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SUBHARMONIC FUNCTIONS AND THEIR RIESZ MEASURE

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ABSTRACT. For subharmonic functions u in $\mathbb{R}^N$, of Riesz measure μ, the growth of the function $s \mapsto \mu(s) = \int_{|\zeta|\leq s} d\mu(\zeta)$ $(s \geq 0)$ is described and compared with the growth of u. It is also shown that, if $\int_{\mathbb{R}^N} u^+(x)\,[-\varphi'(|x|^2)]\,dx < +\infty$ for some decreasing C^1 function $\varphi \geq 0$, then $\int_{\mathbb{R}^N} \frac{1}{|\zeta|^2}\,\varphi(|\zeta|^2+1)\,d\mu(\zeta) < +\infty$. Given two subharmonic functions u_1 and u_2, of Riesz measures μ_1 and μ_2, with a growth like $u_i(x) \leq A + B|x|^\gamma$ $\forall x \in \mathbb{R}^N$ $(i = 1, 2)$, it is proved that $\mu_1 + \mu_2$ is not necessarily the Riesz measure of a subharmonic function u with such a growth as $u(x) \leq A' + B'|x|^\gamma$ $\forall x \in \mathbb{R}^N$ (here $A > 0$, $A' > 0$ and $0 < B' < 2B$).

Key words and phrases: subharmonic functions, order of growth, Riesz measure.

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1. INTRODUCTION

Let μ be the Riesz measure of some subharmonic function u in $\mathbb{R}^N$ ($N \in \mathbb{N}$, $N \geq 2$ and u non identically $-\infty$, see [1, p. 104]) and $\mu(s) = \int_{|\zeta|\leq s} d\mu(\zeta)$ for any $s \geq 0$ (where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$). The function $s \mapsto \mu(s)$ is non–decreasing since μ is a positive measure. The order of the function $s \mapsto s^{2-N}\mu(s)$ is known to coincide with the convergence exponent of μ:

$$\inf\left\{c : \int_1^{+\infty} s^{2-N-c}\,d\mu(s)\right\} = \inf\left\{c : \int_1^{+\infty} s^{1-N-c}\,\mu(s)\,ds\right\}$$

(see [2, p. 66]) and does not exceed γ if u has a growth of the kind:

$$u(x) \leq A + B|x|^\gamma \qquad \forall x \in \mathbb{R}^N \tag{1.1}$$

(with constants $A \in \mathbb{R}$, $B > 0$ and $\gamma > 0$). This estimation of the growth of $\mu(s)$ will be examined below, in Sections 3 and 4.

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Definition 1.1. Given $\gamma > 0$ and $B > 0$, let $SH(\gamma, B)$ stand for the set of all subharmonic functions u in $\mathbb{R}^N$ which are harmonic in some neighbourhood of the origin with $u(0) = 0$ and which satisfy estimate (1.1) for some constant $A \in \mathbb{R}$.

In Proposition 5.2 (see Section 5), a counterexample is produced to show that, given u_1 and u_2 two functions in this set $SH(\gamma, B)$ and $B' \in \,]0, 2B[$, the sum of their respective Riesz measures μ_1 and μ_2 is not necessarily the Riesz measure of a function of $SH(\gamma, B')$.

Of course $\mu_1 + \mu_2$ is the Riesz measure associated with $u_1 + u_2 \in SH(\gamma, 2B)$, but $\mu_1 + \mu_2$ is also the Riesz measure of $u_1 + u_2 - h$ for any harmonic function h in $\mathbb{R}^N$. This proposition means that there does not necessarily exist a harmonic function h such that $u_1 + u_2 - h \in SH(\gamma, B')$.

Let μ denote the Riesz measure of some function of $SH(\gamma, B)$ with growth (1.1). Sections 3 and 4 are devoted to the growth of the repartition function $s \mapsto \mu(s)$. For instance, when $N = 2$, we obtain the inequality: $\mu(s) \le Be\gamma\, s^\gamma\, e^{\frac{A\gamma}{\mu(s)}}$ (see Theorem 3.1 and Corollary 3.2).

Notation 1.1. When $N \ge 3$, throughout the paper we set $C(\gamma, N) = \left(\frac{\gamma+N-2}{\gamma}\right)^{\frac{\gamma+N-2}{N-2}}$ and $D(B, \gamma, N) = \frac{\gamma+N-2}{\gamma}\left(\frac{B\gamma}{N-2}\right)^{\frac{N-2}{\gamma+N-2}}$, sometimes written merely D for brevity.

Note that

$$\frac{\gamma}{N-2}\, C(\gamma, N) = \frac{\gamma}{N-2}\left(\frac{\gamma+N-2}{\gamma}\right)^{\frac{\gamma+N-2}{N-2}} = \frac{\gamma+N-2}{N-2}\left(1 + \frac{N-2}{\gamma}\right)^{\frac{\gamma}{N-2}} \le e\,\frac{\gamma+N-2}{N-2}.$$

For $N \ge 3$, we also obtain inequalities describing the growth of $s \mapsto \mu(s)$ and the constants involved in these estimations are given explicitely in terms of A, B and γ. For example:

$$\mu(s) \le \frac{B\gamma}{N-2}\, C(\gamma, N)\, s^{\gamma+N-2}\left(1 + \frac{A}{D \cdot [\mu(s)]^{\frac{\gamma}{\gamma+N-2}}}\right)^{\frac{\gamma+N-2}{N-2}}$$

(see Theorem 3.4 and Corollary 3.5).

It points out that $\limsup_{s\to+\infty} \frac{\mu(s)}{s^{\gamma+N-2}}$ is not greater than $Be\gamma$ (when $N = 2$) or $\frac{B\gamma}{N-2}\, C(\gamma, N)$ (when $N \ge 3$). Moreover, $\liminf_{s\to+\infty} \frac{\mu(s)}{s^{\gamma+N-2}}$ does not exceed $B\gamma$ (if $N = 2$) or $\frac{B\gamma}{N-2}$ (if $N \ge 3$). This will follow from Theorems 4.2 and 4.5 which assert that the sets:

$$\left\{ s : \mu(s) < B\gamma\, s^\gamma\, e^{\frac{A\gamma}{\mu(s)}} \right\}$$

and

$$\left\{ s \;:\; \mu(s) < \frac{B\gamma}{N-2}\, s^{\gamma+N-2}\left(1 + \frac{A}{D \cdot [\mu(s)]^{\frac{\gamma}{\gamma+N-2}}}\right)^{\frac{\gamma+N-2}{N-2}} \right\}$$

are unbounded in the cases when $N = 2$ and $N \ge 3$ respectively.

The last section studies subharmonic functions u in $\mathbb{R}^N$ (harmonic in some neighbourhood of the origin with $u(0) = 0$) such that the subharmonic function u^+ (defined by $u^+(x) = \max(u(x), 0)\ \forall x \in \mathbb{R}^N$) satisfies a L^1 condition, for example in Theorem 6.1:
$\int_{\mathbb{R}^N} u^+(x)\,[-\varphi'(|x|^2)]\, dx < +\infty$ (see Section 6.1 for more details on the decreasing function φ). The Riesz measure μ of u is then proved to verify: $\int_{\mathbb{R}^N} \frac{\varphi(|\zeta|^2+1)}{|\zeta|^2}\, d\mu(\zeta) < +\infty$. Propositions 6.2 and 6.3 provide similar results under different L^1 conditions.

