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APPROXIMATING THE FINITE HILBERT TRANSFORM VIA AN OSTROWSKI TYPE INEQUALITY FOR FUNCTIONS OF BOUNDED VARIATION

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ABSTRACT. Using the Ostrowski type inequality for functions of bounded variation, an approximation of the finite Hilbert Transform is given. Some numerical experiments are also provided.

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1. INTRODUCTION

Cauchy principal value integrals of the form

$$(Tf)(a,b;t) = PV\int_a^b \frac{f(\tau)}{\tau - t}d\tau := \lim_{\varepsilon\to 0+}\left[\int_a^{t-\varepsilon}\frac{f(\tau)}{\tau - t}d\tau + \int_{t+\varepsilon}^b \frac{f(\tau)}{\tau - t}d\tau\right] \tag{1.1}$$

play an important role in fields like aerodynamics, the theory of elasticity and other areas of the engineering sciences. They are also helpful tools in some methods for the solution of differential equations (cf., e.g. [23]).

For different approaches in approximating the finite Hilbert transform (1.1) including: interpolatory, noninterpolatory, Gaussian, Chebychevian and spline methods, see for example the papers [1] – [12], [14] – [22], [24] – [33] and the references therein.

In contrast with all these methods, we point out here a new method in approximating the finite Hilbert transform by the use of the Ostrowski inequality for functions of bounded variation established in [13].

For a comprehensive list of papers on Ostrowski's inequality, visit the site
http://rgmia.vu.edu.au.

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Estimates for the error bounds and some numerical examples for the obtained approximation are also presented.

2. SOME INEQUALITIES ON THE INTERVAL $[a,b]$

We start with the following lemma proved in [13] dealing with an Ostrowski type inequality for functions of bounded variation.

Lemma 2.1. *Let* $u:[a,b]\to\mathbb{R}$ *be a function of bounded variation on* $[a,b]$. *Then, for all* $x\in[a,b]$, *we have the inequality:*

$$\left|u(x)(b-a)-\int_a^b u(t)\,dt\right|\le\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(u), \tag{2.1}$$

where $\bigvee_a^b(u)$ *denotes the total variation of* u *on* $[a,b]$.
The constant $\frac{1}{2}$ *is the best possible one.*

Proof. For the sake of completeness and since this result will be essentially used in what follows, we give here a short proof.

Using the integration by parts formula for the Riemann-Stieltjes integral we have

$$\int_a^x (t-a)\,du(t)=u(x)(x-a)-\int_a^x u(t)\,dt$$

and

$$\int_x^b (t-b)\,du(t)=u(x)(b-x)-\int_x^b u(t)\,dt.$$

If we add the above two equalities, we get

$$u(x)(b-a)-\int_a^b u(t)\,dt=\int_a^x (t-a)\,du(t)+\int_x^b (t-b)\,du(t) \tag{2.2}$$

for any $x\in[a,b]$.

If $p:[c,d]\to\mathbb{R}$ is continuous on $[c,d]$ and $v:[c,d]\to\mathbb{R}$ is of bounded variation on $[c,d]$, then:

$$\left|\int_c^d p(x)\,dv(x)\right|\le\sup_{x\in[c,d]}|p(x)|\bigvee_c^d(u). \tag{2.3}$$

Using (2.2) and (2.3), we deduce

$$\begin{aligned}\left|u(x)(b-a)-\int_a^b u(t)\,dt\right|&\le\left|\int_a^x (t-a)\,du(t)\right|+\left|\int_x^b (t-b)\,du(t)\right|\\&\le(x-a)\bigvee_a^x(u)+(b-x)\bigvee_x^b(u)\\&\le\max\{x-a,b-x\}\left[\bigvee_a^x(u)+\bigvee_x^b(u)\right]\\&=\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(u)\end{aligned}$$

and the inequality (2.1) is proved.

Now, assume that the inequality (2.2) holds with a constant $c > 0$, i.e.,

$$\left|u(x)(b-a) - \int_a^b u(t)\,dt\right| \leq \left[c(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (u) \tag{2.4}$$

for all $x \in [a,b]$.

Consider the function $u_0 : [a,b] \to \mathbb{R}$ given by

$$u_0(x) = \begin{cases} 0 & \text{if } \quad x \in [a,b] \setminus \left\{\frac{a+b}{2}\right\} \\ 1 & \text{if } \quad x = \frac{a+b}{2}. \end{cases}$$

Then u_0 is of bounded variation on $[a,b]$ and

$$\bigvee_a^b (u_0) = 2, \quad \int_a^b u_0(t)\,dt = 0.$$

If we apply (2.4) for u_0 and choose $x = \frac{a+b}{2}$, then we get $2c \geq 1$ which implies that $c \geq \frac{1}{2}$ showing that $\frac{1}{2}$ is the best possible constant in (2.1). □

The best inequality we can get from (2.1) is the following midpoint inequality.

Corollary 2.2. *With the assumptions in Lemma 2.1, we have*

$$\left|u\left(\frac{a+b}{2}\right)(b-a) - \int_a^b u(t)\,dt\right| \leq \frac{1}{2}(b-a) \bigvee_a^b (u). \tag{2.5}$$

The constant $\frac{1}{2}$ *is best possible.*

Using the above Ostrowski type inequality we may point out the following result in estimating the finite Hilbert transform.

