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POWER-MONOTONE SEQUENCES AND FOURIER SERIES WITH POSITIVE COEFFICIENTS

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ABSTRACT. M. and S. Izumi [2] and the present author [7] have extended certain theorems of R.P. Boas [1] concerning to the Fourier coefficients of functions belonging to the Lipschitz classes. Very recently L. Leindler [6] has given further generalization using the so called quasi power-monotone sequences. The goal of the present work is to prove further theorems similar to those of L. Leindler.

Key words and phrases: Fourier series, Fourier coefficients, modulus of continuity, quasi power-monotone sequences.

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1. INTRODUCTION

In 1967 R.P. Boas [1] proved a series of theorems on the connection between the magnitude of the Fourier-coefficients of a function f and its structural properties described by the modulus of continuity. Namely, he investigated the function classes $\mathrm{Lip}\alpha$ and the Zygmund class from this point of view. In 1969 M. and S. Izumi [2] generalized these results for the case $0 < \alpha < 1$ and for the Zygmund class. They used in the definition of these classes a function $j(t)$, which is more a general function than t^α. In 1990 Boas's results were also generalized by the present author [7] using the so called generalized Lipschitz and Zygmund classes replacing the function t^α $(0 < \alpha \leq 1)$ by the more general function of moduli of continuity: $\omega_\alpha(t)$ $(0 \leq \alpha \leq 1)$.

Very recently, L. Leindler [6] has given generalization of two of our theorems of the type mentioned above, using the so called quasi power-monotone sequences. His results contain our theorems for the case $0 < \alpha < 1$ and in the case $\alpha = 1$ for the sine series. It should be noted that it can easily be proved that Leindler's theorems contain the main results of M. and S. Izumi, too. In other words it turns out that the common root of the two directions of generalizations given by M. and S. Izumi and the present author is tightly connected with the main properties of the quasi power-monotone sequences.

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The object of this paper is to prove two further theorems using Leindler's method for the case $\omega_\alpha(t)$ if $\alpha = 0$ and for the generalized Zygmund class, showing again the utility of the concept of quasi power-monotone sequences in unifying the earlier completely different directions of generalization concerning Boas's results. These results are the generalizations of further theorems of M. and S. Izumi and ours.

The idea of writing this paper originated from L. Leindler's intention drawn up in his recent paper [6].

2. Notions and Notations

Before formulating the known and new results we recall some definitions and notations.

Let $\omega(\delta)$ be a modulus of continuity, i.e. a nondecreasing function on the interval $[0, 2\pi]$ having the properties: $\omega(0) = 0,\ \omega(\delta_1 + \delta_2) \le \omega(\delta_1) + \omega(\delta_2)$.

Denote $\omega(f;\delta)$ and $\omega^{(2)}(f;\delta)$ the modulus of continuity and the modulus of continuity of second order of a function f, respectively.

L. Leindler [3] introduced the following function classes. Let Ω_α $(0 \le \alpha \le 1)$ denote the set of the moduli of continuity $\omega(\delta) = \omega_\alpha(\delta)$ having the following properties:

(1) for any $\alpha' > \alpha$ there exists a natural number $\mu = \mu(\alpha')$ such that

$$2^{\mu\alpha'}\omega_\alpha(2^{-n-\mu}) > 2\omega_\alpha(2^{-n}) \text{ holds for all } n(\ge 1), \tag{2.1}$$

(2) for every natural number ν there exists a natural number $N := N(\nu)$ such that

$$2^{\nu\alpha}\omega_\alpha(2^{-n-\nu}) \le 2\omega_\alpha(2^{-n}),\ \text{if } n > N. \tag{2.2}$$

For any $\omega_\alpha \in \Omega_\alpha$ the classes H^{ω_α}, and $(H^{\omega_1})^*$, i.e.

$$H^{\omega_\alpha} := \{f : \omega(f;\delta) = O(\omega_\alpha(\delta))\}$$

and

$$(H^{\omega_1})^* := \{f : \omega^{(2)}(f;\delta) = O(\omega_1(\delta))\}$$

will be called generalized Lipschitz and Zygmund classes, respectively.

