psi}_u + 6\xi^2 K_n^2 \log H - 1 \right\| + |\Omega|^{1/2} g^{-1}(1) \right] \\ + 2\|\nabla H\|_\infty \left[\left\| H^2 \widetilde{\Psi}_u + 6\xi^2 K_n^2 \log H - 1 \right\| + |\Omega|^{1/2} \log\left(g^{-1}(1)\right) \right] \\ + 12\xi K_n \|\log H \nabla H\| + b^3 \left\| \nabla \widetilde{\Psi}_u \right\|. \end{aligned}$$

□

Now, we will prove the existence of a weak solution for the problem ($\mathcal{P}_u$).

Proposition 2.3. *If the hypotheses (A_1), (A_2) and (2.7) are verified then there exists at least one weak solution for ($\mathcal{P}_u$).*

For the proof we need the following theorem:

Notation 2.1. We denote by $\mathcal{F}(X)$ the family of all non-empty finite subsets of X and by $\mathcal{F}(X, x_0)$ all elements of $\mathcal{F}(X)$ containing x_0. We shall denote by $conv(A)$ the convex hull of A, by $\overline{A}^X$ the closure of A in X and by $int_X(A)$ the interior of A in X.

Theorem 2.4. *Let E be a topological vector space and X be a non-empty convex subset of E; $\Phi_1, \Phi_2 : X \times X \to \overline{\mathbb{R}}$ such that:*

(1) $\Phi_1(\chi, q) \leq \Phi_2(\chi, q)$ *for all* $\chi, q \in X$ *and* $\Phi_2(\chi, \chi) \leq 0$ *for all* $\chi \in X$.
(2) *For all $A \in \mathcal{F}(X)$ and all $\chi \in conv(A)$, $q \mapsto \Phi_1(\chi, q)$ is lower semicontinuous on $conv(A)$.*
(3) *For all $q \in X$, the set $\{\chi \in X,\ \Phi_2(\chi, q) > 0\}$ is convex.*
(4) *For all $A \in \mathcal{F}(X)$ and all $\chi, q \in conv(A)$ and for every net $\{q_\alpha\}$ converging in X to q with $\Phi_1(t\chi + (1-t)q, q_\alpha) \leq 0$ for all α and all $t \in [0, 1]$, we have $\Phi_1(\chi, q) \leq 0$.*
(5) *There exists a non-empty closed and compact K of X and $x_0 \in K$ such that $\Phi_1(x_0, q) > 0\ \forall q \in X \backslash K$.*

Then there exists $\overline{q} \in K$ such that $\Phi_1(\chi, \overline{q}) \leq 0\ \forall \chi \in X$.

Remark 2.5. If the application $q \mapsto \Phi_1(\chi, q)$ is lower semicontinuous on X for all $\chi \in X$, then the conditions (2) and (4) are verified.

Definition 2.3. [6]. $T : X \to 2^E$ is said to be a KKM-application if for all $A \in \mathcal{F}(X)$, $conv(A) \subseteq \underset{\chi \in A}{\cup} T(\chi)$.

First, we recall the following lemma that is a generalization of the Ky-Fan's lemma.

Lemma 2.6. [5]. *Let X a non-empty convex subset $\subseteq E$ (a topological vector space) and $T : X \to 2^E$ is a KKM-application, we suppose that there exists $x_0 \in X$ such that:*

i) $\overline{T(x_0) \cap X}^X$ *is compact on* X.
ii) $\forall A \in \mathcal{F}(X, x_0), \forall \chi \in conv(A),\ T(\chi) \cap conv(A)$ *is closed in* $conv(A)$.
iii) $\forall A \in \mathcal{F}(X, x_0), \overline{X \cap (\underset{\chi \in conv(A)}{\cap} T(\chi))}^X \cap conv(A) = \left(\underset{\chi \in conv(A)}{\cap} T(\chi) \right) \cap conv(A)$.