Theorem 2.3. *Let* $f : [a,b] \to \mathbb{R}$ *be a function such that its derivative* $f' : [a,b] \to \mathbb{R}$ *is of bounded variation on* $[a,b]$. *Then we have the inequality:*

$$\begin{aligned} &\left|(Tf)(a,b;t) - \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) - \frac{b-a}{\pi}[f;\lambda t + (1-\lambda)b, \lambda t + (1-\lambda)a]\right| \\ &\qquad \leq \frac{1}{\pi}\left[\frac{1}{2} + \left|\lambda - \frac{1}{2}\right|\right]\left[\frac{1}{2}(b-a) + \left|t - \frac{a+b}{2}\right|\right] \bigvee_a^b (f'), \end{aligned} \tag{2.6}$$

for any $t \in (a,b)$ *and* $\lambda \in [0,1)$, *where* $[f;\alpha,\beta]$ *is the divided difference, i.e.,*

$$[f;\alpha,\beta] := \frac{f(\alpha) - f(\beta)}{\alpha - \beta}.$$

Proof. Since f' is bounded on $[a,b]$, it follows that f is Lipschitzian on $[a,b]$ and thus the finite Hilbert transform exists everywhere in (a,b).

As for the function $f_0 : (a,b) \to \mathbb{R}$, $f_0(t) = 1$, $t \in (a,b)$, we have

$$(Tf_0)(a,b;t) = \frac{1}{\pi}\ln\left(\frac{b-t}{t-a}\right), \quad t \in (a,b),$$

then obviously

$$(Tf)(a,b;t) - \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) = \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t}\,d\tau. \tag{2.7}$$

Now, if we choose in (2.1), $u = f'$, $x = \lambda c + (1-\lambda) d$, $\lambda \in [0,1]$, then we get

$$\begin{aligned} &\left| f(d) - f(c) - (d-c) f'(\lambda c + (1-\lambda) d) \right| \\ &\qquad \le \left[\frac{1}{2} |d-c| + \left| \lambda c + (1-\lambda) d - \frac{c+d}{2} \right| \right] \left| \bigvee_c^d (f') \right| \end{aligned}$$

where $c, d \in (a,b)$, which is equivalent to

$$\left| \frac{f(d) - f(c)}{d-c} - f'(\lambda c + (1-\lambda) d) \right| \le \left[\frac{1}{2} + \left| \lambda - \frac{1}{2} \right| \right] \left| \bigvee_c^d (f') \right| \tag{2.8}$$

for any $c, d \in (a,b)$, $c \neq d$.

Using (2.8), we may write

$$\begin{aligned} &\left| \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau - \frac{1}{\pi} PV \int_a^b f'(\lambda t + (1-\lambda)\tau) d\tau \right| \\ &\qquad \le \frac{1}{\pi} \left[\frac{1}{2} + \left| \lambda - \frac{1}{2} \right| \right] PV \int_a^b \left| \bigvee_\tau^t (f') \right| dt \\ &\qquad = \frac{1}{\pi} \left[\frac{1}{2} + \left| \lambda - \frac{1}{2} \right| \right] \left[\int_a^t \left(\bigvee_\tau^t (f') \right) dt + \int_t^b \left(\bigvee_t^\tau (f') \right) dt \right] \\ &\qquad \le \frac{1}{\pi} \left[\frac{1}{2} + \left| \lambda - \frac{1}{2} \right| \right] \left[(t-a) \bigvee_a^t (f') + (b-t) \bigvee_t^b (f') \right] \\ &\qquad \le \frac{1}{\pi} \left[\frac{1}{2} + \left| \lambda - \frac{1}{2} \right| \right] \left[\frac{1}{2}(b-a) + \left| t - \frac{a+b}{2} \right| \right] \bigvee_a^b (f'). \end{aligned} \tag{2.9}$$

Since (for $\lambda \neq 1$)

$$\begin{aligned} &\frac{1}{\pi} PV \int_a^b f'(\lambda t + (1-\lambda)\tau) d\tau \\ &\qquad = \frac{1}{\pi} \lim_{\varepsilon \to 0+} \left[\int_a^{t-\varepsilon} + \int_{t+\varepsilon}^b \right] (f'(\lambda t + (1-\lambda)\tau) d\tau) \\ &\qquad = \frac{1}{\pi} \lim_{\varepsilon \to 0+} \left[\frac{1}{1-\lambda} f(\lambda t + (1-\lambda)\tau) \Big|_a^{t-\varepsilon} + \frac{1}{1-\lambda} f(\lambda t + (1-\lambda)\tau) \Big|_{t+\varepsilon}^b \right] \\ &\qquad = \frac{1}{\pi} \cdot \frac{f(t) - f(\lambda t + (1-\lambda) a) + f(\lambda t + (1-\lambda) b) - f(t)}{1-\lambda} \\ &\qquad = \frac{b-a}{\pi} [f; \lambda t + (1-\lambda) b, \lambda t + (1-\lambda) a]. \end{aligned}$$

Using (2.9) and (2.7), we deduce the desired result (2.6). □

It is obvious that the best inequality we can get from (2.6) is the one for $\lambda = \frac{1}{2}$. Thus, we may state the following corollary.

Corollary 2.4. *With the assumptions of Theorem 2.3, we have*

$$\left|(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)-\frac{b-a}{\pi}\left[f;\frac{t+b}{2},\frac{a+t}{2}\right]\right| \leq \frac{1}{2\pi}\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]\bigvee_a^b(f'). \tag{2.10}$$

The above Theorem 2.3 may be used to point out some interesting inequalities for the functions for which the finite Hilbert transforms $(Tf)(a,b;t)$ can be expressed in terms of special functions.