2. SOME PRELIMINARIES

Lemma 2.1. *If* $N = 2$, *then*

$$\int_{|\zeta|\le s} \log \frac{r}{|\zeta|}\, d\mu(\zeta) \le \int_{|\zeta|\le r} \log \frac{r}{|\zeta|}\, d\mu(\zeta)$$

for each $r > 0$ *and each* $s > 0$.

Proof. If $r \le s$, then $h_r(\zeta) := \log \frac{r}{|\zeta|} \le 0$ for $r < |\zeta| \le s$, so that

$$\int_{|\zeta|\le s} h_r(\zeta)\, d\mu(\zeta) = \int_{|\zeta|\le r} h_r(\zeta)\, d\mu(\zeta) + \underbrace{\int_{r<|\zeta|\le s} h_r(\zeta)\, d\mu(\zeta)}_{\le 0} \le \int_{|\zeta|\le r} h_r(\zeta)\, d\mu(\zeta).$$

If $s < r$, then $h_r(\zeta) \ge 0$ for $|\zeta| \le r$, hence

$$\int_{|\zeta|\le r} h_r(\zeta)\, d\mu(\zeta) = \int_{|\zeta|\le s} h_r(\zeta)\, d\mu(\zeta) + \underbrace{\int_{s<|\zeta|\le r} h_r(\zeta)\, d\mu(\zeta)}_{\ge 0} \ge \int_{|\zeta|\le s} h_r(\zeta)\, d\mu(\zeta).$$

□

Lemma 2.2. *When* $N \ge 3$, *the following majoration is valid for all* $r > 0$ *and* $s > 0$*:*

$$\int_{|\zeta|\le s} \left(\frac{1}{|\zeta|^{N-2}} - \frac{1}{r^{N-2}} \right) d\mu(\zeta) \le \int_{|\zeta|\le r} \left(\frac{1}{|\zeta|^{N-2}} - \frac{1}{r^{N-2}} \right) d\mu(\zeta).$$

Proof. As in the previous proof, with $h_r(\zeta) = \frac{1}{|\zeta|^{N-2}} - \frac{1}{r^{N-2}}$ instead of $\log \frac{r}{|\zeta|}$. □

Lemma 2.3. *If* $N = 2$, *then:*

$$\int_0^r \frac{\mu(t)}{t}\, dt = \int_{|\zeta|\le r} \log \frac{r}{|\zeta|}\, d\mu(\zeta),$$

for any $r > 0$.

Proof. It follows from Fubini's theorem that:

$$\int_0^r \frac{\mu(t)}{t}\, dt = \int_0^r \frac{1}{t} \left(\int_{|\zeta|\le t} d\mu(\zeta) \right) dt = \int_{|\zeta|\le r} \left(\int_{|\zeta|}^r \frac{dt}{t} \right) d\mu(\zeta) = \int_{|\zeta|\le r} \log \frac{r}{|\zeta|}\, d\mu(\zeta).$$

□

Lemma 2.4. *When* $N \ge 3$, *then*

$$(N-2) \int_0^r \frac{\mu(t)}{t^{N-1}}\, dt = \int_{|\zeta|\le r} \left(\frac{1}{|\zeta|^{N-2}} - \frac{1}{r^{N-2}} \right) d\mu(\zeta),$$

for any $r > 0$

Proof. As in the previous proof:

$$\begin{aligned}(N-2)\int_0^r \frac{\mu(t)}{t^{N-1}}dt &= \int_0^r \frac{N-2}{t^{N-1}}\left(\int_{|\zeta|\le t} d\mu(\zeta)\right)dt\\ &= \int_{|\zeta|\le r}\left(\int_{|\zeta|}^r \frac{N-2}{t^{N-1}}\,dt\right)d\mu(\zeta)\\ &= \int_{|\zeta|\le r}\left[\frac{-1}{t^{N-2}}\right]_{|\zeta|}^r d\mu(\zeta)\\ &= \int_{|\zeta|\le r}\left(\frac{1}{|\zeta|^{N-2}}-\frac{1}{r^{N-2}}\right)d\mu(\zeta).\end{aligned}$$

□

3. ESTIMATIONS OF THE RIESZ MEASURE

3.1. **Jensen–Privalov formula.** For any function u, subharmonic in $\mathbb{R}^N$, harmonic in some neighbourhood of the origin, the Jensen–Privalov formula (see [2, p. 44]) holds for every $r>0$:

$$\frac{1}{2\pi}\int_0^{2\pi} u(r\,e^{i\theta})\,d\theta = \int_0^r \frac{\mu(t)}{t}\,dt + u(0) \qquad \text{if } N=2$$

$$\frac{1}{\sigma_N}\int_{S_N} u(rx)\,d\sigma_x = (N-2)\int_0^r \frac{\mu(t)}{t^{N-1}}\,dt + u(0) \qquad \text{if } N\ge 3$$

with S_N the unit sphere in $\mathbb{R}^N$, $d\sigma$ the area element on S_N and $\sigma_N=\int_{S_N} d\sigma = \frac{2\,\pi^{N/2}}{\Gamma(N/2)}$ (see [1, p. 29]). In all statements of both Sections 3 and 4, it will be assumed that $u\in SH(\gamma,B)$ and that its growth is indicated by (1.1).

3.2. **The case $N=2$.**

Theorem 3.1. *When $N=2$, the following inequality holds for each $s>0$:*

$$\frac{\mu(s)}{\gamma}\,\log\left(\frac{\mu(s)}{Be\gamma}\right) \le A + \int_{|\zeta|\le s} \log|\zeta|\,d\mu(\zeta).$$

Proof. For each $r>0$ and each $s>0$, it follows from Lemmas 2.1 and 2.3 that

$$\int_{|\zeta|\le s} \log\frac{r}{|\zeta|}\,d\mu(\zeta) \le \frac{1}{2\pi}\int_0^{2\pi} u(r\,e^{i\theta})\,d\theta \le A + B\,r^\gamma,$$

so that

$$\int_{|\zeta|\le s} \log\frac{1}{|\zeta|}\,d\mu(\zeta) \le A + B\,r^\gamma - \mu(s)\log r = A + \frac{\mu(s)}{\gamma}\left(\frac{B\gamma}{\mu(s)}\,r^\gamma - \log r^\gamma\right) := \varphi(r).$$

Consider s constant, the minimum of φ is attained when $B\gamma\,r^\gamma=\mu(s)$, since $\varphi'(r)=\frac{1}{r}(B\gamma\,r^\gamma - \mu(s))$. Finally, for each $s>0$:

$$\int_{|\zeta|\le s} \log\frac{1}{|\zeta|}\,d\mu(\zeta) \le A + \frac{\mu(s)}{\gamma}\left[1-\log\left(\frac{\mu(s)}{B\gamma}\right)\right] = A - \frac{\mu(s)}{\gamma}\log\left(\frac{\mu(s)}{Be\gamma}\right)$$

□

In Corollaries 3.2, 3.3 and 3.5, we set $\varepsilon>0$ such that $\mu(s)>0\ \forall s>\varepsilon$.

