Journal of Inequalities in Pure and Applied Mathematics

ON CEBYSEV-GRÜSS TYPE INEQUALITIES VIA PEČARIĆ'S EXTENSION OF THE MONTGOMERY IDENTITY

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Abstract

In the present note we establish new Čebyšev-Grüss type inequalities by using Pečarić's extension of the Montgomery identity.

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1. Introduction

For two absolutely continuous functions $f, g : [a, b] \to \mathbb{R}$ consider the functional

$$T(f, g) = \frac{1}{b-a} \int_a^b f(x) g(x) \, dx$$
$$- \left(\frac{1}{b-a} \int_a^b f(x) \, dx \right) \left(\frac{1}{b-a} \int_a^b g(x) \, dx \right),$$

where the involved integrals exist. In 1882, Čebyšev [1] proved that if $f', g' \in L_\infty[a, b]$, then
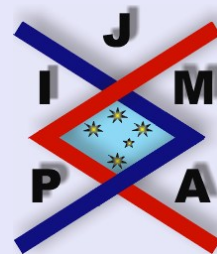
$$(1.1) \qquad |T(f, g)| \leq \frac{1}{12} (b-a)^2 \|f'\|_\infty \|g'\|_\infty.$$

In 1935, Grüss [2] showed that

$$(1.2) \qquad |T(f, g)| \leq \frac{1}{4} (M - m)(N - n),$$

provided m, M, n, N are real numbers satisfying the condition $-\infty < m \leq M < \infty,\ -\infty < n \leq N < \infty$ for $x \in [a, b]$.

Many researchers have given considerable attention to the inequalities (1.1), (1.2) and various generalizations, extensions and variants of these inequalities have appeared in the literature, to mention a few, see [4, 5] and the references cited therein. The aim of this note is to establish two new inequalities similar to those of Čebyšev and Grüss inequalities by using Pečarič's extension of the Montgomery identity given in [6].

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2. Statement of Results

Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$ and $f' : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$. Then the Montgomery identity holds [3]:

$$(2.1) \qquad f(x) = \frac{1}{b-a} \int_a^b f(t)\, dt + \int_a^b P(x, t)\, f'(t)\, dt,$$
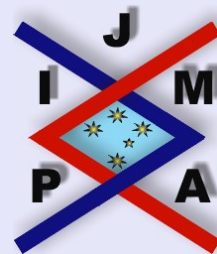
where $P(x, t)$ is the Peano kernel defined by

$$(2.2) \qquad P(x, t) = \begin{cases} \frac{t-a}{b-a}, & a \le t \le x, \\ \frac{t-b}{b-a}, & x < t \le b. \end{cases}$$

Let $w : [a, b] \to [0, \infty)$ be some probability density function, that is, an integrable function satisfying $\int_a^b w(t)\, dt = 1$, and $W(t) = \int_a^t w(x)\, dx$ for $t \in [a, b]$, $W(t) = 0$ for $t < a$, and $W(t) = 1$ for $t > b$. In [6] Pečarić has given the following weighted extension of the Montgomery identity:

$$(2.3) \qquad f(x) = \int_a^b w(t)\, f(t)\, dt + \int_a^b P_w(x, t)\, f'(t)\, dt,$$

where $P_w(x, t)$ is the weighted Peano kernel defined by

$$(2.4) \qquad P_w(x, t) = \begin{cases} W(t), & a \le t \le x, \\ W(t) - 1, & x < t \le b. \end{cases}$$

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We use the following notation to simplify the details of presentation. For some suitable functons $w, f, g : [a, b] \to \mathbb{R}$, we set

$$T(w, f, g) = \int_a^b w(x) f(x) g(x) \, dx$$
$$- \left(\int_a^b w(x) f(x) \, dx \right) \left(\int_a^b w(x) g(x) \, dx \right),$$

and define $\|\cdot\|_\infty$ as the usual Lebesgue norm on $L_\infty[a, b]$ that is, $\|h\|_\infty := ess \sup_{t \in [a,b]} |h(t)|$ for $h \in L_\infty[a, b]$.
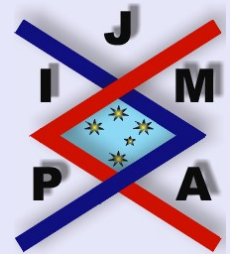
Our main results are given in the following theorems.

Theorem 2.1. *Let* $f, g : [a, b] \to \mathbb{R}$ *be differentiable on* $[a, b]$ *and* $f', g' : [a, b] \to \mathbb{R}$ *are integrable on* $[a, b]$. *Let* $w : [a, b] \to [0, \infty)$ *be an integrable function satisfying* $\int_a^b w(t) \, dt = 1$. *Then*

$$(2.5) \qquad |T(w, f, g)| \leq \|f'\|_\infty \|g'\|_\infty \int_a^b w(x) H^2(x) \, dx,$$

where

$$(2.6) \qquad H(x) = \int_a^b |P_w(x, t)| \, dt$$

for $x \in [a, b]$ *and* $P_w(x, t)$ *is the weighted Peano kernel given by (2.4).*

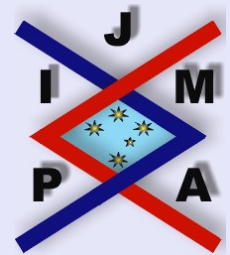*J. Ineq. Pure and Appl. Math.* **7**(1) Art. 11, 2006

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Theorem 2.2. *Let f, g, f', g', w be as in Theorem 2.1. Then*

$$(2.7) \quad |T(w, f, g)| \leq \frac{1}{2} \int_a^b w(x) \left[|g(x)| \, \|f'\|_\infty + |f(x)| \, \|g'\|_\infty \right] H(x) \, dx,$$

where $H(x)$ is defined by (2.6).