For instance, we have:

1) Assume that $f:[a,b]\subset(0,\infty)\to\mathbb{R}$, $f(x)=\frac{1}{x}$. Then

$$(Tf)(a,b;t)=\frac{1}{\pi t}\ln\left[\frac{(b-t)a}{(t-a)b}\right],\quad t\in(a,b),$$

$$\frac{b-a}{\pi}\cdot[f;\lambda t+(1-\lambda)b,\lambda t+(1-\lambda)a]=-\frac{1}{\pi}\cdot\frac{b-a}{[\lambda t+(1-\lambda)b][\lambda t+(1-\lambda)a]},$$

$$\bigvee_a^b(f')=\int_a^b|f''(t)|\,dt=\frac{b^2-a^2}{a^2b^2}.$$

Using the inequality (2.6) we may write that

$$\left|\frac{1}{\pi t}\ln\left[\frac{(b-t)a}{(t-a)b}\right]-\frac{1}{\pi t}\ln\left(\frac{b-t}{t-a}\right)+\frac{b-a}{\pi[\lambda t+(1-\lambda)b][\lambda t+(1-\lambda)a]}\right| \leq \frac{1}{\pi}\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]\cdot\frac{b^2-a^2}{a^2b^2}$$

which is equivalent to:

$$\left|\frac{b-a}{[\lambda t+(1-\lambda)b][\lambda t+(1-\lambda)a]}-\frac{1}{t}\ln\left(\frac{b}{a}\right)\right| \leq \left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]\cdot\frac{b^2-a^2}{a^2b^2}. \tag{2.11}$$

If we use the notations

$$L(a,b):=\frac{b-a}{\ln b-\ln a}\quad\text{(the logarithmic mean)}$$

$$A_\lambda(x,y):=\lambda x+(1-\lambda)y\quad\text{(the weighted arithmetic mean)}$$

$$G(a,b):=\sqrt{ab}\quad\text{(the geometric mean)}$$

$$A(a,b):=\frac{a+b}{2}\quad\text{(the arithmetic mean)}$$

then by (2.11) we deduce

$$\left|\frac{1}{A_\lambda(t,b)A_\lambda(t,a)}-\frac{1}{tL(a,b)}\right| \leq \left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]\left[\frac{1}{2}(b-a)+|t-A(a,b)|\right]\frac{2A(a,b)}{G^4(a,b)},$$

giving the following proposition:

Proposition 2.5. *With the above assumption, we have*

$$\begin{aligned}(2.12)\quad & \left|tL(a,b)-A_{\lambda}(t,b)A_{\lambda}(t,a)\right| \\ & \le \frac{2A(a,b)}{G^4(a,b)}\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]tA_{\lambda}(t,b)A_{\lambda}(t,a)L(a,b)\end{aligned}$$

for any $t\in(a,b)$, $\lambda\in[0,1)$.

In particular, for $t=A(a,b)$ and $\lambda=\frac{1}{2}$, we get

$$\begin{aligned}(2.13)\quad & \left|A(a,b)L(a,b)-\frac{(A(a,b)+a)(A(a,b)+b)}{4}\right| \\ & \le \frac{1}{2}\cdot\frac{A^2(a,b)}{G^4(a,b)}\cdot\frac{(A(a,b)+a)(A(a,b)+b)}{4}L(a,b).\end{aligned}$$

2) Assume that $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$, $f(x)=\exp(x)$. Then

$$(Tf)(a,b;t)=\frac{\exp(t)}{\pi}\left[Ei(b-t)-Ei(a-t)\right],$$

where

$$Ei(z):=PV\int_{-\infty}^{z}\frac{\exp(t)}{t}dt,\qquad z\in\mathbb{R}.$$

Also, we have:

$$\begin{aligned}&\frac{b-a}{\pi}\left[\exp;\lambda t+(1-\lambda)b,\lambda t+(1-\lambda)a\right] \\ &\qquad=\frac{1}{\pi}\cdot\frac{\exp(\lambda t+(1-\lambda)b)-\exp(\lambda t+(1-\lambda)a)}{1-\lambda},\end{aligned}$$

$$\bigvee_a^b(f')=\int_a^b|f''(t)|\,dt=\exp(b)-\exp(a).$$

Using the inequality (2.6) we may write:

$$\begin{aligned}(2.14)\quad & \left|\exp(t)\left[Ei(b-t)-Ei(a-t)-\ln\left(\frac{b-t}{t-a}\right)\right]\right. \\ & \qquad\left.-\frac{\exp(\lambda t+(1-\lambda)b)-\exp(\lambda t+(1-\lambda)a)}{1-\lambda}\right| \\ & \le\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]\left[\exp(b)-\exp(a)\right]\end{aligned}$$

for any $t\in(a,b)$.

If in (2.14) we make $\lambda=\frac{1}{2}$ and $t=\frac{a+b}{2}$, we get

$$\begin{aligned}&\left|\exp\left(\frac{a+b}{2}\right)\mathrm{Ei}\left(\frac{b-a}{2}\right)-2\left[\exp\left(\frac{a+3b}{4}\right)-\exp\left(\frac{3a+b}{4}\right)\right]\right| \\ &\qquad\le\frac{1}{4}(b-a)\left[\exp(b)-\exp(a)\right],\end{aligned}$$

which is equivalent to:

$$\left|Ei\left(\frac{b-a}{2}\right)-2\left[\exp\left(\frac{b-a}{4}\right)-\exp\left(-\frac{b-a}{4}\right)\right]\right| \\ \leq \frac{1}{4}(b-a)\left[\exp\left(\frac{b-a}{2}\right)-\exp\left(-\frac{b-a}{2}\right)\right].$$

If in this inequality we make $\frac{b-a}{2}=z>0$, then we get

$$\left|Ei(z)-2\left[\exp\left(\frac{z}{2}\right)-\exp\left(-\frac{z}{2}\right)\right]\right| \leq \frac{1}{2}z\left[\exp(z)-\exp(-z)\right] \tag{2.15}$$

for any $z>0$.

