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INEQUALITIES BETWEEN THE QUADRATURE OPERATORS AND ERROR BOUNDS OF QUADRATURE RULES

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ABSTRACT. The order structure of the set of six operators connected with quadrature rules is established in the class of 3–convex functions. Convex combinations of these operators are studied and their error bounds for four times differentiable functions are given. In some cases they are obtained for less regular functions as in the classical results.

Key words and phrases: Approximate integration, Quadrature rules, Convex functions of higher order.

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1. INTRODUCTION

For $f:[-1,1]\to\mathbb{R}$ we consider six operators approximating the integral mean value

$$\mathcal{I}(f):=\frac{1}{2}\int_{-1}^{1}f(x)dx.$$

They are

$$\mathcal{C}(f):=\frac{1}{3}\left(f\left(-\frac{\sqrt{2}}{2}\right)+f(0)+f\left(\frac{\sqrt{2}}{2}\right)\right),$$

$$\mathcal{G}_2(f):=\frac{1}{2}\left(f\left(-\frac{\sqrt{3}}{3}\right)+f\left(\frac{\sqrt{3}}{3}\right)\right),$$

$$\mathcal{G}_3(f):=\frac{4}{9}f(0)+\frac{5}{18}\left(f\left(-\frac{\sqrt{15}}{5}\right)+f\left(\frac{\sqrt{15}}{5}\right)\right),$$

$$\mathcal{L}_4(f):=\frac{1}{12}\big(f(-1)+f(1)\big)+\frac{5}{12}\left(f\left(-\frac{\sqrt{5}}{5}\right)+f\left(\frac{\sqrt{5}}{5}\right)\right),$$

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$$\mathcal{L}_5(f) := \frac{16}{45}f(0) + \frac{1}{20}\big(f(-1)+f(1)\big) + \frac{49}{180}\left(f\left(-\frac{\sqrt{21}}{7}\right) + f\left(\frac{\sqrt{21}}{7}\right)\right),$$
$$\mathcal{S}(f) := \frac{1}{6}\big(f(-1)+f(1)\big) + \frac{2}{3}f(0).$$

All of them are connected with very well known rules of the approximate integration: Chebyshev quadrature, Gauss–Legendre quadrature with two and three knots, Lobatto quadrature with four and five knots and Simpson's Rule, respectively (see e.g. [4, 7, 8, 9, 10]).

Our goal is to establish all possible inequalities between the above operators in the class of 3–convex functions and to give the error bounds for convex combinations of the quadratures considered. As a consequence, we obtain the error bound for the quadrature $\mathcal{L}_5$ for four times differentiable functions instead of eight times differentiable functions as in the classical result. We also improve similar results obtained in [6] for the quadratures $\mathcal{G}_3$ and $\mathcal{L}_4$.

Let $I \subset \mathbb{R}$ be an interval. For the function $f : I \to \mathbb{R}$, a positive integer $k \geq 2$ and $x_1, \ldots, x_k \in I$ denote

$$D(x_1,\ldots,x_k;f) := \begin{vmatrix} 1 & \ldots & 1 \\ x_1 & \ldots & x_k \\ \vdots & \ddots & \vdots \\ x_1^{k-2} & \ldots & x_k^{k-2} \\ f(x_1) & \ldots & f(x_k) \end{vmatrix}.$$

Let $V(x_1,\ldots,x_k)$ be the Vandermonde determinant of the terms involved. Then

$$[x_1,\ldots,x_k;f] := \frac{D(x_1,\ldots,x_k;f)}{V(x_1,\ldots,x_k)}$$

is the divided difference of the function f of order k. Recall that f is called n*–convex* if

$$[x_1,\ldots,x_{n+2};f] \geq 0$$

for any $x_1,\ldots,x_{n+2} \in I$. This is obviously equivalent to

$$D(x_1,\ldots,x_{n+2};f) \geq 0$$

for any $x_1,\ldots,x_{n+2} \in I$ such that $x_1 < \cdots < x_{n+2}$. Clearly 1–convex functions are convex in the classical sense. More information on the divided differences, the definition and properties of convex functions of higher order can be found in [1, 2, 3, 5].