Then $\underset{\chi \in X}{\cap} T(\chi) \neq \emptyset$.

Proof of Theorem 2.4. We put for all $\chi \in X$

$$T(\chi) = \{q \in X \ / \ \Phi_1(\chi, q) \leq 0\}.$$

The condition (5) implies that $T(x_0) \subseteq K$, i.e. $\overline{T(x_0)}^X$ is compact on X.
The condition (2) implies that $\forall \chi \in conv(A),\ T(\chi) \cap conv(A)$ is closed on $conv(A)$.
Conditions (1) and (3) imply that T is a $KKM-$application.
Indeed, let us suppose the opposite (T is not a $KKM-$application), then there exists $A \in \mathcal{F}(X)$ and there exists $q_0 \in conv(A)$ such that $q_0 \notin \underset{\chi \in A}{\cup} T(\chi)$, i.e. $\forall \chi \in A$, $\Phi_1(\chi, q_0) > 0$. However $\{\chi \in X \ / \ \Phi_1(\chi, q_0) > 0\}$ is convex, then $conv(A) \subset \{\chi \in X \ / \ \Phi_1(\chi, q_0) > 0\}$. Therefore $\Phi_1(q_0, q_0) > 0$ by following $\Phi_2(q_0, q_0) > 0$ (which is absurd).

It remains to show that

$$\overline{X \cap (\underset{\chi \in conv(A)}{\cap} T(\chi))}^X \cap conv(A) = \left(\underset{\chi \in conv(A)}{\cap} T(\chi)\right) \cap conv(A), \text{ for all } A \in \mathcal{F}(X).$$

Let $q \in \overline{X \cap (\underset{\chi \in conv(A)}{\cap} T(\chi))}^X \cap conv(A)$, then there exists a sequence (q_α) such that $q_\alpha \to q$ and $q_\alpha \in X \cap (\underset{\chi \in conv(A)}{\cap} T(\chi))$. However $q_\alpha \in \underset{\chi \in conv(A)}{\cap} T(\chi)$ implies that $\Phi_1(\chi, q_\alpha) \leq 0$ for all $\chi \in conv(A)$, i.e. $\Phi_1(t\chi + (1-t)q, q_\alpha) \leq 0$, for all $\chi, q \in conv(A)$ and for all $t \in [0,1]$ then (4) implies that $\Phi_1(\chi, q) \leq 0$ for all $\chi \in conv(A)$ i.e. $q \in \left(\underset{\chi \in conv(A)}{\cap} T(\chi)\right) \cap conv(A)$.
By application of Lemma 2.6, there exists $\overline{q} \in K$ such that $\overline{q} \in T(\chi)\ \forall \chi \in X$, i.e. there exists $\overline{q} \in K$ such that $\Phi_1(\chi, \overline{q}) \leq 0\ \forall \chi \in X$. $\square$

Proof of Proposition 2.3. We make a translation for the unknown function to bring it to the same space that functions test. Let $w = u - \widetilde{\Psi}_u \in H_0^1(\Omega)$, then we search $w \in H_0^1(\Omega)$ such that

$$\begin{aligned}(2.10)\quad \int_\Omega H^3 \nabla w \cdot \nabla v dx &= \int_\Omega (\Lambda - 6\xi K_n \nabla H)\, \kappa_1(x, w) \nabla v dx \\ &\quad - \int_\Omega 12\xi^2 K_n^2 \log \kappa_1(x, w) \nabla H \nabla v dx + \int_\Omega 12\xi K_n \log H \nabla H \nabla v dx \\ &\quad - \int_\Omega H^3 \nabla \widetilde{\Psi}_u \cdot \nabla v \ dx \quad \forall v \in H_0^1(\Omega),\end{aligned}$$

with $\kappa_1(x, w) = \kappa(x, w + \widetilde{\Psi}_u)$.