Consequently, we may state the following proposition.

Proposition 2.6. *With the above assumptions, we have*

$$\left|Ei(z)-4\sinh\left(\frac{1}{2}z\right)\right| \leq z\sinh(z) \tag{2.16}$$

for any $z>0$.

The reader may get other similar inequalities for special functions if appropriate examples of functions f are chosen.

3. A QUADRATURE FORMULA FOR EQUIDISTANT DIVISIONS

The following lemma is of interest in itself.

Lemma 3.1. *Let* $u:[a,b]\to\mathbb{R}$ *be a function of bounded variation on* $[a,b]$*. Then for all* $n\geq 1$*,* $\lambda_i\in[0,1)$ $(i=0,\dots,n-1)$ *and* $t,\tau\in[a,b]$ *with* $t\neq\tau$*, we have the inequality:*

$$\left|\frac{1}{\tau-t}\int_t^\tau u(s)\,ds-\frac{1}{n}\sum_{i=0}^{n-1}u\left[t+(i+1-\lambda_i)\frac{\tau-t}{n}\right]\right| \\ \leq \frac{1}{n}\left[\frac{1}{2}+\max_{i=\overline{0,n-1}}\left|\lambda_i-\frac{1}{2}\right|\right]\left|\bigvee_t^\tau(u)\right|. \tag{3.1}$$

Proof. Consider the equidistant division of $[t,\tau]$ (if $t<\tau$) or $[\tau,t]$ (if $\tau<t$) given by

$$E_n: x_i=t+i\cdot\frac{\tau-t}{n},\quad i=\overline{0,n}. \tag{3.2}$$

Then the points $\xi_i=\lambda_i\left[t+i\cdot\frac{\tau-t}{n}\right]+(1-\lambda_i)\left[t+(i+1)\cdot\frac{\tau-t}{n}\right]$ $\left(\lambda_i\in[0,1],\ i=\overline{0,n-1}\right)$ are between x_i and x_{i+1}. We observe that we may write for simplicity $\xi_i=t+(i+1-\lambda_i)\frac{\tau-t}{n}$ $\left(i=\overline{0,n-1}\right)$. We also have

$$\xi_i-\frac{x_i+x_{i+1}}{2}=\frac{\tau-t}{2n}(1-2\lambda_i),$$

$$\xi_i-x_i=(1-\lambda_i)\frac{\tau-t}{n}$$

and

$$x_{i+1}-\xi_i=\lambda_i\cdot\frac{\tau-t}{n}$$

for any $i=\overline{0,n-1}$.

If we apply the inequality (2.1) on the interval $[x_i, x_{i+1}]$ and the intermediate point ξ_i $\left(i = \overline{0, n-1}\right)$, then we may write that

$$\left|\frac{\tau - t}{n} u\left(t + (i + 1 - \lambda_i)\frac{\tau - t}{n}\right) - \int_{x_i}^{x_{i+1}} u(s)\,ds\right| \leq \left[\frac{1}{2} \cdot \frac{|\tau - t|}{n} + \left|\frac{\tau - t}{2n}(1 - 2\lambda_i)\right|\right] \left|\bigvee_{x_i}^{x_{i+1}}(u)\right|. \tag{3.3}$$

Summing, we get

$$\begin{aligned}\left|\int_t^\tau u(s)\,ds - \frac{\tau - t}{n}\sum_{i=0}^{n-1} u\left[t + (i + 1 - \lambda_i)\frac{\tau - t}{n}\right]\right| &\leq \frac{|\tau - t|}{2n}\sum_{i=0}^{n-1}[1 + |1 - 2\lambda_i|]\left|\bigvee_{x_i}^{x_{i+1}}(u)\right| \\ &= \frac{|\tau - t|}{n}\left[\frac{1}{2} + \max_{i=\overline{0,n-1}}\left|\lambda_i - \frac{1}{2}\right|\right]\left|\bigvee_t^\tau(u)\right|,\end{aligned}$$

which is equivalent to (3.1). □

We may now state the following theorem in approximating the finite Hilbert transform of a differentiable function with the derivative of bounded variation on $[a, b]$.

Theorem 3.2. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function such that its derivative f' is of bounded variation on $[a, b]$. If $\lambda = (\lambda_i)_{i=\overline{0,n-1}}$, $\lambda_i \in [0, 1)$ $\left(i = \overline{0, n-1}\right)$ and*

$$S_n(f; \lambda, t) := \frac{b - a}{\pi n}\sum_{i=0}^{n-1}\left[f; (i + 1 - \lambda_i)\frac{b - t}{n} + t, (i + 1 - \lambda_i)\frac{a - t}{n} + t\right], \tag{3.4}$$

then we have the estimate:

$$\begin{aligned}\left|(Tf)(a, b; t) - \frac{f(t)}{\pi}\ln\left(\frac{b - t}{t - a}\right) - S_n(f; \lambda, t)\right| &\leq \frac{b - a}{n\pi}\left[\frac{1}{2} + \max_{i=\overline{0,n-1}}\left|\lambda_i - \frac{1}{2}\right|\right]\left[\frac{1}{2}(b - a) + \left|t - \frac{a + b}{2}\right|\right]\bigvee_a^b(f') \\ &\leq \frac{b - a}{n\pi}\bigvee_a^b(f').\end{aligned} \tag{3.5}$$

Proof. Applying Lemma 3.1 for the function f', we may write that

$$\left|\frac{f(\tau) - f(t)}{\tau - t} - \frac{1}{n}\sum_{i=0}^{n-1} f'\left[t + (i + 1 - \lambda_i)\frac{\tau - t}{n}\right]\right| \leq \frac{1}{n}\left[\frac{1}{2} + \max_{i=\overline{0,n-1}}\left|\lambda_i - \frac{1}{2}\right|\right]\left|\bigvee_t^\tau(f')\right| \tag{3.6}$$

for any $t, \tau \in [a, b]$, $t \neq \tau$.