Corollary 3.2. *If $N=2$, then $\mu(s)\le Be\gamma\,s^\gamma\,e^{\frac{A\gamma}{\mu(s)}}$ for any $s>\varepsilon$.*

Proof. Theorem 3.1 may be rewritten as:

$$\log\left(\frac{\mu(s)}{Be\gamma}\right) \le \frac{A\gamma}{\mu(s)} + \int_{|\zeta|\le s} \log(|\zeta|^\gamma)\,\frac{d\mu(\zeta)}{\mu(s)}. \tag{3.1}$$

The previous integral being $\le \log s^\gamma$, Corollary 3.2 results. □

Corollary 3.3. *When $N = 2$, we have for every $s > \varepsilon$:*

$$[\mu(s)]^2 \le Be\gamma\,\exp\left(\frac{A\gamma}{\mu(s)}\right)\int_{|\zeta|\le s} |\zeta|^\gamma\,d\mu(\zeta).$$

Proof. Jensen's inequality applies to (3.1) since $\int_{|\zeta|\le s}\frac{d\mu(\zeta)}{\mu(s)} = 1$, hence:

$$\begin{aligned}\frac{\mu(s)}{Be\gamma} &\le \exp\left(\frac{A\gamma}{\mu(s)}\right)\cdot\exp\left(\int_{|\zeta|\le s}\log(|\zeta|^\gamma)\,\frac{d\mu(\zeta)}{\mu(s)}\right)\\ &\le \exp\left(\frac{A\gamma}{\mu(s)}\right)\int_{|\zeta|\le s}|\zeta|^\gamma\,\frac{d\mu(\zeta)}{\mu(s)}.\end{aligned}$$

□

3.3. The case $N \ge 3$.

Theorem 3.4. *When $N \ge 3$, the following estimation is valid for each $s > 0$:*

$$\int_{|\zeta|\le s}\frac{1}{|\zeta|^{N-2}}d\mu(\zeta) \le A + D\,[\mu(s)]^{\frac{\gamma}{\gamma+N-2}}.$$

Proof. For all $r > 0$ and $s > 0$, Lemmas 2.2 and 2.4 lead to:

$$\int_{|\zeta|\le s}\left(\frac{1}{|\zeta|^{N-2}} - \frac{1}{r^{N-2}}\right)d\mu(\zeta) \le \frac{1}{\sigma_N}\int_{S_N} u(rx)\,d\sigma_x \le A + B\,r^\gamma,$$

that is

$$\int_{|\zeta|\le s}\frac{1}{|\zeta|^{N-2}}d\mu(\zeta) \le A + B\,r^\gamma + \frac{\mu(s)}{r^{N-2}}$$

whose minimum (with s constant) is attained when $B\gamma\,r^\gamma = (N-2)\,\frac{\mu(s)}{r^{N-2}}$. In other words, this minimum is $A + \left(\frac{N-2}{\gamma}+1\right)\frac{\mu(s)}{r^{N-2}}$ with $\frac{1}{r^{N-2}} = \left(\frac{B\gamma}{N-2}\frac{1}{\mu(s)}\right)^{\frac{N-2}{\gamma+N-2}}$. Finally:

$$\int_{|\zeta|\le s}\frac{1}{|\zeta|^{N-2}}\,d\mu(\zeta) \le A + \left(\frac{N-2}{\gamma}+1\right)\left(\frac{B\gamma}{N-2}\right)^{\frac{N-2}{\gamma+N-2}}(\mu(s))^{\frac{\gamma}{\gamma+N-2}}.$$

□

Corollary 3.5. *When $N \ge 3$, the following estimation holds for every $s > \varepsilon$:*

$$\mu(s) \le \frac{B\gamma}{N-2}\,C(\gamma,N)\,s^{\gamma+N-2}\left(1 + \frac{A}{D\cdot[\mu(s)]^{\frac{\gamma}{\gamma+N-2}}}\right)^{\frac{\gamma+N-2}{N-2}}.$$

Proof. Let $\alpha = \frac{N-2}{\gamma+N-2}$. According to Theorem 3.4, for any $s > \varepsilon$ we have

$$\frac{1}{s^{N-2}} \le \frac{1}{\mu(s)}\int_{|\zeta|\le s}\frac{1}{|\zeta|^{N-2}}\,d\mu(\zeta) \le \frac{A}{\mu(s)} + \frac{D}{\mu(s)^\alpha} = \frac{D}{\mu(s)^\alpha}\left(1 + \frac{A}{D}\,\frac{\mu(s)^\alpha}{\mu(s)}\right).$$

Hence

$$[\mu(s)]^\alpha \le D\,s^{N-2}\left(1 + \frac{A}{D}\,\frac{1}{[\mu(s)]^{1-\alpha}}\right).$$

Now, it is obvious that $1-\alpha = \frac{\gamma}{\gamma+N-2}$ and $D^{1/\alpha} = \frac{B\gamma}{N-2} C(\gamma, N)$. □

Corollary 3.6. *With $N \geq 3$ and $\alpha = \frac{N-2}{\gamma+N-2}$, the following holds for each $s > 0$:*

$$\mu(s)\,\log\left(\frac{\mu(s)^\alpha}{D}\right) - \frac{A}{D}\,\mu(s)^\alpha \leq (N-2)\int_{|\zeta|\leq s} \log|\zeta|\,d\mu(\zeta)$$

Proof. It follows from Jensen's inequality that:

$$\begin{aligned}
\exp\left(\int_{|\zeta|\leq s}\left(\log\frac{1}{|\zeta|^{N-2}}\right)\frac{d\mu(\zeta)}{\mu(s)}\right) &\leq \int_{|\zeta|\leq s}\exp\left(\log\frac{1}{|\zeta|^{N-2}}\right)\frac{d\mu(\zeta)}{\mu(s)} \\
&= \int_{|\zeta|\leq s}\frac{1}{|\zeta|^{N-2}}\,\frac{d\mu(\zeta)}{\mu(s)} \\
&\leq \frac{A}{\mu(s)} + \frac{D}{\mu(s)^\alpha},
\end{aligned}$$

so that:

$$-(N-2)\int_{|\zeta|\leq s}\log|\zeta|\,\frac{d\mu(\zeta)}{\mu(s)} \leq \log\left(\frac{A}{\mu(s)} + \frac{D}{\mu(s)^\alpha}\right) \leq \log\left(\frac{D}{\mu(s)^\alpha}\right) + \frac{A}{D}\,\frac{\mu(s)^\alpha}{\mu(s)}.$$

□

4. GROWTH OF THE REPARTITION FUNCTION

4.1. **A measure on $[0,+\infty[$, image of μ.** Let $\Phi : \mathbb{R}^N \to [0,+\infty[$ be the measurable map defined by $\Phi(\zeta) = \mu(|\zeta|)$ (the function $s \mapsto \mu(s)$ is increasing hence measurable on $[0,+\infty[$). Let $\nu = \Phi * \mu = \mu \circ \Phi^{-1}$ denote the measure image of μ under Φ (see [3, p. 80]):

$$\int_0^{+\infty} f(t)\,d\nu(t) = \int_{\mathbb{R}^N} f(\Phi(\zeta))\,d\mu(\zeta)$$

holds for any nonnegative measurable function f on $[0,+\infty[$ (and for any ν-integrable f)

Remark 4.1. If $s \mapsto \mu(s)$ is continuous on some interval $[a,+\infty[$ with $a \geq 0$, then $\nu(I) = c - b$ for any interval I with bounds b and c ($c > b > \mu(a)$).