In this paper only 3–convex functions are considered. By the above inequalities the function $f : I \to \mathbb{R}$ is 3*–convex* iff

$$[x_1,\ldots,x_5;f] = \frac{D(x_1,\ldots,x_5;f)}{V(x_1,\ldots,x_5)} \geq 0 \tag{1.1}$$

for any $x_1,\ldots,x_5 \in I$, or equivalently, iff

$$D(x_1,\ldots,x_5;f) = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 & x_5^3 \\ f(x_1) & f(x_2) & f(x_3) & f(x_4) & f(x_5) \end{vmatrix} \geq 0$$

for any $x_1,\ldots,x_5 \in I$ such that $x_1 < \cdots < x_5$.

2. INEQUALITIES BETWEEN QUADRATURE OPERATORS

In [6, Lemma 2.1] the inequality

$$v^2\big(f(-u)+f(u)\big) \le u^2\big(f(-v)+f(v)\big) + 2(v^2-u^2)f(0), \quad 0<u<v\le 1$$

was proved for a 3–convex function $f:[-1,1]\to\mathbb{R}$ (this is a simple consequence of the inequality $D(-v,-u,0,u,v;f)\ge 0$ obtained by 3–convexity). Denote by f_e the even part of f, i.e.

$$f_e(x)=\frac{f(x)+f(-x)}{2}.$$

Then we have

Remark 2.1. If $f:[-1,1]\to\mathbb{R}$ is 3–convex then the inequality

$$v^2 f_e(u) \le u^2 f_e(v) + (v^2-u^2)f_e(0) \tag{2.1}$$

holds for any $0<u<v\le 1$.

Let us also record

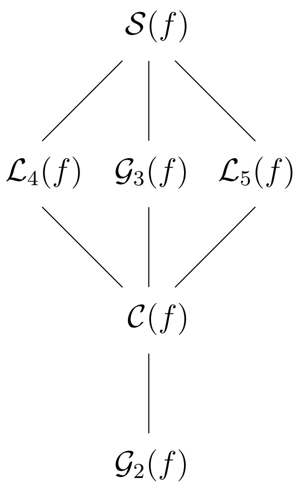
Remark 2.2. If the function $f:[-1,1]\to\mathbb{R}$ is 3–convex then so is f_e.

This property holds in fact for convex functions of any odd order (cf. [3]).

Remark 2.3. If $\mathcal{T}\in\{\mathcal{C},\mathcal{G}_2,\mathcal{G}_3,\mathcal{L}_4,\mathcal{L}_5,\mathcal{S}\}$ then $\mathcal{T}(f)=\mathcal{T}(f_e)$ for any $f:[-1,1]\to\mathbb{R}$.

Now we are ready to establish the inequalities between the operators connected with quadrature rules.

Theorem 2.4. *If $f:[-1,1]\to\mathbb{R}$ is 3–convex then $\mathcal{G}_2(f)\le\mathcal{C}(f)\le\mathcal{T}(f)\le\mathcal{S}(f)$, where $\mathcal{T}\in\{\mathcal{G}_3,\mathcal{L}_4,\mathcal{L}_5\}$. The operators $\mathcal{G}_3,\mathcal{L}_4$ and $\mathcal{L}_5$ are not comparable (see the graph below).*



Proof. Let $f:[-1,1]\to\mathbb{R}$ be a 3–convex function. By Remark 2.2 the function f_e is 3–convex. Then setting in (2.1) the appropriate values of u,v we obtain

(1) $\mathcal{G}_2(f_e)\le\mathcal{C}(f_e)$ for $u=\frac{\sqrt3}{3}$, $v=\frac{\sqrt2}{2}$;
(2) $C(f_e)\le\mathcal{G}_3(f_e)$ for $u=\frac{\sqrt2}{2}$, $v=\frac{\sqrt{15}}{5}$;
(3) $\mathcal{G}_3(f_e)\le\mathcal{S}(f_e)$ for $u=\frac{\sqrt{15}}{5}$, $v=1$ (this inequality was proved in [6, Proposition 2.2]);
(4) $\mathcal{L}_4(f_e)\le\mathcal{S}(f_e)$ for $u=\frac{\sqrt5}{5}$, $v=1$ (this inequality was also proved in [6]);

(5) $\mathcal{L}_5(f_e) \le \mathcal{S}(f_e)$ for $u = \frac{\sqrt{21}}{7}$, $v = 1$.

By Remark 2.3 all the above inequalities hold for f.

