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THE HARDY-LANDAU-LITTLEWOOD INEQUALITIES WITH LESS SMOOTHNESS

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ABSTRACT. One proves Hardy-Landau-Littlewood type inequalities for functions in the Lipschitz space attached to a C_0-semigroup (or to a $C_0^\star$-semigroup).

Key words and phrases: Landau's inequalities, C_0-semigroup, Lipschitz function.

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1. INTRODUCTION

If a function and its second derivative are small, then the first derivative is small too. More precisely, for each $p \in [1, \infty]$ and each of the intervals $I = \mathbb{R}_+$ or $I = \mathbb{R}$, there is a constant $C_p(I) > 0$ such that if $f : I \to \mathbb{R}$ is a twice differentiable function with $f,\ D^2 f \in L^p(I)$, then $Df \in L^p(I)$ and

$$\|Df\|_{L^P} \leq C_p(I) \, \|f\|_{L^p}^{1/2} \left\|D^2 f\right\|_{L^p}^{1/2} . \tag{1.1}$$

We make the convention to denote by $C_p(I)$ the best constant for which the inequality (1.1) holds.

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The investigation of such inequalities was initiated by E. Landau [17] in 1914. He considered the case $p = \infty$ and proved that

$$C_\infty(\mathbb{R}_+) = 2 \quad \text{and} \quad C_\infty(\mathbb{R}) = \sqrt{2}.$$

In 1932, G.H. Hardy and J.E. Littlewood [12] proved (1.1) for $p = 2,$ with best constants

$$C_2(\mathbb{R}_+) = \sqrt{2} \quad \text{and} \quad C_2(\mathbb{R}) = 1.$$

In 1935, G.H. Hardy, E. Landau and J.E. Littlewood [13] showed that

$$C_p(\mathbb{R}_+) \leq 2 \quad \text{for } p \in [1, \infty)$$

which yields $C_p(\mathbb{R}) \leq 2$ for $p \in [1, \infty)$. Actually, $C_p(\mathbb{R}) \leq \sqrt{2}$. See Theorem 1.1 below.

In 1939, A.N. Kolmogorov [16] showed that

$$\left\| D^k f \right\|_{L^\infty} \leq C_\infty(n, k, \mathbb{R}) \; \|f\|_{L^\infty}^{1-k/n} \; \|D^n f\|_{L^\infty}^{k/n} \tag{1.2}$$

for functions f on $\mathbb{R}$ and $1 \leq k < n$ (D^k denotes the kth derivative of f). As above, $C_\infty(n, k, \mathbb{R})$ denote the best constant in (1.2). Their explicit formula was indicated also by A.N. Kolmogorov [16]. An excellent account on inequalities (1.1) (and their relatives) are to be found in the monograph of D. S. Mitrinović, J. E. Pečarić, and A. M. Fink [19].

All these results were extended to C_0-semigroups (subject to different restrictions) by R.R. Kallman and G.-C. Rota [15], E. Hille [14] and Z. Ditzian [5]. We shall consider here the case of *stable* C_0-semigroups on a Banach space E, i.e. of semigroups $(T(t))_{t\geq 0}$ such that

$$\sup_{t\geq 0} \|T(t)\| = M < \infty.$$

Theorem 1.1. *Let $(T(t))_{t\geq 0}$ be a stable C_0-semigroup on E, and let $A : Dom(A) \subset E \to E$ be its infinitesimal generator. Then for each $n = 2, 3, \ldots$ and each integer number $k \in (0, n)$ there exists a constant $K(n, k) > 0$ such that*

$$||A^k f|| \leq K(n, k)||A^n f||^{k/n}||f||^{1-k/n} \quad \text{for all } f \in Dom(A^n). \tag{1.3}$$

Moreover, $K(2, 1) = 2M$ in the case of semigroups, and $K(2, 1) = M\sqrt{2}$ in the case of groups. The other constants $K(n, k)$ can be estimated by recursion.