4.2. **The case $N = 2$.** Up to the end of Section 4, μ stands for the Riesz measure associated with a function of $SH(\gamma, B)$ with growth (1.1).

Theorem 4.2. *If $N = 2$ and $A > \frac{2}{\gamma}$, then the set of those $s > 0$ which satisfy $\mu(s) < B\gamma\, s^\gamma\, e^{\frac{A\gamma}{\mu(s)}}$ is unbounded.*

A proof is required only in the case where $\lim_{s\to+\infty}\mu(s) = +\infty$ (otherwise, Theorem 4.2 is obvious). When the function $s \mapsto \mu(s)$ is continuous, at least on some interval $[a,+\infty[$ with $a > 0$, there is a direct proof which is quoted below in Subsection 4.3. In this case, the assumption $A > \frac{2}{\gamma}$ is no longer required. The proof in the general case is the subject of Subsection 4.5.

4.3. **Proof of Theorem 4.2 in the case of a continuous repartition function.**

Proof. Let us suppose that the set $\left\{s > 0 \;:\; \mu(s) < B\gamma\, s^\gamma e^{\frac{A\gamma}{\mu(s)}}\right\}$ is bounded and let s_0 be one of its majorants, chosen in such a way that $s \mapsto \mu(s)$ is continuous on some neighbourhood of $[s_0,+\infty[$.

Thus $\mu(s) \geq B\gamma\, s^{\gamma} e^{\frac{A\gamma}{\mu(s)}}$ for all $s \geq s_0$, that is: $\log s \leq \frac{1}{\gamma}\, \log\left(\frac{\mu(s)}{B\gamma}\right) - \frac{A}{\mu(s)}$, such that:

$$\begin{aligned}
\int_{s_0 \leq |\zeta| \leq s} \log|\zeta|\, d\mu(\zeta) &\leq \int_{s_0 \leq |\zeta| \leq s} \left(\frac{1}{\gamma}\, \log\left(\frac{\mu(|\zeta|)}{B\gamma}\right) - \frac{A}{\mu(|\zeta|)}\right) d\mu(\zeta) \\
&= \int_{\mu(s_0)}^{\mu(s)} \left(\frac{1}{\gamma}\, \log\left(\frac{t}{B\gamma}\right) - \frac{A}{t}\right) d\nu(t) \\
&= \int_{\mu(s_0)}^{\mu(s)} \left(\frac{1}{\gamma}\, \log\left(\frac{t}{B\gamma}\right) - \frac{A}{t}\right) dt \\
&= B\, \left[x\, \log\left(\frac{x}{e}\right)\right]_{\mu(s_0)/B\gamma}^{\mu(s)/B\gamma} - A\, [\log t]_{\mu(s_0)}^{\mu(s)} \\
&= \frac{\mu(s)}{\gamma}\, \log\left(\frac{\mu(s)}{Be\gamma}\right) - A\, \log\mu(s) + K(s_0),
\end{aligned}$$

where $K(s_0)$ stands for $A\, \log\mu(s_0) - \frac{\mu(s_0)}{\gamma}\, \log\left(\frac{\mu(s_0)}{Be\gamma}\right)$. It follows from Theorem 3.1 that:

$$\frac{\mu(s)}{\gamma}\, \log\left(\frac{\mu(s)}{Be\gamma}\right) \leq A + \int_{|\zeta|<s_0} \log|\zeta|\, d\mu(\zeta) + \frac{\mu(s)}{\gamma}\, \log\left(\frac{\mu(s)}{Be\gamma}\right) - A\, \log\mu(s) + K(s_0).$$

Finally: $A\, \log\mu(s) \leq A + K(s_0) + \mu(s_0)\, \log s_0$ for all $s \geq s_0$. When s tends to $+\infty$, a contradiction arises. □

4.4. **Splitting measure μ.** Now, in order to prove Theorem 4.2 in the general case, we will introduce some notations which will also be useful in proving Theorem 4.5 (where $N \geq 3$). That is why these notations are already given in $\mathbb{R}^N$ for any $N \in \mathbb{N}$, $N \geq 2$.

It is still assumed that $\lim_{s\to+\infty} \mu(s) = +\infty$. Let $(s_n)_n$ be the non–decreasing sequence defined by: $s_n = \inf\{s > 0 : \mu(s) \geq n\}$. As the function $s \mapsto \mu(s)$ is right–continuous, we have $\mu(s_n) \geq n$ for all $n \in \mathbb{N}$. If this function is continuous at some point s_n, then $\mu(s_n) = n$.

If $s_n < s_{n+1}$, then $\mu(s_n) < n + 1$. There are infinitely many integers n such that $s_n < s_{n+1}$ because the measure $d\mu$ is finite on compact subsets of $\mathbb{R}^N$ (see [1, p. 81]).

For any $s > 0$, let $\mu^-(s) = \int_{|\zeta|<s} d\mu(\zeta)$. The discontinuity points of $s \mapsto \mu(s)$ are thus characterized by $\mu(s) > \mu^-(s)$. For every $n \in \mathbb{N}$, let $c_n = 0$ if the function $s \mapsto \mu(s)$ is continuous at point s_n, and $c_n = \frac{\mu(s_n)-n}{\mu(s_n)-\mu^-(s_n)}$ if this function is discontinuous at s_n. Note that $1 - c_n = \frac{n-\mu^-(s_n)}{\mu(s_n)-\mu^-(s_n)}$ in case of discontinuity at s_n.