Now we will prove the inequality $\mathcal{C}(f) \le \mathcal{L}_5(f)$. Let p be the polynomial of degree at most 3 interpolating f at four knots -1, $-\frac{\sqrt{21}}{7}$, $\frac{\sqrt{21}}{7}$, 1. Since $[x_1, \dots, x_5; p] = 0$ for any $x_1, \dots, x_5 \in [-1, 1]$, the function $g := f - p$ is also 3–convex and

$$g(-1) = g\left(-\frac{\sqrt{21}}{7}\right) = g\left(\frac{\sqrt{21}}{7}\right) = g(1) = 0.$$

It is easy to observe that $\mathcal{C}(p) = \mathcal{L}_5(p) = \mathcal{I}(p)$. Then by linearity

$$\mathcal{C}(f) \le \mathcal{L}_5(f) \iff \mathcal{C}(g) \le \mathcal{L}_5(g).$$

By Remark 2.3 it is enough to prove $\mathcal{C}(g_e) \le \mathcal{L}_5(g_e)$, which is equivalent to

$$g_e\left(\frac{\sqrt{2}}{2}\right) \le \frac{1}{30} g_e(0). \tag{2.2}$$

By 3–convexity of g_e we get $D\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{21}}{7}, 0, \frac{\sqrt{21}}{7}, \frac{\sqrt{2}}{2}; g_e\right) \ge 0$. Expanding this determinant by the last row we arrive at

$$\left(V\left(-\frac{\sqrt{21}}{7}, 0, \frac{\sqrt{21}}{7}, \frac{\sqrt{2}}{2}\right) + V\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{21}}{7}, 0, \frac{\sqrt{21}}{7}\right)\right) g_e\left(\frac{\sqrt{2}}{2}\right) + V\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{21}}{7}, \frac{\sqrt{21}}{7}, \frac{\sqrt{2}}{2}\right) g_e(0) \ge 0.$$

By computing the Vandermonde determinants we obtain

$$6 g_e\left(\frac{\sqrt{2}}{2}\right) + g_e(0) \ge 0. \tag{2.3}$$

Similarly, by $D\left(-\frac{\sqrt{21}}{7}, 0, \frac{\sqrt{21}}{7}, \frac{\sqrt{2}}{2}, 1; g_e\right) \ge 0$ we get

$$-6 g_e\left(\frac{\sqrt{2}}{2}\right) - \left(1 - \frac{\sqrt{2}}{2}\right) g_e(0) \ge 0. \tag{2.4}$$

The inequalities (2.3) and (2.4) now imply $g_e\left(\frac{\sqrt{2}}{2}\right) \le 0 \le g_e(0)$, which proves (2.2).

The last inequality to prove is $\mathcal{C}(f) \le \mathcal{L}_4(f)$. It seems to be more complicated than the other inequalities. In the proof it is not enough to consider divided differences containing only the knots of the quadratures involved. We need to consider some other points. Let $u = \frac{\sqrt{5}}{5}$, $v = \frac{\sqrt{2}}{2}$. Arguing similarly as in the previous part of the proof we may assume that

$$f\left(-\frac{v}{2}\right) = f\left(-\frac{u}{2}\right) = f\left(\frac{u}{2}\right) = f\left(\frac{v}{2}\right) = 0.$$

Furthermore, by Remarks 2.2 and 2.3 it is enough to prove $\mathcal{C}(f_e) \le \mathcal{L}_4(f_e)$.

By 3–convexity and (1.1)

$$\left[-\frac{u}{2}, 0, u, v, 1; f_e\right] \ge 0, \ \left[-\frac{v}{2}, 0, u, v, 1; f_e\right] \ge 0,$$
$$\left[0, \frac{u}{2}, u, v, 1; f_e\right] \ge 0, \ \left[0, \frac{v}{2}, u, v, 1; f_e\right] \ge 0.$$

Using the above inequalities and the determinantal formula (1.1) we obtain after some simplifications

$$0 \le x := -\frac{5f_e(0)}{v} + \frac{5f_e(u)}{3(v-u)(1-u)} - \frac{f_e(v)}{v(u+2v)(v-u)(1-v)} + \frac{f_e(1)}{(u+2)(1-u)(1-v)},$$

$$0 \le y := -\frac{2f_e(0)}{u} + \frac{f_e(u)}{u(2u+v)(v-u)(1-u)} - \frac{2f_e(v)}{3(v-u)(1-v)} + \frac{f_e(1)}{(2+v)(1-u)(1-v)},$$

$$0 \le z := \frac{5f_e(0)}{v} + \frac{5f_e(u)}{(v-u)(1-u)} - \frac{f_e(v)}{v(2v-u)(v-u)(1-v)} + \frac{f_e(1)}{(2-u)(1-u)(1-v)},$$

$$0 \le t := \frac{2f_e(0)}{u} + \frac{f_e(u)}{u(2u-v)(v-u)(1-u)} - \frac{2f_e(v)}{(v-u)(1-v)} + \frac{f_e(1)}{(2-v)(1-u)(1-v)}.$$

