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AN APPLICATION OF VAN DER CORPUT'S INEQUALITY

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ABSTRACT. In this note we give a short and elegant proof of the result $\sum_{t=1}^{n} e^{\imath(\omega t+\alpha t^2)} = o(n)$ for α not a rational multiple of π, uniformly in ω. This was first proved by Hardy and Littlewood, in 1938. The main ingredient of our proof is Van der Corput's inequality. We then generalize this to obtain $\sum_{t=1}^{n} t^{\beta} e^{\imath(\omega t+\alpha t^2)} = o(n^{\beta+1})$, where β is a nonnegative constant.

Key words and phrases: Van der Corput's inequality, Hardy and Littlewood.

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1. INTRODUCTION

Hardy and Littlewood [1] studied the series of the form $\sum_{t=1}^{n} e^{\imath(\omega t+\alpha t^2)}$ and other similar series associated with the elliptic Theta functions. It was noted there that the behavior is interesting and difficult when α is not a rational multiple of π. The main result proved in [1] can essentially be stated as $\sum_{t=1}^{n} e^{\imath(\omega t+\alpha t^2)} = o(n)$ for α not a rational multiple of π uniformly in ω.

We became interested in this result, rather a generalization of it, while working on a problem of estimation of parameters of a chirp-type statistical model. Although we hoped to find an easy proof of this result in the literature, we were unable to find one. The purpose of this note is to give an easy proof of it. We then generalize this to obtain a similar result for $\sum_{t=1}^{n} t^{\beta} e^{\imath(\omega t+\alpha t^2)}$, where β is a positive constant.

The main ingredient of our proof is the following remarkable inequality.

Theorem 1.1 (Van der Corput's Fundamental Inequality)**.** [2, p. 25]*:*

Let $u_1 \cdots u_n$ be complex numbers, and let H be an integer with $1 \le H \le N$. Then

$$H^2 \left| \sum_{n=1}^{N} u_n \right|^2 \le H(N+H-1) \sum_{n=1}^{N} |u_n|^2 + 2(N+H-1) \sum_{h=1}^{H-1} (H-h) \left| \sum_{n=1}^{N-h} u_n \bar{u}_{n+h} \right|.$$

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2. Main Result

Theorem 2.1. *Let β be a nonnegative real number. Then $\sum_{t=1}^{n} t^{\beta} e^{\imath(\omega t+\alpha t^2)} = o(n^{\beta+1})$, for α not a rational multiple of π, uniformly in ω.*

Proof. By using Van der Corput's inequality with fixed H and $u_t = e^{\imath(\omega t+\alpha t^2)}$, we obtain

$$H^2 \left| \sum_{t=1}^{n} e^{\imath(\omega t+\alpha t^2)} \right|^2 \leq H(n+H-1) \sum_{t=1}^{n} \left| e^{\imath(\omega t+\alpha t^2)} \right|^2 + 2 \sum_{h=1}^{H-1} (n+H-1)(H-h) \left| \sum_{t=1}^{n-h} u_t \bar{u}_{t+h} \right| \tag{2.1}$$

for all $n \geq H$. But $u_t \bar{u}_{t+h} = e^{-\imath\omega h - \imath\alpha h^2 - 2\imath\alpha t h}$, and so

$$\left| \sum_{t=1}^{n-h} u_t \bar{u}_{t+h} \right| = \left| \sum_{t=1}^{n-h} e^{-2\imath\alpha t h} \right|.$$

Substituting this in the inequality (2.1), we obtain

$$H^2 \left| \sum_{t=1}^{n} e^{\imath(\omega t+\alpha t^2)} \right|^2 \leq H(n+H-1)n + 2 \sum_{h=1}^{H-1} (n+H-1)(H-h) \left| \sum_{t=1}^{n-h} e^{-2\imath\alpha t h} \right|.$$

Thus

$$\left| \frac{1}{n} \sum_{t=1}^{n} e^{\imath(\omega t+\alpha t^2)} \right|^2 \leq \frac{1}{H} + \frac{1}{n} - \frac{1}{nH} + 2 \sum_{h=1}^{H-1} \frac{(n+H-1)(H-h)}{n^2 H^2} \left| \sum_{t=1}^{n-h} e^{-2\imath\alpha t h} \right|. \tag{2.2}$$

Let

$$M_n(\alpha, h) = \sum_{t=1}^{n-h} e^{-2\imath\alpha t h}.$$

Thus if α is not a rational multiple of π we can write $M_n(\alpha, h)$ in the following form.

$$M_n(\alpha, h) = e^{-\imath\alpha h(n-h+1)} \frac{\sin[(n-h)\alpha h]}{\sin(\alpha h)}.$$