For all $0 < t < s$, let I_t and $I_{t,s}$ be defined in $\mathbb{R}^N$ by:

$$I_t(\zeta) = \begin{cases} 1 \text{ if } |\zeta| = t \\ 0 \text{ otherwise} \end{cases} \qquad I_{t,s}(\zeta) = \begin{cases} 1 \text{ if } t < |\zeta| < s \\ 0 \text{ otherwise} \end{cases}$$

Let us write $\mu = \mu_1 + \mu_2 + \cdots + \mu_n + \dots$, where measures μ_k are defined such that

$$\int_{\mathbb{R}^N} d\mu_k(\zeta) = \int_{s_{k-1} \leq |\zeta| \leq s_k} d\mu_k(\zeta) = 1$$

in the following way:

$$d\mu_k = (\, c_{k-1}\, I_{s_{k-1}} + I_{s_{k-1},s_k} + (1 - c_k)\, I_{s_k}\,)\, d\mu \qquad \text{if } s_{k-1} < s_k$$

$$d\mu_k = \frac{1}{\mu(s_k) - \mu^-(s_k)}\, I_{s_k}\, d\mu \qquad \text{if } s_{k-1} = s_k.$$

Remark 4.3. If $s_{k-1} < s_k = s_{k+1} = \cdots = s_{k+l} < s_{k+l+1}$, then $\mu^-(s_k) \le k < k+l \le \mu(s_k)$ and it is easy to check that

$$(1-c_k)\, I_{s_k} \; + \sum_{j=k+1}^{k+l} \frac{1}{\mu(s_j)-\mu^-(s_j)}\, I_{s_j} \; + \, c_{k+l}\, I_{s_{k+l}} \; = \; I_{s_k}.$$

In addition, notice that $\sum_{k=1}^{n} \mu_k(s) = \min[n, \mu(s)]$ and that, for any integrable function $h \ge 0$:

$$\int_{|\zeta|\le s_n} h(\zeta)\, d\mu \ge \sum_{k=1}^{n} \int h(\zeta)\, d\mu_k$$

$$\int_{|\zeta|\le s_n} h(\zeta)\, d\mu \le \sum_{k=1}^{n+1} \int h(\zeta)\, d\mu_k \qquad\qquad \text{if } s_n < s_{n+1}$$

4.5. **A reformulation of Theorem 4.2.**

Proposition 4.4. *If $N = 2$ and $A > \frac{2}{\gamma}$, then $n < B\gamma(s_n)^\gamma e^{\frac{A\gamma}{n}}$ for infinitely many $n \in \mathbb{N}^*$.*

Proof. Suppose that there exists some integer $m \in \mathbb{N}^*$ such that $n \ge B\gamma(s_n)^\gamma e^{\frac{A\gamma}{n}}$ for each $n \ge m$. It may be assumed that $s_m > s_{m-1} \ge 1$. For any $n \ge m$ satisfying $s_n < s_{n+1}$, we have:

$$\begin{aligned}
\int_{s_m\le|\zeta|\le s_n} \log|\zeta|\, d\mu(\zeta) \;&\le\; \sum_{k=m}^{n+1} \int \log|\zeta|\, d\mu_k(\zeta) \\
&\le\; \sum_{k=m}^{n+1} \log s_k \\
&\le\; \sum_{k=m}^{n+1} \left(\frac{1}{\gamma}\, \log\left(\frac{k}{B\gamma}\right) - \frac{A}{k} \right) \\
&\le\; \int_m^{n+2} \left(\frac{1}{\gamma}\, \log\left(\frac{t}{B\gamma}\right) - \frac{A}{t} \right) dt \\
&=\; \frac{n+2}{\gamma}\, \log\left(\frac{n+2}{Be\gamma}\right) - A\, \log(n+2) + K_m
\end{aligned}$$

with a constant K_m independent from n. Since $\mu(s_n) \ge n$, Theorem 3.1 leads to:

$$\frac{n}{\gamma}\, \log\left(\frac{n}{Be\gamma}\right) \le A + (\log s_m)\mu(s_m) + \frac{n+2}{\gamma}\, \log\left(\frac{n+2}{Be\gamma}\right) - A\, \log(n+2) + K_m$$

hence

$$\left(A - \frac{2}{\gamma}\right) \log(n+2) \le A + \underbrace{\frac{n}{\gamma}\, \log\left(\frac{n+2}{n}\right)}_{\le \frac{2}{\gamma}} - \frac{2}{\gamma} \log(Be\gamma) + K_m + (\log s_m)\mu(s_m)$$

The contradiction stems from the fact that there exists infinitely many $n > m$ with $s_n < s_{n+1}$. □

Proof of Theorem 4.2 in the general case. Obviously, function $s \mapsto B\gamma\, s^\gamma$ is increasing. Thus, for any n such that $n\, e^{-\frac{A\gamma}{n}} < B\gamma(s_n)^\gamma$, there exists an open non–empty interval J_n (with upper bound s_n) such that $n\, e^{-\frac{A\gamma}{n}} < B\gamma\, s^\gamma < B\gamma(s_n)^\gamma$ $\forall s \in J_n$. Moreover $\mu(s)\, e^{-\frac{A\gamma}{\mu(s)}} < n\, e^{-\frac{A\gamma}{n}}$ $\forall s \in J_n$ (because $\mu(s) < n$ for every $s < s_n$). Hence Theorem 4.2. □

4.6. **The case $N \geq 3$.**

Theorem 4.5. *When $N \geq 3$, the set of those $s > 0$ such that*

$$\mu(s) < \frac{B\gamma}{N-2} s^{\gamma+N-2} \left(1 + \frac{A}{D \cdot [\mu(s)]^{\frac{\gamma}{\gamma+N-2}}}\right)^{\frac{\gamma+N-2}{N-2}} \tag{4.1}$$

is unbounded.

Inequalities (4.1) and (4.2) are equivalent, with

$$\frac{1}{s^{N-2}} < \frac{\gamma}{\gamma+N-2} \left(\frac{A}{\mu(s)} + \frac{D}{\mu(s)^\alpha}\right) \tag{4.2}$$

and $\alpha = \frac{N-2}{\gamma+N-2}$ as in Section 3.3. Indeed, (4.2) may be rewritten

$$\mu(s)^\alpha < s^{N-2} \frac{\gamma\, D}{\gamma+N-2} \left(1 + \frac{A}{D[\mu(s)]^{1-\alpha}}\right).$$

Now $\frac{\gamma\, D}{\gamma+N-2} = \left(\frac{B\gamma}{N-2}\right)^\alpha$ so that formula (4.1) arises.

To prove Theorem 4.5, we can still assume $\lim_{s\to+\infty} \mu(s) = +\infty$. The case where function $s \mapsto \mu(s)$ is continuous (at least on some interval $[a, +\infty[$ with $a > 0$) is proved in Subsection 4.7 and the general case is proved in Subsection 4.8.

4.7. **Proof of Theorem 4.5 in the case of a continuous repartition function.**

Proof. Let us assume that there exists some $s_0 > 0$ such that $s \mapsto \mu(s)$ is continuous on some neighbourhood of $[s_0, +\infty[$ and that $\frac{1}{s^{N-2}} \geq \frac{\gamma}{\gamma+N-2} \left(\frac{A}{\mu(s)} + \frac{D}{\mu(s)^\alpha}\right)$ for all $s \geq s_0$. It follows that:

$$\begin{aligned}
\int_{|\zeta|\leq s} \frac{d\mu(\zeta)}{|\zeta|^{N-2}} &\geq \int_{s_0\leq|\zeta|\leq s} \frac{d\mu(\zeta)}{|\zeta|^{N-2}} \\
&\geq \frac{\gamma}{\gamma+N-2} \int_{s_0\leq|\zeta|\leq s} \left(\frac{A}{\mu(|\zeta|)} + \frac{D}{\mu(|\zeta|)^\alpha}\right) d\mu(\zeta) \\
&= \frac{\gamma}{\gamma+N-2} \int_{\mu(s_0)}^{\mu(s)} \left(\frac{A}{t} + \frac{D}{t^\alpha}\right) d\nu(t) \\
&= \frac{\gamma}{\gamma+N-2} \int_{\mu(s_0)}^{\mu(s)} \left(\frac{A}{t} + \frac{D}{t^\alpha}\right) dt \\
&= \frac{\gamma}{\gamma+N-2} \left[A \log t + \frac{D}{1-\alpha} t^{1-\alpha}\right]_{\mu(s_0)}^{\mu(s)} \\
&= \frac{A\gamma \log \mu(s)}{\gamma+N-2} + D\, \mu(s)^{1-\alpha} - K'(s_0),
\end{aligned}$$

with

$$K'(s_0) = \frac{A\gamma}{\gamma+N-2} \log \mu(s_0) + D\, \mu(s_0)^{1-\alpha}.$$