Let us consider the space $E := H_0^1(\Omega)$ endowed with its weak topology and

$$X := \left\{\varphi \in E \ / \ \|\varphi\|_{H_0^1(\Omega)} \leq C + 1\right\}$$

(C is the constant given in Proposition 2.2). Consider the following applications:

$$\Phi_1(\chi, q) := \Phi_2(\chi, q) := \int_\Omega H^3 \nabla q \nabla (q - \chi) dx - \int_\Omega F(q) \nabla (q - \chi) dx$$

such that

$$\begin{aligned}F(q) := (\Lambda - 6\xi K_n \nabla H)\, \kappa_1(x, q) - 12\xi^2 K_n^2 \log \kappa_1(x, q) \nabla H \\ + 12\xi K_n \log H \nabla H - H^3 \nabla \widetilde{\Psi}_u\end{aligned}$$

for all χ, q in $H_0^1(\Omega)$.

We will show that conditions of the theorem 2.4 are satisfied.

Condition (1) is evidently satisfied. Since the application $\chi \to \Phi_1(\chi, q)$ is linear then condition (3) is also verified. For condition (5) it is sufficient to take

$$K := X = \left\{\varphi \in E \;/\; \|\varphi\|_{H_0^1(\Omega)} \leq C + 1\right\}.$$

According to Remark 2.5, it is sufficient to demonstrate that the application $q \mapsto \Phi_1(\chi, q)$ is weakly lower semicontinuous in $H_0^1(\Omega)$ to conclude that conditions (2) and (4) are satisfied. Indeed, let $q_n \rightharpoonup q$ in $H_0^1(\Omega)$, then there exists a subsequence q_{n_k} such that $q_{n_k} \to q$ in $L^2(\Omega)$ and $\nabla q_{n_k} \rightharpoonup \nabla q$ in $L^2(\Omega)$. Therefore while using the Lebesgue dominated convergence theorem and estimations (2.9), we have

$$\begin{aligned}\int_\Omega a_2 F(q_{n_k})\nabla q_{n_k} - \chi &= \int_\Omega a_2 F(q_{n_k})\nabla q_{n_k} - \int_\Omega a_2 F(q_{n_k})\nabla\chi \\ &\to \int_\Omega a_2 F(q)\nabla q - \int_\Omega a_2 F(q)\nabla\chi.\end{aligned}$$

For the other term of $\Phi_1(\chi, q_{n_k})$ we have

$$\int_\Omega H^3.\nabla q_{n_k}\nabla(q_{n_k} - \chi) = \int_\Omega H^3 \cdot \nabla q_{n_k}\nabla q_{n_k} - \int_\Omega H^3 \cdot \nabla q_{n_k}\nabla\chi.$$

However $\nabla q_{n_k} \rightharpoonup \nabla q$ in $L^2(\Omega)$, then $\int_\Omega H^3 \cdot \nabla q_{n_k}\nabla\chi \to \int_\Omega H^3 \cdot \nabla q\nabla\chi$. It remains to show that $q \mapsto \int_\Omega H^3 \cdot (\nabla q_{n_k})^2$ is weakly lower semicontinuous in $H_0^1(\Omega)$.

We consider the application $T : L^2(\Omega) \to \mathbb{R},\; z \mapsto \int_\Omega H^3 \cdot z^2$ which is convex and strongly semi continuous in $L^2(\Omega)$ therefore weakly semi continuous in $L^2(\Omega)$. However $\nabla q_{n_k} \rightharpoonup \nabla q$ in $L^2(\Omega)$ then $\underline{lim}\,(H^3 \cdot ((\nabla q_{nk})^2 - (\nabla q)^2)) \geq 0$, from where we obtain the result.