Then

$$[x, y, z, t]^T = A\,[f_e(0), f_e(u), f_e(v), f_e(1)]^T,$$

where

$$A = \begin{bmatrix} -5\sqrt{2} & \frac{25(2+5\sqrt{2}+2\sqrt{5}+\sqrt{10})}{36} & -\frac{10(8+8\sqrt{2}+2\sqrt{5}+\sqrt{10})}{27} & \frac{5(18+9\sqrt{2}+2\sqrt{5}+\sqrt{10})}{76} \\ -2\sqrt{5} & \frac{25(-1+5\sqrt{2}-\sqrt{5}+\sqrt{10})}{18} & -\frac{4(5+5\sqrt{2}+2\sqrt{5}+\sqrt{10})}{9} & \frac{15+5\sqrt{2}+3\sqrt{5}+\sqrt{10}}{14} \\ 5\sqrt{2} & \frac{25(2+5\sqrt{2}+2\sqrt{5}+\sqrt{10})}{12} & -\frac{10(4+4\sqrt{2}+2\sqrt{5}+\sqrt{10})}{9} & \frac{5(22+11\sqrt{2}+6\sqrt{5}+3\sqrt{10})}{76} \\ 2\sqrt{5} & \frac{25(3+5\sqrt{2}+3\sqrt{5}+\sqrt{10})}{6} & -\frac{4(5+5\sqrt{2}+2\sqrt{5}+\sqrt{10})}{3} & \frac{25+15\sqrt{2}+5\sqrt{5}+3\sqrt{10}}{14} \end{bmatrix}.$$

Using the elementary properties of determinants we can compute

$$\det A = -\frac{320000}{10773}(7 + 6\sqrt{2} + 3\sqrt{5} + 2\sqrt{10}).$$

Hence

$$\big[f_e(0), f_e(u), f_e(v), f_e(1)\big]^T = A^{-1}\big[x, y, z, t\big]^T$$

and

$$6\big(\mathcal{L}_4(f_e) - \mathcal{C}(f_e)\big) = -2f(0) + 5f(u) - 4f(v) + f(1) = ax + by + cz + dt$$

for some a, b, c, d. Notice that the approximate values of the entries of the matrices A, A^{-1} are

$$A \approx \begin{bmatrix} -7.0711 & 11.6010 & -9.9808 & 2.5238 \\ -4.4721 & 9.7184 & -8.7580 & 2.2815 \\ 7.0711 & 34.8031 & -19.2125 & 3.9776 \\ 4.4721 & 83.0898 & -26.2740 & 4.7772 \end{bmatrix},$$

$$A^{-1} \approx \begin{bmatrix} -0.2847 & 0.2710 & 0.0313 & -0.0050 \\ -0.1708 & 0.2154 & -0.0563 & 0.0343 \\ -1.8470 & 2.4389 & -0.3906 & 0.1362 \\ -6.9203 & 9.4143 & -1.1984 & 0.3671 \end{bmatrix}.$$

Then the constants a, b, c, d can be approximately computed:

$$6\big(\mathcal{L}_4(f_e) - \mathcal{C}(f_e)\big) \approx 0.1831x + 0.1937y + 0.0199z + 0.0038t \ge 0,$$

by $x, y, z, t \ge 0$ and we infer $\mathcal{C}(f_e) \le \mathcal{L}_4(f_e)$.

We finish the proof with examples showing that the quadratures $\mathcal{L}_4$, $\mathcal{L}_5$ and $\mathcal{G}_3$ are not comparable in the class of 3–convex functions. The table below contains the approximate values of these operators.

f	$\mathcal{L}_4(f)$	$\mathcal{L}_5(f)$	$\mathcal{G}_3(f)$
exp	1.17524	1.17520	1.17517
cos	0.84143	0.84147	0.84150

The functions $\exp$ and $\cos$ are 3–convex on $[-1,1]$ since their derivatives of the fourth order are nonnegative on $[-1,1]$ (cf. [1, 2, 3], cf. also [6, Theorems A, B]). $\square$

3. Error Bounds of Convex Combinations of Quadrature Rules

Recall that $\mathcal{I}(f)=\frac{1}{2}\int_{-1}^{1} f(x)dx$. For $f\in\mathcal{C}^4\big([-1,1]\big)$ denote

$$M(f):=\sup\Big\{\big|f^{(4)}(x)\big| : x\in[-1,1]\Big\}.$$

We start with two lemmas.