The majoration of $\int_{|\zeta|\leq s} \frac{1}{|\zeta|^{N-2}}\, d\mu(\zeta)$ (Theorem 3.4) leads, after cancellation of $D\, \mu(s)^{1-\alpha} = D\, \mu(s)^{\frac{\gamma}{\gamma+N-2}}$, to: $\frac{A\gamma \log \mu(s)}{\gamma+N-2} \leq A + K'(s_0)$ for any $s \geq s_0$. A contradiction arises as $s \to +\infty$. □

4.8. **A reformulation of Theorem 4.5.**

Proposition 4.6. *With $N \geq 3$ and $\alpha = \frac{N-2}{\gamma+N-2}$, infinitely many $n \in \mathbb{N}^*$ satisfy:*

$$\frac{1}{s_n^{N-2}} < \frac{\gamma}{\gamma+N-2}\left(\frac{A}{n} + \frac{D}{n^\alpha}\right). \tag{4.3}$$

Proof. Suppose that there exists some $m \in \mathbb{N}$ such that $\frac{1}{s_n^{N-2}} \geq \frac{\gamma}{\gamma+N-2}\left(\frac{A}{n} + \frac{D}{n^\alpha}\right)$ $\forall n > m$. It then follows for all $n > m$:

$$\begin{aligned}
\int_{s_m \leq |\zeta| \leq s_n} \frac{1}{|\zeta|^{N-2}}\, d\mu(\zeta) &\geq \sum_{k=m+1}^{n} \int \frac{1}{|\zeta|^{N-2}}\, d\mu_k(\zeta) \\
&\geq \sum_{k=m+1}^{n} \frac{1}{s_k^{N-2}} \\
&\geq \frac{\gamma}{\gamma+N-2} \sum_{k=m+1}^{n} \left(\frac{A}{k} + \frac{D}{k^\alpha}\right) \\
&\geq \frac{\gamma}{\gamma+N-2} \int_{m+1}^{n+1} \left(\frac{A}{t} + \frac{D}{t^\alpha}\right) dt \\
&= \frac{\gamma A \log(n+1)}{\gamma+N-2} + D(n+1)^{1-\alpha} - K'_m
\end{aligned}$$

where the constant K'_m does not depend on n. For those $n > m$ such that $s_n < s_{n+1}$ we have $\mu(s_n) < n+1$ and Theorem 3.4 provides us with:

$$\int_{s_m \leq |\zeta| \leq s_n} \frac{1}{|\zeta|^{N-2}}\, d\mu(\zeta) \leq \int_{|\zeta| \leq s_n} \frac{1}{|\zeta|^{N-2}}\, d\mu(\zeta) \leq A + D(n+1)^{1-\alpha}$$

hence $\frac{\gamma A \log(n+1)}{\gamma+N-2} \leq A + K'_m$. A contradiction arises as $n \to +\infty$. □

Proof of Theorem 4.5 in the general case. Since the function $s \mapsto \frac{1}{s^{N-2}}$ is decreasing, for each $n \in \mathbb{N}^*$ satisfying (4.3) there exists an open interval $J_n \neq \emptyset$ (with right bound s_n) where

$$\frac{1}{s_n^{N-2}} < \frac{1}{s^{N-2}} < \frac{\gamma}{\gamma+N-2}\left(\frac{A}{n} + \frac{D}{n^\alpha}\right) \quad (\forall s \in J_n).$$

Now, $\mu(s) < n$ for each $s < s_n$, so that $\frac{A}{n} + \frac{D}{n^\alpha} < \frac{A}{\mu(s)} + \frac{D}{\mu(s)^\alpha}$. Hence $\frac{1}{s^{N-2}} < \frac{\gamma}{\gamma+N-2}\left(\frac{A}{\mu(s)} + \frac{D}{\mu(s)^\alpha}\right)$ $\forall s \in J_n$ and Theorem 4.5 follows. □

5. SUM OF TWO RIESZ MEASURES

Lemma 5.1. *Given $\gamma > 0$, $B > 0$ and $\varepsilon \in]0,1[$, let u_ε be defined in $\mathbb{R}^N$ by :*

$$u_\varepsilon(x) = \max\{0, \varphi_\varepsilon(|x|)\} \qquad \forall x \in \mathbb{R}^N$$

with $\varphi_\varepsilon(r) = B\, r^\gamma - B\, \varepsilon^\gamma$ $\forall r \geq 0$. Then $u_\varepsilon \in SH(\gamma, B)$. Let μ_ε denote its Riesz measure, then: $\mu_\varepsilon(s) = \frac{B\gamma}{\tau_N} s^{\gamma+N-2} + k_\varepsilon$ $\forall s \geq 1$, where $\tau_N = \max(1, N-2)$ and k_ε is a constant depending only on B, γ, N and ε.

Proof. Subharmonicity of $u_\varepsilon = \max(u_1, u_2)$ will follow (see [1, p. 41]) from the subharmonicity of both functions u_1 and u_2 defined in $\mathbb{R}^N$ by $u_1(x) = \varphi_\varepsilon(|x|)$ and $u_2(x) \equiv 0$: it is easy to verify that $\Delta u_1(x) = \varphi''_\varepsilon(r) + \frac{N-1}{r}\varphi'_\varepsilon(r) = B\gamma r^{\gamma-2}(\gamma+N-2) \geq 0$ (see [1, p. 26]). Obviously, u_ε has a growth of the kind (1.1), $u_\varepsilon(0) = 0$ and u_ε is harmonic in the neighbourhood $\{x \in \mathbb{R}^N : |x| < \varepsilon\}$ of the origin. □

Let $\theta_N = (N-2)\sigma_N$ when $N \geq 3$ and $\theta_2 = 2\pi$ (see [2, p. 43]), since $d\mu_\varepsilon = \frac{1}{\theta_N}\Delta u_\varepsilon\, dx = \frac{1}{\theta_N}\Delta u_\varepsilon\, r^{N-1}\, dr\, d\sigma$, it is possible for all $s \geq 1$ to compute

$$\mu_\varepsilon(s) = \mu_\varepsilon(1) + \int_1^s \frac{\sigma_N}{\theta_N} B\gamma(\gamma+N-2) r^{\gamma+N-3} dr = \mu_\varepsilon(1) + \frac{1}{\tau_N} B\gamma \left[r^{\gamma+N-2}\right]_1^s$$