The aim of this paper is to prove similar inequalities with less smoothness assumptions, i.e. outside $Dom\,(A^2)$. See Theorem 2.1 below. The idea is to replace twice differentiability by the membership of the first differential to the Lipschitz class. In the simplest case our result is equivalent with the following fact: *Let $f : \mathbb{R} \to \mathbb{R}^n$ be a differentiable bounded function, whose derivative is Lipschitz. Then Df is bounded and*

$$\|Df\|_{L^\infty}^2 \leq 2 \, \|f\|_{L^\infty} \cdot \|Df\|_{Lip} \,. \tag{1.4}$$

See Section 3 for details.

An important question concerning the above inequalities is their significance. One possible physical interpretation of the inequality studied by Landau is as follows: Suppose that a mass m particle moves along a curve $\mathbf{r} = \mathbf{r}(t),\ t \geq 0,$ under the presence of a continuous force $\mathbf{F}$, according to Newton's Law of motion,

$$m\ddot{\mathbf{r}} = \mathbf{F}.$$

If the entire evolution takes place in a ball $B_R(0)$, then the kinetic energy of the particle,

$$E = \frac{m \, \|\dot{\mathbf{r}}\|^2}{2},$$

satisfies an estimate of the form

$$E \leq \begin{cases} 2R\,\|\mathbf{F}\|_{L^\infty}\,, & \text{if the temporal interval is } \mathbb{R}_+ \\ \\ R\,\|\mathbf{F}\|_{L^\infty}\,, & \text{if the temporal interval is } \mathbb{R}, \end{cases}$$

which relates the level of energy and the size of ambient space where motion took place.

The same inequality of Landau reveals an obstruction concerning the extension properties of smooth functions outside a given compact interval I. Does there exist a constant $C > 0$ such that for each function $f \in C^2(I)$ there is a corresponding function $F \in C^2(\mathbb{R})$ such that

$$F = f \quad \text{on } I$$

and

$$\sup_{x\in\mathbb{R}} |D^k F(x)| \leq C \sup_{x\in I} |D^k f(x)| \quad \text{for } k = 0, 1, 2\,?$$

By assuming a positive answer, an immediate consequence would be the relation

$$\sup_{x\in I} |f'(x)|^2 \leq 2C^2 \left(\sup_{x\in I} |f(x)|\right)\left(\sup_{x\in I} |f''(x)|\right).$$

Or, simple examples (such as that one at the end of section 3 below) show the impossibility of such a universal estimate.

A recent paper by G. Ramm [21] describes still another obstruction derived from (1.1), concerning the stable approximation of f'.

2. TAYLOR'S FORMULA AND THE EXTENSION OF THE HARDY-LANDAU-LITTLEWOOD INEQUALITY

Throughout this section we shall deal with $\sigma(E, X)$-continuous semigroups of linear operators on a Banach space E, where X is a (norm) closed subspace of $E^\star$ which satisfies the following three technical conditions:

S1) $\|x\| = \sup\{|x^\star(x)|;\ x^\star \in X,\ \|x^\star\| = 1\}$.
S2) The $\sigma(E, X)$-closed convex hull of every $\sigma(E, X)$-compact subset of E is $\sigma(E, X)$-compact as well.
S3) The $\sigma(X, E)$-closed convex hull of every $\sigma(X, E)$-compact subset of X is $\sigma(X, E)$-compact as well.

For example, these conditions are verified when X is the dual space of E or its predual (if any), so that our approach will include both the case of C_0-semigroups and of $C_0^\star$-semigroups. See [3], Section 3.1.2, for details.

$(A, Dom(A))$ will always denote the generator of such a semigroup $\mathcal{T} = (T(t))_{t\geq 0}$.

The *Lipschitz space* of order $\alpha \in (0, 1]$ attached to A is defined as the set $\Lambda^\alpha(A)$ of all $x \in E$ such that

$$\|x\|_{\Lambda^\alpha} = \sup_{s>0} \frac{\|T(s)x - x\|}{s^\alpha} < \infty.$$

This terminology is (partly) motivated by the case of $A = d/dt$, with domain

$$Dom(A) = \{f \in L^\infty(\mathbb{R});\ f \text{ is absolutely continuous and } f' \in L^\infty(\mathbb{R})\},$$

which generates the $C_0^\star$-semigroup of translations on $L^\infty(\mathbb{R})$:

$$T(t)\, f(s) = f(s + t), \quad \text{for every } f \in L^\infty(\mathbb{R}).$$

In this case, the elements of $\Lambda^\alpha(A)$ are the usual Lipschitz mappings $f : \mathbb{R} \to \mathbb{C}$ of order α (which are essentially bounded).