M. and S. Izumi [2], introduced the following function classes. Let $j(t)$ be a positive and nondecreasing function defined on the interval $(0,1)$. The $\mathrm{Lip}j(t)$ and $\Lambda(j(t))$ classes are defined as follows:

$$\mathrm{Lip}j(t) \ :\ = \left\{ f : \sup_{t,x} \left(\frac{|f(x+t) - f(x)|}{j(t)} \right) < \infty \right\};$$
$$\Lambda(j(t)) \ :\ = \left\{ f : \sup_{t,x} \left(\frac{|f(x+t) - 2f(x) + f(x-t)|}{j(t)} \right) < \infty \right\}.$$

(Further conditions required on $j(t)$ will be detailed later in the next paragraph.)

We shall say that a sequence $\gamma := \{\gamma_n\}$ of positive terms is quasi β-power-monotone increasing (decreasing) if there exists a natural number $N := N(\beta,\gamma)$ and constant $K := K(\beta,\gamma) \ge 1$ such that

$$Kn^\beta\gamma_n \ge m^\beta\gamma_m, \quad (n^\beta\gamma_n \le Km^\beta\gamma_m) \tag{2.3}$$

holds for any $n \ge m \ge N$.

Here and in what follows K and K_i denote positive constants that are not necessarily the same at each occurrence.

If (2.3) holds with $\beta = 0$ then we omit the attribute "β-power" in the inequality.

Furthermore, we shall say that a sequence $\gamma := \{\gamma_n\}$ of positive terms is quasi geometrically increasing (decreasing) if there exist natural numbers $\mu := \mu(\gamma),\ N := N(\gamma)$ and a constant $K := K(\gamma) \geq 1$ such that

$$\gamma_{n+\mu} \geq 2\gamma_n \text{ and } \gamma_n \leq K\gamma_{n+1}, \ \left(\gamma_{n+\mu} \leq \frac{1}{2}\gamma_n \text{ and } \gamma_{n+1} \leq K\gamma_n\right) \tag{2.4}$$

hold for all $n \geq N$.

Finally a sequence $\{\gamma_n\}$ will be called bounded by blocks if the inequalities

$$\alpha_1 \Gamma_m^{(k)} \leq \gamma_n \leq \alpha_2 \Gamma_M^{(k)}, \quad 0 < \alpha_1 \leq \alpha_2 < \infty$$

hold for any $2^k \leq n \leq 2^{k+1},\ k = 1, 2, \ldots,$ where

$$\Gamma_m^{(k)} := \min(\gamma_{2^k}, \gamma_{2^{k+1}}) \text{ and } \Gamma_M^{(k)} := \max(\gamma_{2^k}, \gamma_{2^{k+1}}).$$

3. THEOREMS

To begin with, we recall one theorem of M. and S. Izumi [2], two of ours [7] and finally one of Leindler's theorems [6].

Throughout the rest of this paper $g(x), f(x), \varphi(x)$ will denote continuous 2π periodic functions; furthermore $g(x)$ and $f(x)$ always denote odd and even functions, respectively. $\varphi(x)$ will denote either an odd or an even function while λ_n will denote the Fourier coefficients of $g(x), f(x)$ or $\varphi(x)$.

Theorem 3.1. *([2]). Let $\lambda_n \geq 0$ and let $j(t)$ be a positive and nondecreasing function in the interval $(0, 1)$, satisfying the conditions*

$$\int_0^t j(u)u^{-1}du \leq Kj(t) \quad \textit{as} \quad t \to 0, \tag{3.1}$$

and

$$\int_t^1 j(u)u^{-3}du \leq Kj(t)t^{-2} \quad \textit{as} \quad t \to 0. \tag{3.2}$$

Then $\varphi \in \Lambda(j(t))$ if and only if

$$\sum_{k=n/2}^{n} \lambda_k \leq Kj\left(\frac{1}{n}\right) \quad \textit{as} \quad n \to \infty. \tag{3.3}$$

It should be noted that by (3.1) the condition (3.3) is equivalent to

$$\sum_{k=n}^{\infty} \lambda_k \leq Kj\left(\frac{1}{n}\right) \quad \text{as} \quad n \to \infty. \tag{3.4}$$

Furthermore, in its original form this theorem seems to be slightly more general, since the continuity of φ is not mentioned, although in the definition of $\Lambda(t)$ given by Zygmund [9] this additional condition is assumed.