Consequently, we have

$$\left|\frac{1}{\pi}PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}d\tau-\frac{1}{\pi n}\sum_{i=0}^{n-1}PV\int_a^b f'\left[t+(i+1-\lambda_i)\frac{\tau-t}{n}\right]d\tau\right| \tag{3.7}$$
$$\le\frac{1}{n\pi}\left[\frac{1}{2}+\max_{i=\overline{0,n-1}}\left|\lambda_i-\frac{1}{2}\right|\right]PV\int_a^b\left|\bigvee_t^\tau(f')\right|d\tau$$
$$\le\frac{1}{n\pi}\left[\frac{1}{2}+\max_{i=\overline{0,n-1}}\left|\lambda_i-\frac{1}{2}\right|\right]\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]\bigvee_a^b(f').$$

On the other hand

$$PV\int_a^b f'\left[t+(i+1-\lambda_i)\frac{\tau-t}{n}\right]d\tau \tag{3.8}$$
$$=\lim_{\varepsilon\to0+}\left[\int_a^{t-\varepsilon}+\int_{t+\varepsilon}^b\right]\left(f'\left[t+(i+1-\lambda_i)\frac{\tau-t}{n}\right]d\tau\right)$$
$$=\lim_{\varepsilon\to0+}\left[\frac{n}{i+1-\lambda_i}f\left(t+(i+1-\lambda_i)\frac{\tau-t}{n}\right)\Big|_a^{t-\varepsilon}\right.$$
$$\left.+\frac{n}{i+1-\lambda_i}f\left(t+(i+1-\lambda_i)\frac{\tau-t}{n}\right)\Big|_{t+\varepsilon}^b\right]$$
$$=\frac{n}{i+1-\lambda_i}\left[f\left(t+(i+1-\lambda_i)\frac{b-t}{n}\right)-f\left(t+(i+1-\lambda_i)\frac{a-t}{n}\right)\right]$$
$$=(b-a)\left[f;t+(i+1-\lambda_i)\frac{b-t}{n},(i+1-\lambda_i)\frac{a-t}{n}+t\right].$$

Since (see for example (2.7)),

$$(Tf)(a,b;t)=\frac{1}{\pi}PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}d\tau+\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)$$

for $t\in(a,b)$, then by (3.7) and (3.8) we deduce the desired estimate (3.5). □

Remark 3.3. For $n=1$, we recapture the inequality (2.6).

Corollary 3.4. *With the assumptions of Theorem 3.2, we have*

$$(Tf)(a,b;t)=\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)+\lim_{n\to\infty}S_n(f;\lambda,t) \tag{3.9}$$

uniformly by rapport of $t\in(a,b)$ *and* λ *with* $\lambda_i\in[0,1)$ $(i\in\mathbb{N})$.

Remark 3.5. If one needs to approximate the finite Hilbert Transform $(Tf)(a,b;t)$ in terms of

$$\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)+S_n(f;\lambda,t)$$

with the accuracy $\varepsilon>0$ (ε small), then the theoretical minimal number n_ε to be chosen is:

$$n_\varepsilon:=\left[\frac{b-a}{\varepsilon\pi}\bigvee_a^b(f')\right]+1 \tag{3.10}$$

where $[\alpha]$ is the integer part of α.

It is obvious that the best inequality we can get in (3.5) is for $\lambda_i = \frac{1}{2}\ \left(i = \overline{0, n-1}\right)$ obtaining the following corollary.

Corollary 3.6. *Let f be as in Theorem 3.2. Define*

$$M_n(f;t) := \frac{b-a}{\pi n} \sum_{i=0}^{n-1} \left[f; \left(i + \frac{1}{2}\right) \frac{b-t}{n} + t, \left(i + \frac{1}{2}\right) \frac{a-t}{n} + t \right]. \tag{3.11}$$

Then we have the estimate

$$\left| (Tf)(a,b;t) - \frac{f(t)}{\pi} \ln\left(\frac{b-t}{t-a}\right) - M_n(f;t) \right| \leq \frac{b-a}{2n\pi} \left[\frac{1}{2}(b-a) + \left| t - \frac{a+b}{2} \right| \right] \bigvee_a^b (f') \tag{3.12}$$

for any $t \in (a,b)$.

This rule will be numerically implemented in Section 5 for different choices of f and n.

4. A More General Quadrature Formula

We may state the following lemma.

Lemma 4.1. *Let $u : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$, $0 = \mu_0 < \mu_1 < \cdots < \mu_{n-1} < \mu_n = 1$ and $\nu_i \in [\mu_i, \mu_{i+1}]$, $i = \overline{0, n-1}$. Then for any $t, \tau \in [a,b]$ with $t \neq \tau$, we have the inequality:*

$$\left| \frac{1}{\tau - t} \int_t^\tau u(s)\, ds - \sum_{i=0}^{n-1} (\mu_{i+1} - \mu_i)\, u[(1-\nu_i)t + \nu_i \tau] \right| \leq \left[\frac{1}{2}\Delta_n(\mu) + \max_{i=\overline{0,n-1}} \left| \nu_i - \frac{\mu_i + \mu_{i+1}}{2} \right| \right] \left| \bigvee_t^\tau (u) \right|, \tag{4.1}$$

where $\Delta_n(\mu) := \max_{i=\overline{0,n-1}} (\mu_{i+1} - \mu_i)$.