Coming back to the general case, notice that

$$T(t)x = x + tAx + \int_0^t (T(s) - I)Ax\,ds, \quad \text{for } x \in Dom(A) \text{ and } t > 0 \tag{2.1}$$

(possibly, in the weak* sense, if the given semigroup is $C_0^\star$-continuous). In the classical approach, the remainder is estimated via "higher derivatives", i.e. via A^2. In the framework of semigroups, we need the inequality

$$\left\| \int_0^t (T(s) - I)Ax\,ds \right\| \leq \frac{t^{\alpha+1}}{\alpha+1} \|Ax\|_{\Lambda^\alpha},$$

which works for every $x \in Dom(A)$ with $Ax \in \Lambda^\alpha(A)$ and every $t > 0$. Then, from Taylor's formula (2.1), we can infer immediately the relation

$$\|Ax\| \leq \frac{(1 + \|T(t)\|)\,\|x\|}{t} + \frac{t^\alpha}{\alpha+1} \|Ax\|_{\Lambda^\alpha},$$

for every $x \in Dom(A)$ with $Ax \in \Lambda^\alpha(A)$ and every $t > 0$. Taking in the right-hand side the infimum over $t > 0$, we arrive at the following generalization of the Hardy-Landau-Littlewood inequality:

Theorem 2.1. *If $(A, Dom(A))$ is the generator of a C_0-(or of a $C_0^\star$-) semigroup $(T(t))_{t\geq 0}$ such that*

$$\sup_{t\geq 0} \|T(t)\| \leq M < \infty,$$

then

$$\|Ax\| \leq M_{sg}(A)\,\|x\|^{\alpha/(1+\alpha)} \cdot \|Ax\|_{\Lambda^\alpha}^{1/(1+\alpha)},$$

for every $x \in Dom(A)$ with $Ax \in \Lambda^\alpha(A)$, where

$$M_{sg}(A) = (1+M)^{\alpha/(1+\alpha)} \left[\left(\frac{\alpha}{1+\alpha}\right)^{1/(1+\alpha)} + \frac{1}{1+\alpha} \cdot \left(\frac{1+\alpha}{\alpha}\right)^{\alpha/(1+\alpha)} \right].$$

In the case of (C_0- or $C_0^\star$-continuous) groups of isometries, again by Taylor's formula (2.1),

$$T(-t)x = x - tAx + \int_{-t}^0 (T(s) - I)Axds, \quad \text{for } x \in Dom(A) \text{ and } t > 0 \tag{2.2}$$

so that subtracting (2.2) from (2.1) we get

$$\|Ax\| \leq \frac{(\|T(t)\| + \|T(-t)\|)\,\|x\|}{2t} + \frac{t^\alpha}{\alpha+1} \|Ax\|_{\Lambda^\alpha}$$

which leads to a better constant in the Hardy-Landau-Littlewood inequality, more precisely, the bound M_{sg} should be replaced by

$$M_g(A) = M^{\alpha/(1+\alpha)} \left[\left(\frac{\alpha}{1+\alpha}\right)^{1/(1+\alpha)} + \frac{1}{1+\alpha} \cdot \left(\frac{1+\alpha}{\alpha}\right)^{\alpha/(1+\alpha)} \right].$$

The problem of finding the best constants in the Hardy-Landau-Littlewood inequality is left open. Notice that even the best values of $C_p(I)$, for $1 < p < \infty$, are still unknown; an interesting conjecture concerning this particular case appeared in a paper by J.A. Goldstein and F. Räbiger [9], but only a little progress has been made since then. See [7].

The generalization of Taylor's formula for higher order of differentiability is straightforward (and it allows us to extend A.N. Kolmogorov's interpolating inequalities to the case of semigroups).