By application of Theorem 2.4, there exists $w \in K$ such that $\Phi_1(\chi, w) \leq 0$ for all $\chi \in X$, however $w \in int_E(X)$ (according to Proposition 2.2), then $\Phi_1(\chi, w) \leq 0$. In particular, for $\chi = w + \sigma \cdot \xi \in X$, for all $\xi \in \mathcal{D}(\Omega)$ and σ appropriately chosen, we deduct that $\Phi_1(\xi, w) = 0$, for all $\xi \in H_0^1(\Omega)$ (by density of $\mathcal{D}(\Omega)$ in $H_0^1(\Omega)$) which implies that there exists $w \in H_0^1(\Omega)$ satisfying the equation (2.10). □

It follows that we have solutions for the problems ($\mathcal{P}_u$) and ($\mathcal{P}$).

2.2. **Uniqueness.**

In the next lemma we give a general monotonicity and uniqueness result for a class of semi-linear elliptic problems.

Lemma 2.7. *Let $I \subseteq \mathbb{R}$ and $l : \Omega \times I \to \mathbb{R}^n$ an uniform Lipschitz function in the following sense:*

$$\exists N > 0,\; |l(x, u_1) - l(x, u_2)| \leq N\,|u_1 - u_2|\,,\; \forall x \in \Omega \textit{ and } u_1, u_2 \in \mathbb{R}. \tag{2.11}$$

Let $j : \Omega \to \mathbb{R}$ be a function satisfying $j(x) \geq \alpha_0 > 0$ a.e. $x \in \Omega$. Suppose that $u_i,\; i = 1, 2$, is a weak solution to

$$\begin{cases} -\nabla \cdot (j(x)\nabla u_i) = \nabla \cdot l(x, u_i),\; x \in \Omega \\ u_i = \varphi_i,\; x \in \partial\Omega. \end{cases} \tag{2.12}$$

If $\varphi_1 \geq \varphi_2$ a.e. on $\partial\Omega$, then $u_1 \geq u_2$ a.e. on Ω.

Proof. We take $u_3 = u_1 - u_2$ which satisfies the problem

$$\begin{cases} u_3 \in \varphi_1 - \varphi_2 + H_0^1(\Omega) \\ \int_\Omega j(x)\nabla u \cdot \nabla v dx = \int_\Omega (l(x, u_1) - l(x, u_2)) \cdot \nabla v dx. \end{cases} \tag{2.13}$$

We have that $u_3^+ \in H_0^1(\Omega)$, so we can take (as in [8]) $v = \frac{u_3^+}{u_3^+ + \delta}$ as a test function in (2.13) with $\delta > 0$, which gives

$$\int_\Omega j(x) \nabla u_3^+ \cdot \nabla \left(\frac{u_3^+}{u_3^+ + \delta} \right) dx = \int_\Omega \left(l(x,u_1) - l(x,u_2) \right) \cdot \nabla \left(\frac{u_3^+}{u_3^+ + \delta} \right) dx. \tag{2.14}$$

However

$$\nabla \left(\frac{u_3^+}{u_3^+ + \delta} \right) = \delta \frac{\nabla u_3^+}{\left(u_3^+ + \delta \right)^2}, \qquad \nabla \log \left(1 + \frac{u_3^+}{\delta} \right) = \frac{\nabla u_3^+}{\left(u_3^+ + \delta \right)},$$

which implies

$$\int_\Omega j(x) \nabla u_3^+ \cdot \nabla \left(\frac{u_3^+}{u_3^+ + \delta} \right) dx = \delta \int_\Omega j(x) \left| \nabla \log \left(1 + \frac{u_3^+}{\delta} \right) \right|^2 dx. \tag{2.15}$$