Lemma 3.1. *Let $\mathcal{T}$ be a linear operator acting on functions mapping $[-1,1]$ into $\mathbb{R}$ such that $\mathcal{T}(g)=\mathcal{I}(g)$ for $g(x)=x^4$ and $\mathcal{G}_2(f)\le\mathcal{T}(f)$ for any 3–convex function $f:[-1,1]\to\mathbb{R}$. Then*

$$\big|\mathcal{T}(f)-\mathcal{I}(f)\big|\le\frac{M(f)}{135}$$

for any $f\in\mathcal{C}^4\big([-1,1]\big)$.

Proof. Let $f\in\mathcal{C}^4\big([-1,1]\big)$. It is well known (cf. [4, 8]) that $\mathcal{I}(f)=\mathcal{G}_2(f)+\frac{f^{(4)}(\xi)}{270}$ for some $\xi\in(-1,1)$.

Assume for a while that f is 3–convex. Then $\mathcal{I}(f)-\frac{f^{(4)}(\xi)}{270}=\mathcal{G}_2(f)\le\mathcal{T}(f)$. Therefore

$$\mathcal{I}(f)-\mathcal{T}(f)\le\frac{M(f)}{270}. \tag{3.1}$$

Now let $f\in\mathcal{C}^4\big([-1,1]\big)$ be an arbitrary function and let $g(x):=\frac{M(f)x^4}{24}$. Then $\big|f^{(4)}(x)\big|\le g^{(4)}(x)$, $x\in[-1,1]$, whence $(g-f)^{(4)}\ge 0$ and $(g+f)^{(4)}\ge 0$ on $[-1,1]$. This implies that $g-f$ and $g+f$ are 3–convex on $[-1,1]$ (cf. [1, 2, 3], cf. also [6, Theorem B]). It is easy to see that $M(g-f)\le 2M(f)$ and $M(g+f)\le 2M(f)$. We infer by 3–convexity and (3.1) that

$$\mathcal{I}(g-f)-\mathcal{T}(g-f)\le\frac{M(g-f)}{270}\le\frac{M(f)}{135}\quad\text{and}\quad\mathcal{I}(g+f)-\mathcal{T}(g+f)\le\frac{M(g+f)}{270}\le\frac{M(f)}{135}.$$

Since the operators $\mathcal{T},\mathcal{I}$ are linear and $\mathcal{T}(g)=\mathcal{I}(g)$ by the assumption, then

$$-\mathcal{I}(f)+\mathcal{T}(f)\le\frac{M(f)}{135}\qquad\text{and}\qquad\mathcal{I}(f)-\mathcal{T}(f)\le\frac{M(f)}{135},$$

which concludes the proof. $\square$

Lemma 3.2. *Let $\mathcal{T}$ be a linear operator acting on functions mapping $[-1,1]$ into $\mathbb{R}$ such that $\mathcal{T}(g)=\mathcal{I}(g)$ for $g(x)=x^4$ and $\mathcal{C}(f)\le\mathcal{T}(f)$ for any 3–convex function $f:[-1,1]\to\mathbb{R}$. Then*

$$\big|\mathcal{T}(f)-\mathcal{I}(f)\big|\le\frac{M(f)}{360}$$

for any $f\in\mathcal{C}^4\big([-1,1]\big)$.

Proof. Let $f\in\mathcal{C}^4\big([-1,1]\big)$. It is well known (cf. [4, 7]) that $\mathcal{I}(f)=\mathcal{C}(f)+\frac{f^{(4)}(\xi)}{720}$ for some $\xi\in(-1,1)$. The rest of the proof is exactly the same as above. $\square$