Proposition 5.2. *Given $\gamma > 0$, $B > 0$ and $0 < B' < 2B$, let μ_1 and μ_2 be the Riesz measures of two functions, respectively u_1 and u_2, belonging to $SH(\gamma, B)$. Then $\mu_1 + \mu_2$ is not necessarily the Riesz measure associated with a function of $SH(\gamma, B')$.*

Proof. Given ε_1 and $\varepsilon_2 \in]0,1[$, let u_{ε_1} and $u_{\varepsilon_2} \in SH(\gamma, B)$ be defined as in the previous lemma and $\mu = \mu_{\varepsilon_1} + \mu_{\varepsilon_2}$ be the sum of their Riesz measures. Thus $\mu(s) = \frac{2B\gamma}{\tau_N} s^{\gamma+N-2} + k_{\varepsilon_1} + k_{\varepsilon_2}$ $\forall s \geq 1$. Note that $\lim_{s\to+\infty} \frac{\mu(s)}{s^{\gamma+N-2}} = \frac{2B\gamma}{\tau_N}$.

Suppose that μ is the Riesz measure of some function $u \in SH(\gamma, B')$ with an estimate such as: $u(x) \leq A + B'|x|^\gamma$ ($\forall x \in \mathbb{R}^N$) for some constant $A \in \mathbb{R}$. In Theorems 4.2 and 4.5, one asserts that $\liminf_{s\to+\infty} \frac{\mu(s)}{s^{\gamma+N-2}} \leq \frac{B'\gamma}{\tau_N}$, which leads to $2B \leq B'$, hence a contradiction. ☐

6. SUBHARMONIC FUNCTIONS SUBJECT TO CONDITIONS OF L^1 TYPE

6.1. A weighted integral condition for subharmonic functions.

Theorem 6.1. *Given $N \in \mathbb{N}$ ($N \geq 2$) and a positive non–increasing C^1 function φ on $[0, +\infty[$ such that $\lim_{s\to+\infty} (\log s)\varphi(s) = 0$ (when $N = 2$) or $\lim_{s\to+\infty} s^{\frac{N}{2}-1}\varphi(s) = 0$ (when $N \geq 3$), let u be a subharmonic function in $\mathbb{R}^N$, harmonic in some neighbourhood of the origin with $u(0) = 0$, such that:*

$$\int_{\mathbb{R}^N} u^+(x)\, [-\varphi'(|x|^2)]\, dx < +\infty$$

where the subharmonic function u^+ is defined by $u^+(x) = \max(u(x), 0)$ $\forall x \in \mathbb{R}^N$. Then the Riesz measure μ of u verifies:

$$\int_{\mathbb{R}^N} \frac{\varphi(|\zeta|^2+1)}{|\zeta|^2}\, d\mu(\zeta) < +\infty.$$

Example 6.1. With $N \geq 2$, $\beta > 0$ and φ defined by $\varphi(s) = e^{-\beta s}$ $\forall s > 0$, obviously

$$\lim_{s\to+\infty} (\log s)\varphi(s) = \lim_{s\to+\infty} s^{\frac{N}{2}-1}\varphi(s) = 0.$$

If a subharmonic function u in $\mathbb{R}^N$ (harmonic in some neighbourhood of the origin, with $u(0) = 0$) satisfies $\int_{\mathbb{R}^N} u^+(x)\, e^{-\beta|x|^2}\, dx < +\infty$ then its Riesz measure μ verifies $\int_{\mathbb{R}^N} \frac{e^{-\beta|\zeta|^2}}{|\zeta|^2}\, d\mu(\zeta) < +\infty$. One thus encounters a result of [4, p. 88] for holomorphic functions in $\mathbb{C}$.

6.2. Proof of Theorem 6.1 in the case $N = 2$.

Proof. Abiding by Jensen's formula (Subsection 3.1) and by Lemma 2.3:

$$\int_{|\zeta|\leq r} \log\frac{r}{|\zeta|}\, d\mu(\zeta) \leq \frac{1}{2\pi}\int_0^{2\pi} u^+(r\, e^{i\theta})\, d\theta \qquad \forall r > 0.$$

Since $-\varphi'(r^2) \geq 0$, it follows that:

$$\int_0^{+\infty} \left(\int_{|\zeta|\leq r} \log\frac{r}{|\zeta|}\, d\mu(\zeta)\right) [-\varphi'(r^2)]\, r\, dr\; < +\infty.$$

Fubini's theorem transforms the above integral into:

$$\int_{\mathbb{R}^2} \underbrace{\left(\int_{|\zeta|}^{+\infty} \log \frac{r}{|\zeta|} \, [-\varphi'(r^2)] \, r \, dr\right)}_{:=I(\zeta)\geq 0} d\mu(\zeta).$$

Now,

$$I(\zeta) = \frac{1}{4} \int_{|\zeta|^2}^{+\infty} \log \frac{s}{|\zeta|^2} [-\varphi'(s)] \, ds$$

for any $\zeta \in \mathbb{R}^2$ and an integration by parts leads to: $4\, I(\zeta) = \int_{|\zeta|^2}^{+\infty} \frac{\varphi(s)}{s} \, ds$ since $\lim_{s\to+\infty} (\log s) \, \varphi(s) = 0$ and $\lim_{s\to+\infty} \varphi(s) = 0$ as well. The positive function $f : s \mapsto \frac{\varphi(s)}{s}$ decreases for $s > 0$ so that $\int_b^{+\infty} f(s) \, ds \geq f(b+1)$ for all $b > 0$, hence: $4\, I(\zeta) \geq \frac{\varphi(|\zeta|^2+1)}{|\zeta|^2+1}$ for all $\zeta \in \mathbb{R}^2$. If $|\zeta| \geq 1$, then $\frac{1}{|\zeta|^2+1} \geq \frac{1}{2\,|\zeta|^2}$ and $8\, I(\zeta) \geq \frac{\varphi(|\zeta|^2+1)}{|\zeta|^2} \geq 0$. Because of the harmonicity of u in a neighbourhood of the origin, $\int_{|\zeta|<1} \frac{\varphi(|\zeta|^2+1)}{|\zeta|^2} \, d\mu(\zeta) < +\infty$. The conclusion follows from $\int_{|\zeta|\geq 1} I(\zeta) \, d\mu(\zeta) < +\infty$. □

6.3. Proof of Theorem 6.1 in the case $N \geq 3$.

Proof. Jensen–Privalov formula together with Lemma 2.4 lead to:

$$\int_{|\zeta|\leq r} \left(\frac{1}{|\zeta|^{N-2}} - \frac{1}{r^{N-2}}\right) d\mu(\zeta) \leq \frac{1}{\sigma_N} \int_{S_N} u(rx) \, d\sigma_x \qquad \forall r > 0.$$

Hence:

$$\int_0^{+\infty} \left(\int_{|\zeta|\leq r} \left(\frac{1}{|\zeta|^{N-2}} - \frac{1}{r^{N-2}}\right) d\mu(\zeta)\right) [-\varphi'(r^2)] \, r^{N-1} \, dr \ < +\infty.$$