Then

$$|M_n(\alpha, h)| \leq \frac{1}{|\sin(\alpha h)|}.$$

If h_0 is the member of $\{1, 2, \cdots, (H-1)\}$ for which

$$|\sin(\alpha h_0)| = \min_{1 \leq h \leq H-1} |\sin(\alpha h)|,$$

then

$$|M_n(\alpha, h)| \leq \frac{1}{|\sin(\alpha h_0)|}.$$

Substituting this in equation (2.2) we get

$$\begin{aligned}\left|\frac{1}{n}\sum_{t=1}^{n} e^{\imath(\omega t+\alpha t^2)}\right|^2 &\leq \frac{1}{H}+\frac{1}{n}-\frac{1}{nH}\\ &\quad +2\sum_{h=1}^{H-1}\frac{(n+H-1)(H-h)}{n^2H^2}\frac{1}{|\sin(\alpha,h_0)|}\\ &\leq \frac{1}{H}+\frac{1}{n}-\frac{1}{nH}+\frac{2}{n\,|\sin(\alpha h_0)|}.\end{aligned}$$

Since this is true for all $n \geq H$, we obtain

$$\lim_{n\to\infty}\left|\frac{1}{n}\sum_{t=1}^{n} e^{\imath(\omega t+\alpha t^2)}\right|^2 \leq \frac{1}{H}.$$

Since this is true for all $H \geq 1$, it follows that

$$\lim_{n\to\infty}\left|\frac{1}{n}\sum_{t=1}^{n} e^{\imath(\omega t+\alpha t^2)}\right|^2 = 0,$$

uniformly in ω. That is, if α is not a rational multiple of π,

$$\sum_{t=1}^{n} e^{\imath(\omega t+\alpha t^2)} = o(n), \tag{2.3}$$

uniformly in ω.

Now we will show that $\sum_{t=1}^{n} t^\beta e^{\imath(\omega t+\alpha t^2)} = o(n^{\beta+1})$ provided α is not a rational multiple of π. Let

$$Q_0(\omega,\alpha)=0,$$

and, for $n \geq 1$,

$$Q_n(\omega,\alpha)=\sum_{t=1}^{n} e^{\imath(\omega t+\alpha t^2)} \text{ and } S_n(\omega,\alpha)=\sum_{t=1}^{n} t^\beta e^{\imath(\omega t+\alpha t^2)}.$$

Then

$$\begin{aligned}S_n(\omega,\alpha) &= \sum_{t=1}^{n} t^\beta[Q_t(\omega,\alpha)-Q_{t-1}(\omega,\alpha)]\\ &= n^\beta Q_n(\omega,\alpha)-Q_0(\omega,\alpha)-\sum_{t=1}^{n-1}[(t+1)^\beta-t^\beta]Q_t(\omega,\alpha)\\ &= n^\beta Q_n(\omega,\alpha)-\sum_{t=1}^{n-1} f_t(\omega,\alpha), \qquad (2.4)\end{aligned}$$

where

$$f_n(\omega,\alpha)=[(n+1)^\beta-n^\beta]Q_n(\omega,\alpha) \quad \text{for} \quad n=1,2,\dots$$

By the mean-value theorem we have

$$(n+1)^\beta-n^\beta=\beta\tilde{n}^{\beta-1} \qquad \text{where} \quad n\leq\tilde{n}\leq n+1.$$

If $0\leq\beta\leq 1$, then $\tilde{n}^{\beta-1}\leq n^{\beta-1}$, while if $\beta\geq 1$, then $\tilde{n}^{\beta-1}\leq(n+1)^{\beta-1}\leq(2n)^{\beta-1}$. It follows that, for $\beta\geq 0$,

$$(n+1)^\beta-n^\beta\leq c_\beta n^{\beta-1}, \tag{2.5}$$

where c_β is a constant. Hence

$$|f_n(\omega,\alpha)| \leq c_\beta n^{\beta-1} |Q_n(\omega,\alpha)| .$$

But by (2.3), if α is not a rational multiple of π, then $Q_n(\omega,\alpha) = o(n)$ uniformly in ω. Thus if α is not a rational multiple of π, then

$$|f_n(\omega,\alpha)| \leq c_\beta n^{\beta-1} o(n),$$

so

$$f_n(\omega,\alpha) = o(n^\beta).$$

uniformly in ω. However, $f_n(\omega,\alpha) = o(n^\beta)$ implies that the mean $\frac{1}{n-1}\sum_{t=1}^{n-1} f_t(\omega,\alpha)$ is $o(n^\beta)$. Hence, if α is not a rational multiple of π,

$$\sum_{t=1}^{n-1} f_t(\omega,\alpha) = o(n^{\beta+1}),$$

uniformly in ω. But by (2.4)

$$S_n(\omega,\alpha) = n^\beta Q_n(\omega,\alpha) - \sum_{t=1}^{n-1} f_t(\omega,\alpha).$$

It follows that $S_n(\omega,\alpha) = o(n^{\beta+1})$, for α not a rational multiple of π, uniformly in ω. This completes the proof of the Theorem. □

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