Let

$$\mathcal{T} := a\mathcal{G}_2 + b\mathcal{C} + c\mathcal{S} + \lambda_1\mathcal{L}_4 + \lambda_2\mathcal{L}_5 + \lambda_3\mathcal{G}_3$$

be an arbitrary convex combination of the operators considered in this paper. Observe that it can be also written as

$$\mathcal{T} = a\mathcal{G}_2 + b\mathcal{C} + c\mathcal{S} + d\mathcal{U},$$

where $a, b, c, d \geq 0$, $a + b + c + d = 1$ and $\mathcal{U}$ is a convex combination of the operators $\mathcal{L}_4$, $\mathcal{L}_5$ and $\mathcal{G}_3$. For $g(x) = x^4$ we compute

$$\mathcal{G}_2(g) = \frac{1}{9}, \quad \mathcal{C}(g) = \frac{1}{6}, \quad \mathcal{S}(g) = \frac{1}{3} \quad \text{and}$$
$$\mathcal{L}_4(g) = \mathcal{L}_5(g) = \mathcal{G}_3(g) = \mathcal{I}(g) = \frac{1}{5}.$$

Then $\mathcal{T}(g) = \mathcal{I}(g)$ if and only if

$$\frac{a}{9} + \frac{b}{6} + \frac{c}{3} + \frac{d}{5} = \frac{1}{5}.$$

By $a, b, c, d \geq 0$, $a + b + c + d = 1$, the solution of this inequality is the following

$$\begin{cases} a = -\frac{3}{5} + 3c + \frac{3}{5}d, \\ b = \frac{8}{5} - 4c - \frac{8}{5}d, \\ 0 \leq c \leq \frac{2}{5}, \\ 0 \leq d \leq 1, \\ 1 - 5c \leq d \leq 1 - \frac{5}{2}c. \end{cases} \tag{3.2}$$

For $a = 0$ we get by Theorem 2.4 $\mathcal{C}(f) \leq \mathcal{T}(f)$ for any 3–convex function $f : [-1, 1] \to \mathbb{R}$ and by the above inequalities

$$b = \frac{4}{5}(1 - d), \qquad c = \frac{1}{5}(1 - d), \qquad 0 \leq d \leq 1.$$

Then by Lemma 3.2 we obtain:

Corollary 3.3. *Let* $0 \leq d \leq 1$ *and*

$$\mathcal{T}(f) = \frac{4}{5}(1 - d)\mathcal{C}(f) + \frac{1}{5}(1 - d)\mathcal{S}(f) + d\mathcal{U}(f),$$

where $\mathcal{U}$ *is an arbitrary convex combination of the operators* $\mathcal{L}_4$, $\mathcal{L}_5$ *and* $\mathcal{G}_3$. *If* $f \in C^4\big([-1, 1]\big)$ *then*

$$\big|\mathcal{T}(f) - \mathcal{I}(f)\big| \leq \frac{M(f)}{360}.$$

For $a > 0$ we get by Theorem 2.4 $\mathcal{G}_2(f) \leq \mathcal{T}(f)$ for any 3–convex function $f : [-1, 1] \to \mathbb{R}$ and the inequality $\mathcal{T}(f) < \mathcal{C}(f)$ is possible. Then by Lemma 3.1 we obtain

Corollary 3.4. *Let* $a > 0$, b, c, d *fulfil the inequalities* (3.2) *and*

$$\mathcal{T} = a\mathcal{G}_2 + b\mathcal{C} + c\mathcal{S} + d\mathcal{U},$$

where $\mathcal{U}$ *is an arbitrary convex combination of the operators* $\mathcal{L}_4$, $\mathcal{L}_5$ *and* $\mathcal{G}_3$. *If* $f \in C^4\big([-1, 1]\big)$ *then*

$$\big|\mathcal{T}(f) - \mathcal{I}(f)\big| \leq \frac{M(f)}{135}.$$

By Corollary 3.3 we obtain immediately (for $d = 1$):

Corollary 3.5. *If $\mathcal{T}$ is an arbitrary convex combination of the operators $\mathcal{L}_4$, $\mathcal{L}_5$ and $\mathcal{G}_3$ then*

$$\big|\mathcal{T}(f) - \mathcal{I}(f)\big| \le \frac{M(f)}{360}$$

for any $f \in \mathcal{C}^4\big([-1,1]\big)$.

This result improves the error bounds obtained in [6] for the quadratures $\mathcal{L}_4$ and $\mathcal{G}_3$, where the error bound was $\frac{M(f)}{90}$. Observe that the above corollary applies to the quadrature $\mathcal{L}_5$.

Corollary 3.6. *If $f \in \mathcal{C}^4\big([-1,1]\big)$ then $\big|\mathcal{L}_5(f) - \mathcal{I}(f)\big| \le \frac{M(f)}{360}$.*

This new result gives the error bound for the quadrature $\mathcal{L}_5$ for four times differentiable functions instead of eight times differentiable functions as in the classical result (see [4, 9]).

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