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3. Proofs of Theorems 2.1 and 2.2

Proof of Theorem 2.1. From the hypotheses the following identities hold [6]:

$$(3.1) \qquad f(x) = \int_a^b w(t) f(t) \, dt + \int_a^b P_w(x,t) f'(t) \, dt,$$

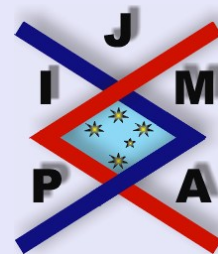$$(3.2) \qquad g(x) = \int_a^b w(t) g(t) \, dt + \int_a^b P_w(x,t) g'(t) \, dt,$$

From (3.1) and (3.2) we observe that

$$\left[f(x) - \int_a^b w(t) f(t) \, dt \right] \left[g(x) - \int_a^b w(t) g(t) \, dt \right]$$
$$= \left[\int_a^b P_w(x,t) f'(t) \, dt \right] \left[\int_a^b P_w(x,t) g'(t) \, dt \right],$$

i.e.,

$$(3.3) \quad f(x) g(x) - f(x) \int_a^b w(t) g(t) \, dt - g(x) \int_a^b w(t) f(t) \, dt$$
$$+ \left(\int_a^b w(t) f(t) \, dt \right) \left(\int_a^b w(t) g(t) \, dt \right)$$
$$= \left[\int_a^b P_w(x,t) f'(t) \, dt \right] \left[\int_a^b P_w(x,t) g'(t) \, dt \right].$$

Multiplying both sides of (3.3) by $w(x)$ and then integrating both sides of the resulting identity with respect to x from a to b and using the fact that

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$\int_a^b w(t)\, dt = 1$, we have

$$(3.4) \qquad T(w, f, g) = \int_a^b w(x) \left[\int_a^b P_w(x, t) f'(t)\, dt \right]$$
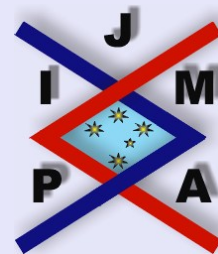$$\times \left[\int_a^b P_w(x, t) g'(t)\, dt \right] dx.$$

From (3.4) and using the properties of modulus we observe that

$$|T(w, f, g)| \leq \int_a^b w(x) \left[\int_a^b |P_w(x, t)|\, |f'(t)|\, dt \right] \left[\int_a^b |P_w(x, t)|\, |g'(t)|\, dt \right] dx$$
$$\leq \|f'\|_\infty \|g'\|_\infty \int_a^b w(x) H^2(x)\, dx.$$

This completes the proof of Theorem 2.1. $\qquad\qquad\qquad\qquad\qquad$ $\square$

Proof of Theorem 2.2. Multiplying both sides of (3.1) and (3.2) by $w(x)g(x)$ and $w(x)f(x)$, adding the resulting identities and rewriting we have

$$(3.5) \quad w(x) f(x) g(x)$$
$$= \frac{1}{2} \left[w(x) g(x) \int_a^b w(t) f(t)\, dt + w(x) f(x) \int_a^b w(t) g(t)\, dt \right]$$
$$+ \frac{1}{2} \left[w(x) g(x) \int_a^b P_w(x, t) f'(t)\, dt \right.$$
$$\left. + w(x) f(x) \int_a^b P_w(x, t) g'(t)\, dt \right].$$

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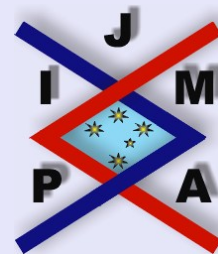
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Integrating both sides of (3.5) with respect to x from a to b and rewriting we have

$$(3.6) \quad T(w, f, g) = \frac{1}{2} \int_a^b \left[w(x) g(x) \int_a^b P_w(x, t) f'(t) dt \right.$$
$$\left. + w(x) f(x) \int_a^b P_w(x, t) g'(t) dt \right] dx.$$

From (3.6) and using the properties of modulus we observe that

$$|T(w, f, g)|$$
$$\leq \frac{1}{2} \int_a^b w(x) \left[|g(x)| \int_a^b |P_w(x, t)| |f'(t)| dt \right.$$
$$\left. + |f(x)| \int_a^b |P_w(x, t)| |g'(t)| dt \right] dx$$
$$\leq \frac{1}{2} \int_a^b w(x) \left[|g(x)| \|f'(t)\|_\infty + |f(x)| \|g'(t)\|_\infty \right] H(x) dx.$$

The proof of Theorem 2.2 is complete. □

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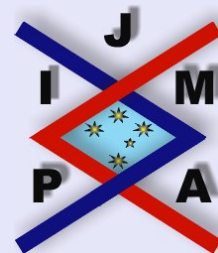
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