Theorem 2.1 outlined the *Sobolev-Lipschitz space of order* $1 + \alpha$,

$$W\Lambda^{\alpha}(A) = \{x \in Dom(A);\ Ax \in \Lambda^{\alpha}(A)\}.$$

which can be endowed with the norm

$$\|x\|_{W\Lambda^{\alpha}} = \|x\|_{W^1} + \|Ax\|_{\Lambda^{\alpha}}.$$

Clearly,

$$Dom(A^2) \subset W\Lambda^1(A) \subset D(A)$$

and the following example shows that the above inclusions can be strict.

Let $\mathfrak{X} = C_0(\mathbb{R}_+)$ be the Banach space of all continuous functions $f : \mathbb{R}_+ \to \mathbb{R}$ such that $\lim_{t\to\infty} f(t) = 0$ (endowed with the sup-norm). The generator of the translation semigroup on $\mathfrak{X}$ is

$$A = \frac{d}{dt} \text{ with } Dom(A) = \{f \in \mathfrak{X};\ f \text{ differentiable and } f' \in \mathfrak{X}\}.$$

See [20]. Then we have

$$Dom(A^2) = \{f \in Dom(A);\ f'' \in \mathfrak{X}\}$$

and

$$W\Lambda^1(A) = \{f \in Dom(A);\ f' \text{ is a Lipschitz function}\}.$$

3. THE INEQUALITIES OF HADAMARD

When I is $\mathbb{R}_+$ or $\mathbb{R}$, the following result (essentially due to J. Hadamard [11]) is a straightforward consequence of Theorem 2 above, applied to the semigroup generated by $\frac{d}{dx}$ on $L^{\infty}_{\mathbb{R}^n}(I)$:

Theorem 3.1. *Let I be an interval and let $f : I \to \mathbb{R}^n$ be a differentiable bounded function, whose derivative is Lipschitz, of order 1. Then f' is bounded and*

$$\|f'\|_{L^\infty} \le \begin{cases} \dfrac{4\|f\|_{L^\infty}}{\ell(I)} + \dfrac{\ell(I)}{4}\|f'\|_{Lip}, & \text{if } \ell(I) \le 4\sqrt{\|f\|_{L^\infty}/\|f'\|_{Lip}} \\ 2\sqrt{\|f\|_{L^\infty}\cdot\|f'\|_{Lip}}, & \text{if } \ell(I) \ge 4\sqrt{\|f\|_{L^\infty}/\|f'\|_{Lip}} \text{ and } I \ne \mathbb{R} \\ \sqrt{2\|f\|_{L^\infty}\cdot\|f'\|_{Lip}}, & \text{if } I = \mathbb{R}. \end{cases}$$

Furthermore, these inequalities are sharp. Here $\ell(I)$ denotes the length of I.

Proof. Of course, Theorem 3.1 admits a direct argument. Notice first that we can restrict ourselves to the case of real functions.

According to our hypotheses, f' satisfies on I an estimate of the form

$$|f'(t) - f'(s)| \le \|f'\|_{Lip}|t - s|$$

where $\|f'\|_{Lip} = \|f'\|_{\Lambda^1}$ is the best constant for which this relation holds. As

$$f(t) = f(a) + f'(a)(t - a) + \int_a^t [f'(t) - f'(a)]\,dt,$$

we have

$$\begin{aligned} |f(t) - f(a) - f'(a)(t - a)| &\le \left|\int_a^t [f'(t) - f'(a)]dt\right| \\ &\le \frac{1}{2}\|f'\|_{Lip}|t - a|^2 \end{aligned}$$

for every $t, a \in I$, $t \neq a$. The integrability is meant here in the sense of Henstock-Kurzveil [2], [10]. Consequently,

$$\begin{aligned} |f'(a)| &\leq \frac{|f(t)-f(a)|}{|t-a|} + \frac{1}{2}\|f'\|_{Lip}|t-a| \\ &\leq \frac{2\|f\|_{L^\infty}}{|t-a|} + \frac{1}{2}\|f'\|_{Lip}|t-a| \end{aligned}$$

for every $t, a \in I$, $t \neq a$. Now, the problem is how much room is left to t. In the worse case, i.e., when I is bounded and $\ell(I) \leq 4\sqrt{\|f\|_{L^\infty} / \|f'\|_{Lip}}$, the infimum over t in the right side hand is $\dfrac{4\|f\|_{L^\infty}}{\ell(I)} + \dfrac{\ell(I)}{4}\|f'\|_{Lip}$.