Theorem 3.2. *([7]). Let $\lambda_n \geq 0$. Then*

$$\varphi \in (H^{\omega_1})^* \tag{3.5}$$

if and only if

$$\sum_{k=n}^{\infty} \lambda_k = O\left(\omega_1\left(\frac{1}{n}\right)\right). \tag{3.6}$$

Theorem 3.3. *([7]). Let $\lambda_n \geq 0$. Then*

$$f \in H^{\omega_0} \tag{3.7}$$

if and only if

$$\sum_{k=n}^{\infty} \lambda_n = O\left(\omega_0\left(\frac{1}{n}\right)\right). \tag{3.8}$$

Furthermore,

$$g \in H^{\omega_0} \tag{3.9}$$

implies

$$\sum_{k=n}^{n} k\lambda_k = O\left(n\omega_0\left(\frac{1}{n}\right)\right), \tag{3.10}$$

and from

$$\sum_{k=n}^{\infty} \lambda_k = O\left(\omega_0\left(\frac{1}{n}\right)\right), \tag{3.11}$$

$$g \in H^{\omega_0} \tag{3.12}$$

follows.

Theorem 3.4. *([6]). Assume that a given positive sequence $\{\gamma_n\}$ has the following properties. There exists a positive ε such that:*

(P_+) *the sequence $\{n^{\varepsilon}\gamma_n\}$ is quasi monotone decreasing and*
(P_-) *the sequence $\{n^{1-\varepsilon}\gamma_n\}$ is quasi monotone increasing.*

If $\lambda_n \geq 0$, then

$$\omega\left(\varphi, \frac{1}{n}\right) = O(\gamma_n) \tag{3.13}$$

if and only if

$$\sum_{k=n}^{\infty} \lambda_k = O(\gamma_n) \tag{3.14}$$

or, equivalently,

$$\sum_{k=1}^{n} k\lambda_k = O(n\gamma_n). \tag{3.15}$$

As we mentioned earlier, Theorem 3.4 contains one of theorems of M. and S. Izumi (see [2, Theorem 1]) and two of ours (see [7, Theorems 1 and 2]). We now proceed to formulate our new theorems.

Theorem 3.5. *Let $\lambda_n \geq 0$ and let γ_n have the properties:*

(P_+) *the sequence $\{n^{\varepsilon}\gamma_n\}$ is quasi monotone decreasing and*
$(\hat{P})$ *the sequence $\{n^{2-\varepsilon}\gamma_n\}$ is quasi monotone increasing for some positive ε. Then*

$$\omega^{(2)}\left(\varphi; \frac{1}{n}\right) = O(\gamma_n) \tag{3.16}$$

if and only if

$$\sum_{k=n}^{\infty} \lambda_k = O(\gamma_n). \tag{3.17}$$

Theorem 3.6. *Assume that γ_n has the following property:*

(P_-) *the sequence $\{n^{1-\varepsilon}\gamma_n\}$ is quasi monotone increasing for some positive ε. If $\lambda_n \geq 0$, then*

$$\omega\left(f; \frac{1}{n}\right) = O(\gamma_n) \tag{3.18}$$

if and only if

$$\sum_{k=n}^{\infty} \lambda_k = O(\gamma_n). \tag{3.19}$$

Furthermore,

$$\omega\left(g; \frac{1}{n}\right) = O(\gamma_n) \tag{3.20}$$

implies

$$\sum_{k=1}^{n} k\lambda_k = O(n\gamma_n), \tag{3.21}$$

and from

$$\sum_{k=n}^{\infty} \lambda_k = O(\gamma_n), \tag{3.22}$$

$$\omega\left(g; \frac{1}{n}\right) = O(\gamma_n) \tag{3.23}$$

follows.