Taking Fubini's theorem into account, this integral becomes:

$$\int_{I\!R^N} \underbrace{\left(\int_{|\zeta|}^{+\infty} \left(\frac{1}{|\zeta|^{N-2}} - \frac{1}{r^{N-2}}\right) [-\varphi'(r^2)] \, r^{N-1} \, dr\right)}_{:=J(\zeta)} d\mu(\zeta).$$

Now, for any $\zeta \in \mathbb{R}^N$:

$$0 \leq J(\zeta) = \int_{|\zeta|}^{+\infty} \left(\frac{r^{N-2}}{|\zeta|^{N-2}} - 1\right) [-\varphi'(r^2)] \, r \, dr = \frac{1}{2} \int_{|\zeta|^2}^{+\infty} \left(\frac{s^{\frac{N}{2}-1}}{|\zeta|^{N-2}} - 1\right) [-\varphi'(s)] \, ds.$$

Since $\lim_{s\to+\infty} \left(s^{\frac{N}{2}-1} - |\zeta|^{N-2}\right) \varphi(s) = 0$, an integration by parts leads to:

$$2\, J(\zeta) = \frac{N-2}{2} \int_{|\zeta|^2}^{+\infty} \frac{s^{\frac{N}{2}-2}}{|\zeta|^{N-2}} \, \varphi(s) \, ds.$$

Obviously, $s^{\frac{N}{2}-2} \geq |\zeta|^{N-4}$ for all $s \geq |\zeta|^2$, so that:

$$\frac{4}{N-2} J(\zeta) \geq \frac{1}{|\zeta|^2} \int_{|\zeta|^2}^{+\infty} \varphi(s) \, ds \geq \frac{\varphi(|\zeta|^2+1)}{|\zeta|^2} \geq 0.$$

□

Propositions 6.2 and 6.3 will be proved by using the same method.

Proposition 6.2. *Let φ be a positive C^1 non–increasing function on $[0,+\infty[$ such that $\lim_{r\to+\infty} r\,\varphi(r)\,\log r = 0$. If a subharmonic function u in $\mathbb{R}^2$ (harmonic in some neighbourhood of the origin with $u(0)=0$) verifies:*

$$\int_{\mathbb{R}^2} u^+(x)\,[-\varphi'(|x|)]\,dx < +\infty$$

then its Riesz measure μ satisfies: $\int_{\mathbb{R}^2} \varphi(|\zeta|+1)\,d\mu(\zeta) < +\infty$ and

$$\int_{|\zeta|\geq 1} \varphi(|\zeta|^\alpha+1)\,\log|\zeta|\,d\mu(\zeta) < +\infty$$

holds for each $\alpha > 1$.

Proof. As in Section 6.2: $\int_{\mathbb{R}^2} I(\zeta)\,d\mu(\zeta) < +\infty$, here with

$$I(\zeta) = \int_{|\zeta|}^{+\infty} r\,\log\frac{r}{|\zeta|}\,[-\varphi'(r)]\,dr$$

which turns into $I(\zeta) = \int_{|\zeta|}^{+\infty} \varphi(r)\,\log\frac{er}{|\zeta|}\,dr$ after an integration by parts which uses $\lim_{r\to+\infty} r\,\varphi(r)\,\log r = 0$ (this garantees that $\lim_{r\to+\infty} r\,\varphi(r) = 0$ as well). Since φ is non–increasing and $\log\frac{er}{|\zeta|} \geq 1$ for each $r \geq |\zeta|$, it follows that $I(\zeta) \geq \varphi(|\zeta|+1)\ \forall\zeta\in\mathbb{R}^2$.

Given $\alpha > 1$, obviously $|\zeta|^\alpha \geq |\zeta|$ as soon as $|\zeta| \geq 1$, so that

$$I(\zeta) \geq \int_{|\zeta|^\alpha}^{+\infty} \varphi(r)\,\log\frac{er}{|\zeta|}\,dr \geq (\alpha-1)\int_{|\zeta|^\alpha}^{+\infty} \varphi(r)\,\log|\zeta|\,dr \geq (\alpha-1)\varphi(|\zeta|^\alpha+1)\,\log|\zeta| \geq 0.$$

The conclusion proceeds from $\int_{|\zeta|\geq 1} I(\zeta)\,d\mu(\zeta) < +\infty$. □

Proposition 6.3. *Given $N\in\mathbb{N}$, $N\geq 3$, let φ be a positive non–increasing C^1 function in $[0,+\infty[$ such that $\lim_{r\to+\infty} r^{N-1}\,\varphi(r) = 0$. If a subharmonic function u in $\mathbb{R}^N$ (harmonic in some neighbourhood of the origin with $u(0)=0$) verifies:*

$$\int_{\mathbb{R}^N} u^+(x)\,[-\varphi'(|x|)]\,dx < +\infty$$

then its Riesz measure μ satisfies

$$\int_{\mathbb{R}^N} \varphi(|\zeta|^\alpha+1)\,|\zeta|^{(\alpha-1)(N-2)}\,d\mu(\zeta) < +\infty$$

for any $\alpha \geq 1$.

Remark 6.4. When $\alpha = 1$, we encounter $\int_{\mathbb{R}^N} \varphi(|\zeta|+1)\,d\mu(\zeta) < +\infty$ again.

Proof. As in Section 6.3: $\int_{\mathbb{R}^N} J(\zeta)\,d\mu(\zeta) < +\infty$, here with

$$J(\zeta) = \int_{|\zeta|}^{+\infty} \left(\frac{r^{N-1}}{|\zeta|^{N-2}} - r\right)[-\varphi'(r)]\,dr = \int_{|\zeta|}^{+\infty} \left((N-1)\frac{r^{N-2}}{|\zeta|^{N-2}} - 1\right)\varphi(r)\,dr$$

after an integration by parts. Obviously, $\frac{r^{N-2}}{|\zeta|^{N-2}} \geq 1$ for every $r \geq |\zeta|$, so that:

$$(N-1)\frac{r^{N-2}}{|\zeta|^{N-2}} - 1 \geq (N-2)\frac{r^{N-2}}{|\zeta|^{N-2}}$$

and

$$J(\zeta) \geq (N-2)\int_{|\zeta|}^{+\infty} \frac{r^{N-2}}{|\zeta|^{N-2}}\,\varphi(r)\,dr \qquad \forall\zeta\in\mathbb{R}^N.$$

If $|\zeta| \geq 1$, then $|\zeta|^\alpha \geq |\zeta|$ since $\alpha \geq 1$, hence

$$\begin{aligned} J(\zeta) &\geq (N-2)\int_{|\zeta|^\alpha}^{+\infty} \frac{r^{N-2}}{|\zeta|^{N-2}}\,\varphi(r)\,dr \\ &\geq (N-2)\,|\zeta|^{(\alpha-1)(N-2)}\int_{|\zeta|^\alpha}^{+\infty}\varphi(r)\,dr \\ &\geq (N-2)\,|\zeta|^{(\alpha-1)(N-2)}\varphi(|\zeta|^\alpha+1). \end{aligned}$$

□

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