If I is unbounded, then the infimum is at most $2\sqrt{\|f\|_{L^\infty}\|f'\|_{Lip}}$, or even

$$\sqrt{2\|f\|_{L^\infty}\|f'\|_{Lip}},$$

for $I = \mathbb{R}$).

In order to prove that the bounds indicated in Theorem 3.1 above are sharp it suffices to exhibit some appropriate examples. The critical case is that of bounded intervals, because for half-lines, as well as for $\mathbb{R}$, the sharpness is already covered by Landau's work.

Restricting to the case of $I = [0,1]$, we shall consider the following example, borrowed from [4]. Let $a \in [0,4]$. The function

$$f_a(t) = -\frac{at^2}{2} + \left(2 + \frac{a}{2}\right)t - 1, \quad t \in I = [0,1]$$

verifies $\|f_a\|_{L^\infty} = 1$, $\|f'_a\|_{L^\infty} = 2 + a/2$ and $\|f'_a\|_{Lip} = a$. As $\ell(I) = 1$, the relation given by Theorem 3.1 becomes

$$2 + \frac{a}{2} \leq \frac{2 \cdot 1}{1} + \frac{1}{2}a.$$

On the other hand, no estimate of the form

$$\|f'\|_{L^\infty} \leq C\sqrt{\|f\|_{L^\infty}\|f'\|_{Lip}}$$

can work for all functions $f \in C^2(I)$, because, taking into account the case of the functions f_a, we are led to

$$\left(2 + \frac{a}{2}\right)^2 \leq Ca \quad \text{for every } a \in [0,4]$$

a fact which contradicts the finiteness of C. □

4. THE CASE OF NONLINEAR SEMIGROUPS

We shall discuss here the case of one of the most popular nonlinear semigroup of contractions, precisely, that generated by the p-Laplacian $(p \in (2,\infty))$,

$$Au = \Delta_p u = div\left(|\nabla u|^{p-2} \cdot \nabla u\right), \quad (p \in (2,\infty))$$

acting on $H = L^2(\Omega)$ and having as its domain

$$Dom(A) = \left\{u \in W_0^{1,p}(\Omega);\ \Delta_p u \in H\right\}.$$

Here Ω denotes a bounded open subset of $\mathbb{R}^N$, with sufficiently smooth boundary.

Put $V = W_0^{1,p}(\Omega)$ and denote by $j : V \to H$ and $j' : H \to V'$ the canonical embeddings.

Clearly, A is a dissipative operator. It is also maximal dissipative i.e., the image of $I_H - A$ equals H. In fact, let $f \in H$. Since A is dissipative, hemicontinuous and coercive as an operator from V into V', it follows that $\operatorname{Im} A = V'$, so that $\operatorname{Im}(j'j - A) = V'$. Therefore the equation

$$u - Au = f$$

has a unique solution $u \in V$. This shows that $u \in Dom(A)$ i.e., A is maximal dissipative and thus it generates a nonlinear semigroup of contractions on H. See [1].

Suppose there exists a positive constant C such that

$$||Ax||_H^2 \leq C||A^2x||_H \cdot ||x||_H \quad \text{for every } x \in Dom(A^2).$$

As $||Ax||_{V'} = ||x||_V^{p-1}$, it would follow that

$$\begin{aligned} ||x||_V^{2(p-1)} &\leq C_1||A^2x||_H \cdot ||x||_H \\ &\leq C_2\,||A^2x||_H \cdot ||x||_V \end{aligned}$$

i.e., $||x||_V^{2p-3} \leq C_2||A^2x||_H$ for every $x \in Dom(A^2)$. Letting $x = \varepsilon y$, where $\varepsilon > 0$ and $y \in Dom(A^2)$, $y \neq 0$, we are led to

$$\varepsilon^{2p-3}||y||_V^{2p-3} \leq C_2\varepsilon^{(p-1)^2}\,||A^2y||_H$$

i.e., to $||y||_V^{2p-3} \leq C_2\varepsilon^{(p-2)^2}||A^2y||_H$, which constitutes a contradiction for y fixed and ε small enough.

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