The right-hand side of (2.14) can be estimated as

$$\begin{aligned} &\left| \int_\Omega \left(l(x,u_1) - l(x,u_2) \right) \cdot \nabla \left(\frac{u_3^+}{u_3^+ + \delta} \right) dx \right| \\ &\qquad \le \sum_{i=1}^n \int_\Omega \left| l_i(x,u_1) - l_i(x,u_2) \right| \left| \frac{\partial}{\partial x_i} \left(\frac{u_3^+}{u_3^+ + \delta} \right) \right| dx \\ &\qquad \le N \sum_{i=1}^n \int_\Omega |u_3| \left| \frac{\partial}{\partial x_i} \left(\frac{u_3^+}{u_3^+ + \delta} \right) \right| dx \\ &\qquad = N \sum_{i=1}^n \int_\Omega \left| u_3^+ \frac{\partial}{\partial x_i} \left(\frac{u_3^+}{u_3^+ + \delta} \right) \right| dx. \end{aligned} \tag{2.16}$$

However

$$\begin{aligned} \left| u_3^+ \frac{\partial}{\partial x_i} \left(\frac{u_3^+}{u_3^+ + \delta} \right) \right| &= \delta \left| \frac{\partial u_3^+}{\partial x_i} \frac{u_3^+}{\left(u_3^+ + \delta \right)^2} \right| \le \delta \left| \frac{\partial u_3^+}{\partial x_i} \left(\frac{1}{u_3^+ + \delta} \right) \right| \\ &= \delta \left| \frac{\partial}{\partial x_i} \log \left(1 + \frac{u_3^+}{\delta} \right) \right| \le \delta \left| \nabla \log \left(1 + \frac{u_3^+}{\delta} \right) \right|. \end{aligned} \tag{2.17}$$

So, from (2.14) we obtain using also (2.15) – (2.17),

$$\alpha_0 \int_\Omega \left| \nabla \log \left(1 + \frac{u_3^+}{\delta} \right) \right|^2 dx \le N \cdot n \int_\Omega \left| \nabla \log \left(1 + \frac{u_3^+}{\delta} \right) \right| dx. \tag{2.18}$$

Since $\log \left(1 + \frac{u_3^+}{\delta} \right) \in H_0^1(\Omega)$, from the Poincaré inequality we deduce

$$\int_\Omega \left| \log \left(1 + \frac{u_3^+}{\delta} \right) \right|^2 dx \le C_2, \tag{2.19}$$

where C_2 is independent on δ.

Then we have $u_3^+ = 0$ a.e. $x \in \Omega$ and the proof is ended. $\square$

Proposition 2.8. *Under the hypotheses (A_1) and (A_2), we have uniqueness among all weak solutions of problem ($\mathcal{P}$).*

Lemma 2.9. *We suppose that u_i is a weak solution to ($\mathcal{P}_u$) corresponding to the boundary data $\Psi_u^i,\ i = 1, 2$. If $\Psi_u^1 \ge \Psi_u^2$ a.e. on $\partial\Omega$, then $u_1 \ge u_2$ a.e. on Ω. Further, we have uniqueness among all weak solutions of problem ($\mathcal{P}_u$).*

Proof. We apply Lemma 2.7 with $j = H^3$ and

$$l = (\Lambda - 6\xi K_n \nabla H)\,\kappa(x,u) - 12\xi^2 K_n^2 \log \kappa(x,u)\nabla H + 12\xi K_n \log H \nabla H.$$

Due to the fact that, for all $s \in \mathbb{R}$,

$$0 \le \frac{dg^{-1}}{ds}(s) = \frac{g^{-1}(s)}{(g^{-1})^2(s) + 6\xi K_n g^{-1}(s) + 6\xi^2 K_n^2} \le \frac{1}{6\xi K_n},$$

$$0 \le \frac{d}{ds}\log(g^{-1}(s)) = \frac{1}{(g^{-1})^2(s) + 6\xi K_n g^{-1}(s) + 6\xi^2 K_n^2} \le \frac{1}{6\xi^2 K_n^2}$$

and the fact that $H \in W^{1,\infty}(\Omega)$, the Lipschitz condition (2.11) is satisfied for l. □

Proof of Proposition 2.8. The proof is a consequence of Lemmas 2.1 and 2.9 and the fact that g^{-1} is an increasing function. □

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