Proof. Consider the division of $[t,\tau]$ (if $t < \tau$) or $[\tau, t]$ (if $\tau < t$) given by

$$I_n : x_i := (1-\mu_i)t + \mu_i \tau \quad \left(i = \overline{0,n}\right). \tag{4.2}$$

Then the points $\xi_i := (1-\nu_i)t + \nu_i\tau \ \left(i = \overline{0, n-1}\right)$ are between x_i and x_{i+1}. We have

$$x_{i+1} - x_i = (\mu_{i+1} - \mu_i)(\tau - t) \quad \left(i = \overline{0, n-1}\right)$$

and

$$\xi_i - \frac{x_i + x_{i+1}}{2} = \left(\nu_i - \frac{\mu_i + \mu_{i+1}}{2}\right)(\tau - t) \quad \left(i = \overline{0, n-1}\right).$$

Applying the inequality (2.1) on $[x_i, x_{i+1}]$ with the intermediate points ξ_i $\left(i = \overline{0, n-1}\right)$, we get

$$\left| \int_{x_i}^{x_{i+1}} u(s)\, ds - (\mu_{i+1} - \mu_i)(\tau - t)\, u[(1-\nu_i)t + \nu_i\tau] \right| \leq \left[\frac{1}{2}(\mu_{i+1} - \mu_i)|\tau - t| + |\tau - t| \left| \nu_i - \frac{\mu_i + \mu_{i+1}}{2} \right| \right] \left| \bigvee_{x_i}^{x_{i+1}} (u) \right|$$

for any $i=\overline{0,n-1}$. Summing over i, using the generalised triangle inequality and dividing by $|t-\tau|>0$, we obtain

$$\left|\frac{1}{\tau-t}\int_a^b u(s)\,ds-\sum_{i=0}^{n-1}(\mu_{i+1}-\mu_i)\,u\left[(1-\nu_i)\,t+\nu_i\tau\right]\right|$$
$$\le\sum_{i=0}^{n-1}\left[\frac{1}{2}(\mu_{i+1}-\mu_i)+\left|\nu_i-\frac{\mu_i+\mu_{i+1}}{2}\right|\right]\left|\bigvee_{x_i}^{x_{i+1}}(u)\right|$$
$$\le\left[\frac{1}{2}\Delta_n(\mu)+\max_{i=\overline{0,n-1}}\left|\nu_i-\frac{\mu_i+\mu_{i+1}}{2}\right|\right]\left|\bigvee_{t}^{\tau}(u)\right|$$

and the inequality (4.1) is proved. □

The following theorem holds.

Theorem 4.2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function such that its derivative f' is of bounded variation on $[a,b]$. If $0=\mu_0<\mu_1<\cdots<\mu_{n-1}<\mu_n=1$ and $\nu_i\in[\mu_i,\mu_{i+1}]$, $\left(i=\overline{0,n-1}\right)$, then*

$$(Tf)(a,b;t)=\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)+\frac{1}{\pi}Q_n(\mu,\nu,t)+W_n(\mu,\nu,t) \tag{4.3}$$

for any $t\in(a,b)$, where

$$Q_n(\mu,\nu,t):=\mu_1 f'(t)(b-a)+(b-a)\sum_{i=1}^{n-2}\Big\{(\mu_{i+1}-\mu_i)\times\left[f;(1-\nu_i)\,t+\nu_i b,(1-\nu_i)\,t+\nu_i a\right]\Big\}+(1-\mu_{n-1})\left[f(b)-f(a)\right] \tag{4.4}$$

if $\nu_0=0$, $\nu_{n-1}=1$,

$$Q_n(\mu,\nu,t):=\mu_1 f'(t)(b-a)+(b-a)\sum_{i=1}^{n-1}(\mu_{i+1}-\mu_i)\times\left[f;(1-\nu_i)\,t+\nu_i b,(1-\nu_i)\,t+\nu_i a\right] \tag{4.5}$$

if $\nu_0=0$, $\nu_{n-1}<1$,

$$Q_n(\mu,\nu,t):=(b-a)\sum_{i=1}^{n-2}(\mu_{i+1}-\mu_i)\times\left[f;(1-\nu_i)\,t+\nu_i b,(1-\nu_i)\,t+\nu_i a\right]+(1-\mu_{n-1})\left[f(b)-f(a)\right] \tag{4.6}$$

if $\nu_0>0$, $\nu_{n-1}=1$ and

$$Q_n(\mu,\nu,t):=(b-a)\sum_{i=1}^{n-1}(\mu_{i+1}-\mu_i)\left[f;(1-\nu_i)\,t+\nu_i b,(1-\nu_i)\,t+\nu_i a\right] \tag{4.7}$$

if $\nu_0 > 0$, $\nu_{n-1} < 1$.
In all cases, the remainder satisfies the estimate:

$$
\begin{aligned}
(4.8)\qquad |W_n(\mu,\nu,t)| &\le \frac{1}{\pi}\left[\frac{1}{2}\Delta_n(\mu) + \max_{i=\overline{0,n-1}}\left|\nu_i - \frac{\mu_i+\mu_{i+1}}{2}\right|\right] \\
&\quad\times\left[\frac{1}{2}(b-a) + \left|t-\frac{a+b}{2}\right|\right]\bigvee_a^b(f') \\
&\le \frac{1}{\pi}\Delta_n(\mu)\left[\frac{1}{2}(b-a) + \left|t-\frac{a+b}{2}\right|\right]\left|\bigvee_a^b(f')\right| \\
&\le \frac{1}{\pi}\Delta_n(\mu)(b-a)\bigvee_a^b(f').
\end{aligned}
$$

Proof. If we apply Lemma 4.1 for the function f', we may write that

$$
\begin{aligned}
&\left|\frac{f(\tau)-f(t)}{\tau-t} - \sum_{i=0}^{n-1}(\mu_{i+1}-\mu_i) f'[(1-\nu_i)t+\nu_i\tau]\right| \\
&\qquad\le \left[\frac{1}{2}\Delta_n(\mu) + \max_{i=\overline{0,n-1}}\left|\nu_i - \frac{\mu_i+\mu_{i+1}}{2}\right|\right]\left|\bigvee_t^\tau(f')\right|
\end{aligned}
$$

for any $t,\tau\in[a,b]$, $t\neq\tau$.