Remark 3.7. We shall prove that Theorem 3.5 includes Theorems 3.1 and 3.2. Additionally, Theorem 3.6 implies Theorem 3.3.

4. LEMMAS

To prove our theorems we require the following lemmas.

Lemma 4.1. *(*[5]*). A positive sequence $\{\delta_n\}$ bounded by blocks is quasi ε-power monotone increasing (decreasing) with a certain negative (positive) exponent ε if and only if the sequence $\{\delta_{2^n}\}$ is quasi geometrically increasing (decreasing).*

Lemma 4.2. *(*[4]*). For any positive sequence $\gamma := \{\gamma_n\}$ the inequalities*

$$\sum_{n=m}^{\infty} \gamma_n \leq K\gamma_m \quad (m = 1, 2, \ldots; K \geq 1),$$

or

$$\sum_{n=1}^{m} \gamma_n \leq K\gamma_m \quad (m = 1, 2, \ldots; K \geq 1),$$

hold if and only if the sequence γ is quasi geometrically decreasing or increasing, respectively.

Lemma 4.3. *([6]). Let $\mu_n \geq 0,\ \beta_n > 0$ and $\delta > 0$. Assume that there exists a positive ε such that the sequence*

$$(i) \qquad \{n^{-\varepsilon}\beta_n\} \text{ is quasi monotone increasing},$$

and the sequence

$$(ii) \qquad \{n^{\varepsilon-\delta}\beta_n\} \text{ is quasi monotone decreasing}.$$

Then

$$\sum_{k=1}^{n} k^{\delta}\mu_k = O(\beta_n) \tag{4.1}$$

is equivalent to

$$\sum_{k=n}^{\infty} \mu_k = O(\beta_n n^{-\delta}). \tag{4.2}$$

Lemma 4.4. *([6]). Let $\mu_k \geq 0,\ \sum_{k=1}^{\infty}\mu_k$ be convergent and $0 \leq \alpha \leq 1$. Moreover, assume that a given positive sequence $\{\delta_n\}$ has the following properties. There exists a positive ε such that:*

$$(iii) \qquad \text{the sequence } \{n^{\varepsilon-\alpha}\delta_n\} \text{ is quasi monotone decreasing, and}$$

$$(iv) \qquad \text{the sequence } \{n^{2-\alpha-\varepsilon}\delta_n\} \text{ is quasi monotone increasing.}$$

Finally let

$$\delta(x) := \begin{cases} \delta_n \ \text{ if } \ x = \frac{1}{n}, \quad n \geq 1; \\ \text{linear on the interval } \ \left[\frac{1}{n+1}, \frac{1}{n}\right]. \end{cases}$$

Then

$$\sum_{k=1}^{\infty} \mu_k(1-\cos kx) = O(x^{\alpha}\delta(x)) \quad (x \to 0) \tag{4.3}$$

if and only if

$$\sum_{k=n}^{\infty} \mu_k = O(n^{-\alpha}\delta_n). \tag{4.4}$$

5. PROOF OF THE THEOREMS

Proof of Theorem 3.5. Firstly we prove the theorem for the cosine series. Suppose that (3.16) holds. This implies that

$$|f(x+h) + f(x-h) - 2f(x)| \leq K\gamma(h), \tag{5.1}$$

where

$$\gamma(x) := \begin{cases} \gamma_n \ \text{ if } \ x = \frac{1}{n}, \quad n \geq 1; \\ \text{linear on the interval } \ \left[\frac{1}{n+1}, \frac{1}{n}\right]. \end{cases} \tag{5.2}$$

From (5.1) it follows that

$$|f(h) - f(0)| \leq K\gamma(h). \tag{5.3}$$

Since f is continuous and $\lambda_n \geq 0$, from a theorem of Paley (see [8]) it follows that

$$\sum_{k=1}^{\infty} \lambda_k < \infty,$$

whence

$$\sum_{k=1}^{\infty} \lambda_k (1 - \cos kh) = O(\gamma(h)) \tag{5.4}$$

follows.