Taking the PV in both sides, we may write that

$$
\begin{aligned}
(4.9)\qquad &\left|\frac{1}{\pi}PV\int_a^b \frac{f(\tau)-f(t)}{\tau-t}d\tau \right.\\
&\qquad\left.-\frac{1}{\pi}PV\int_a^b\left(\sum_{i=0}^{n-1}(\mu_{i+1}-\mu_i) f'[(1-\nu_i)t+\nu_i\tau]\right)d\tau\right| \\
&\qquad\le \frac{1}{\pi}\left[\frac{1}{2}\Delta_n(\mu) + \max_{i=\overline{0,n-1}}\left|\nu_i - \frac{\mu_i+\mu_{i+1}}{2}\right|\right] PV\int_a^b\left|\bigvee_t^\tau(f')\right|d\tau.
\end{aligned}
$$

If $\nu_0 = 0$, $\nu_{n-1} = 1$, then

$$
\begin{aligned}
&PV\int_a^b\left(\sum_{i=0}^{n-1}(\mu_{i+1}-\mu_i) f'[(1-\nu_i)t+\nu_i\tau]\right)d\tau \\
&= PV\int_a^b \mu_1 f'(t)\,d\tau + \sum_{i=1}^{n-2}(\mu_{i+1}-\mu_i)PV\int_a^b f'[(1-\nu_i)t+\nu_i\tau]\,d\tau \\
&\quad + (1-\mu_{n-1})PV\int_a^b f'(\tau)\,d\tau \\
&= \mu_1 f'(t)(b-a) + (b-a)\sum_{i=1}^{n-2}(\mu_{i+1}-\mu_i)[f;(1-\nu_i)t+\nu_i b,(1-\nu_i)t+\nu_i a] \\
&\quad + (1-\mu_{n-1})[f(b)-f(a)].
\end{aligned}
$$

If $\nu_0 = 0$, $\nu_{n-1} < 1$, then

$$PV \int_a^b \left(\sum_{i=0}^{n-1} (\mu_{i+1} - \mu_i) f' [(1-\nu_i) t + \nu_i \tau] \right) d\tau$$
$$= \mu_1 f'(t)(b-a) + (b-a) \sum_{i=1}^{n-1} (\mu_{i+1} - \mu_i) [f; (1-\nu_i) t + \nu_i b, (1-\nu_i) t + \nu_i a].$$

If $\nu_0 > 0$, $\nu_{n-1} = 1$, then

$$PV \int_a^b \left(\sum_{i=0}^{n-1} (\mu_{i+1} - \mu_i) f' [(1-\nu_i) t + \nu_i \tau] \right) d\tau$$
$$= (b-a) \sum_{i=1}^{n-2} (\mu_{i+1} - \mu_i) [f; (1-\nu_i) t + \nu_i b, (1-\nu_i) t + \nu_i a] + (1-\mu_{n-1}) [f(b) - f(a)].$$

and, finally, if $\nu_0 > 0$, $\nu_{n-1} < 1$, then

$$PV \int_a^b \left(\sum_{i=0}^{n-1} (\mu_{i+1} - \mu_i) f' [(1-\nu_i) t + \nu_i \tau] \right) d\tau$$
$$= (b-a) \sum_{i=1}^{n-1} (\mu_{i+1} - \mu_i) [f; (1-\nu_i) t + \nu_i b, (1-\nu_i) t + \nu_i a].$$

Since

$$PV \int_a^b \left| \bigvee_t^\tau (f') \right| d\tau \le \left[\frac{1}{2}(b-a) + \left| t - \frac{a+b}{2} \right| \right] \bigvee_a^b (f')$$

and

$$(Tf)(a,b;t) = \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau + \frac{f(t)}{\pi} \ln\left(\frac{b-t}{t-a} \right),$$

then by (4.9) we deduce (4.3). □

5. NUMERICAL EXPERIMENTS

For a function $f : [a,b] \to \mathbb{R}$, we may consider the quadrature formula

$$E_n(f;a,b,t) := \frac{f(t)}{\pi} \ln\left(\frac{b-t}{t-a} \right) + M_n(f;t), t \in [a,b].$$

As shown above in Section 4, $E_n(f;a,b,t)$ provides an approximation for the Finite Hilbert Transform $(Tf)(a,b;t)$ and the error estimate fulfils the bound described in (2.3).

If we consider the function $f : [-1,1] \to \mathbb{R}$, $f(x) = \exp(x)$, then the exact value of the Hilbert transform is

$$(Tf)(a,b;t) = \frac{\exp(t) Ei(1-t) - \exp(t) Ei(-1-t)}{\pi}, \ t \in [-1,1]$$

and the plot of this function is embodied in Figure 5.1.

If we implement the quadrature formula provided by $E_n(f;a,b,t)$ using Maple 6 and chose the value of $n = 100$, then the error $Er(f;a,b,t) := (Tf)(a,b;t) - E_n(f;a,b,t)$ has the variation described in the Figure 5.2.

For n =1,000, the plot of $Er(f;a,b,t)$ is embodied in the following Figure 5.3.