Using Lemma 4.4 for $\alpha = 0,\ \mu_k = \lambda_k,\ \delta_n = \gamma_n$ we have (3.17), that was to be proved. Now we assume (3.17) and estimate the following difference by using again Lemma 4.4 in the last step (for $\alpha = 0,\ \mu_k = \lambda_k$ and $\delta_n = \gamma_n$)

$$\begin{aligned} |f(x+2h) + f(x-2h) - 2f(x)| &= 4 \left| \sum_{k=1}^{\infty} \lambda_k \sin^2 kh \cos kx \right| \\ &\leq 4 \sum_{k=1}^{\infty} \lambda_k \sin^2 kh = 2 \sum_{k=1}^{\infty} \lambda_k (1 - \cos 2kh) \\ &= O(\gamma(h)). \end{aligned}$$

Thus the proof of Theorem 3.5 is completed for the cosine series.

The proof for the sine series in the direction from (3.17) to (3.16) can be done in the same way as for the cosine series, since

$$|g(x+2h) + g(x-2h) - 2g(x)| = 4 \left| \sum_{k=1}^{\infty} \lambda_k \sin kx \sin^2 kh \right|. \tag{5.5}$$

So we detail only the other direction. Suppose (3.16), that is

$$|g(x+h) + g(x-h) - 2g(x)| = O(\gamma(h)). \tag{5.6}$$

Writing (5.6) in the following form (using again Paley's theorem cited before):

$$2 \left| \sum_{k=1}^{\infty} \lambda_k \sin kx (1 - \cos kh) \right| = O(\gamma(h)). \tag{5.7}$$

By integrating term by term on $(0, x)$ in (5.7) we get

$$\sum_{k=1}^{\infty} \lambda_k \frac{1 - \cos kx}{k} (1 - \cos kh) = O(x\gamma(h)). \tag{5.8}$$

From (5.8) we have

$$\sum_{k=1}^{\infty} x^2 k \lambda_k \frac{1 - \cos kx}{k^2 x^2} (1 - \cos kh) = O(x\gamma(h)). \tag{5.9}$$

Since $K \geq t^{-2}(1 - \cos t) \downarrow$ on $(0, 1)$, from (5.9) it follows that

$$\sum_{k=1}^{[1/x]} x k \lambda_k (1 - \cos kh) = O(\gamma(h)). \tag{5.10}$$

Putting $h = x$ in (5.10)

$$\sum_{k=1}^{[1/h]} hk\lambda_k(1-\cos kh) = O(\gamma(h)) \tag{5.11}$$

can be obtained which gives

$$\sum_{k=1}^{[1/h]} h^3k^3\lambda_k \frac{1-\cos kh}{k^2h^2} = O(\gamma(h)). \tag{5.12}$$

From (5.12) taking $h = \frac{1}{n}$

$$\sum_{k=1}^{n} k^3\lambda_k = O(n^3\gamma_n) \tag{5.13}$$

follows.

By using Lemma 4.3 for $\beta_n = n^3\gamma_n,\ \delta = 3$ (5.13) implies (3.17) which was to be proved. It can easily be verified that the conditions (i) and (ii) of Lemma 4.3 follow from properties $\hat{P}$ and P^+ of γ_n, respectively.

Thus Theorem 3.5 is completely proved. □

Proof of Theorem 3.6. Let $f(x) = \sum_{k=1}^{\infty} \lambda_k \cos kx$ and suppose that (3.18) is valid. Then we have $|f(h) - f(0)| \leq K\gamma(h)$ (for the definition $\gamma(x)$ see (5.2)).

That is,

$$\sum_{k=1}^{\infty} \lambda_k(1-\cos kh) \leq K\gamma(h).$$

Integrating both sides on $(0, x)$ we have

$$\sum_{k=1}^{\infty} \frac{\lambda_k}{k}(kx - \sin kx) \leq Kx\gamma(x). \tag{5.14}$$

Since $kx - \sin kx \geq 0$, we have from (5.14)

$$\sum_{k=2n}^{\infty} \frac{\lambda_k}{k}(kx - \sin kx) \leq Kx\gamma(x). \tag{5.15}$$

Putting $1/n$ for x and taking into account that

$$\frac{k}{n} - \sin\left(\frac{k}{n}\right) \geq \frac{1}{2}\frac{k}{n} \quad \text{for} \quad k \geq 2n$$

we get

$$\sum_{k=2n}^{\infty} \lambda_k \leq K\gamma_n, \tag{5.16}$$

which gives (3.19).

Now we suppose that (3.19) holds and we prove (3.18).

Let us consider the following difference:

$$|f(x+2h)-f(x)| = \left|\sum_{k=1}^{\infty}\lambda_k[\cos k(x+2h)-\cos kx]\right| = 2\left|\sum_{k=1}^{\infty}\lambda_k \sin k(x+h)\sin kh\right| \leq 2\sum_{k=1}^{[1/h]}\lambda_k \sin kh + \sum_{k=[1/h]}^{\infty}\lambda_k = I + II. \tag{5.17}$$

Using (3.19) we have that $II = O(\gamma(h))$. Now we estimate I.

$$I = 2\cdot\sum_{k=1}^{[1/h]}\lambda_k \sin kh = 2h\sum_{k=1}^{[1/h]}k\lambda_k\frac{\sin kh}{kh} \leq K\cdot h\sum_{k=1}^{[1/h]}k\lambda_k = I'. \tag{5.18}$$

But by using the property P_- and Lemma 1, Lemma 2 we show that (3.19) implies

$$\sum_{k=1}^{n}k\lambda_k = O(n\gamma_n). \tag{5.19}$$

Indeed, let $2^\nu < n \leq 2^{\nu+1}$ then we have

$$\sum_{k=2}^{n}k\lambda_k \leq \sum_{m=0}^{\nu}\sum_{k=2^m+1}^{2^{m+1}}k\lambda_k \leq K\sum_{m=0}^{\nu}2^m\sum_{k=2^m+1}^{2^{m+1}}\lambda_k \leq K\sum_{m=0}^{\nu}2^m\gamma_{2^m} \leq Kn\gamma_n,$$

which gives (5.19). Finally (5.17), (5.18) and (5.19) give (3.18), which was to be proved.

Now we prove (3.21) from (3.20). Using the estimation

$$|g(x)| \leq K\gamma(x), \tag{5.20}$$

term by term integration on $(0,x)$ gives from (5.20) that

$$\sum_{k=1}^{\infty}\frac{\lambda_k}{k}(1-\cos kx) \leq Kx\gamma(x), \tag{5.21}$$

that is

$$\sum_{k=1}^{[1/x]}k\lambda_k\frac{1-\cos kx}{k^2x^2} \leq K\frac{\gamma(x)}{x} \tag{5.22}$$

holds for any positive x. As before from (5.22) it follows that

$$\sum_{k=1}^{[1/x]}k\lambda_k \leq K\frac{\gamma(x)}{x}, \tag{5.23}$$

which taking $x = \frac{1}{n}$ gives (3.21).

The proof of (3.23) from (3.22) can be done in the very same way as (3.18) from (3.19), so we omit it. Theorem 3.6 is completed. □

Proof of Remark 3.7. For the implication Theorem 3.6 ⇒Theorem 3.1 let $\gamma_n := j\left(1/n\right)$. Then using Lemma 1 and Lemma 2 from (3.1), property P_+ follows, while (3.2) implies property $\hat{P}$ of γ_n.

To show that Theorem 3.5 includes Theorem 3.2 it is enough to take $\gamma_n := \omega_1 (1/n)$ and to take into account that using Lemma 1 property (2.1) of $\omega_1(\delta)$ implies property $\hat{P}$ while from condition (2.2) the property P_+ of γ_n follows.

Similarly, to prove the conclusion Theorem 3.6 ⇒Theorem 3.3 it is enough to use Lemma 1 to show that the condition (2.1) of $\omega_0(\delta)$ implies that $\omega_0 (1/n)$ satisfies the property P_-, so choosing $\gamma_n := \omega_0 (1/n)$ the proof is completed. □

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