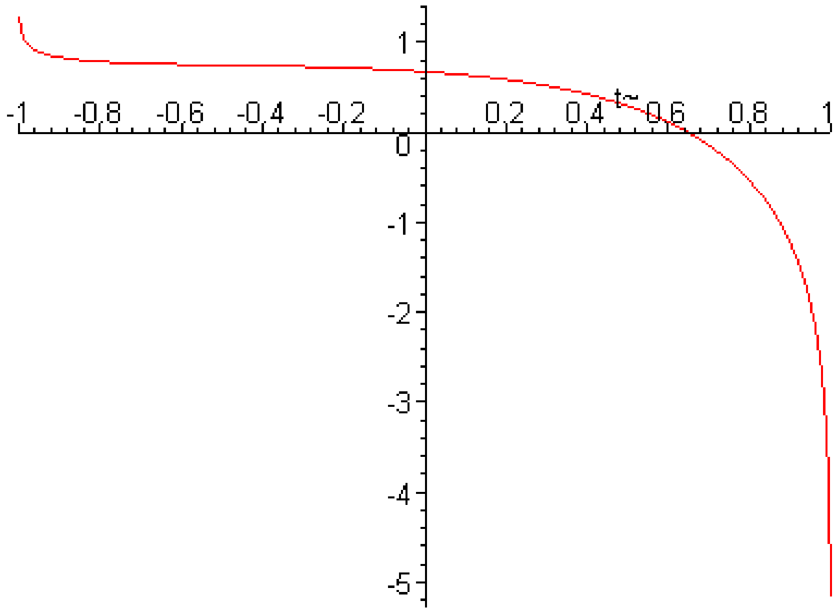

Figure 5.1:

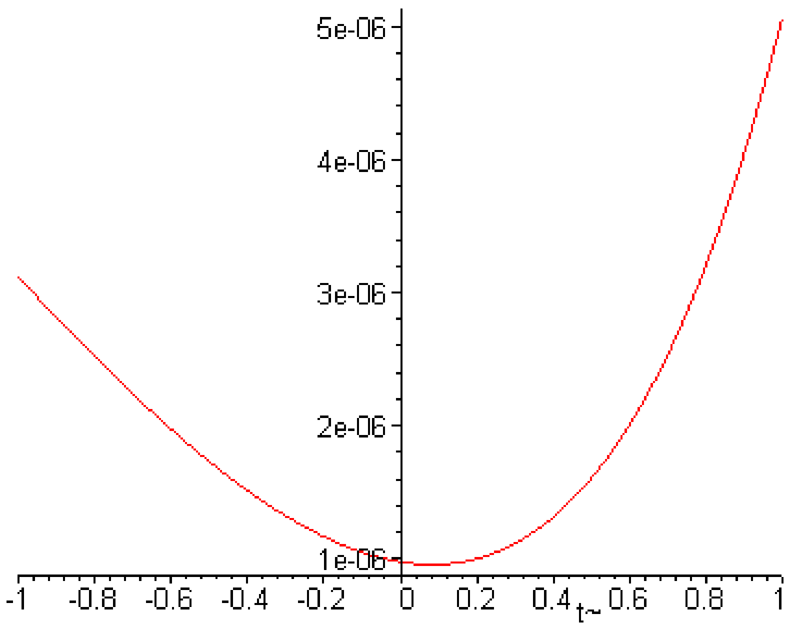

Figure 5.2:

Now, if we consider another function, $f : [-1, 1] \to \mathbb{R}, \ f(x) = \sin x$, then the exact value of the Hilbert transform is

$$(Tf)(a, b; t) = \frac{-Si(-1+t)\cos(t) + Ci(1-t)\sin(t)}{\pi} + \frac{Si(t+1)\cos(t) - \sin(t)Ci(t+1))}{\pi}, \ t \in [-1, 1];$$

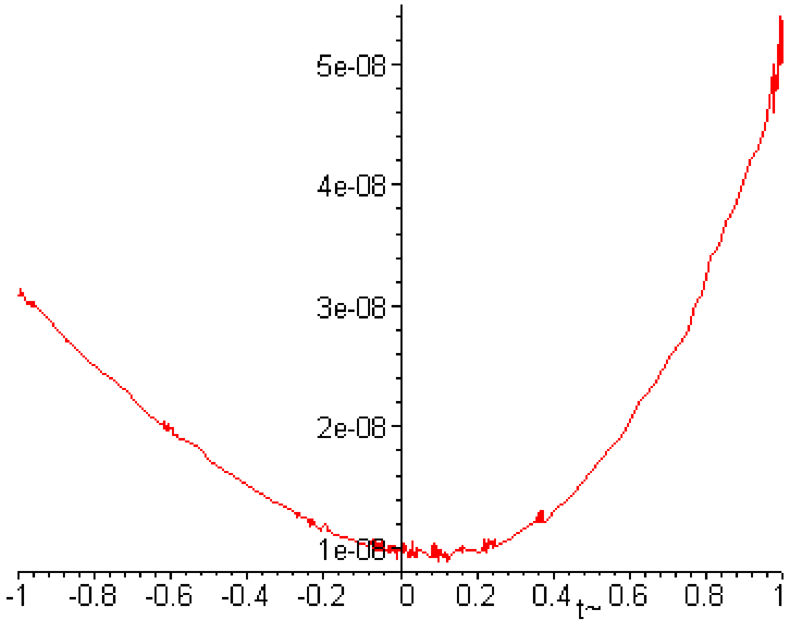

Figure 5.3:

where

$$Si(x) = \int_0^x \frac{\sin(t)}{t} dt, \qquad Ci(x) = \gamma + \ln x + \int_0^x \frac{\cos(t) - 1}{t} dt;$$

having the plot embodied in the following Figure 5.4.

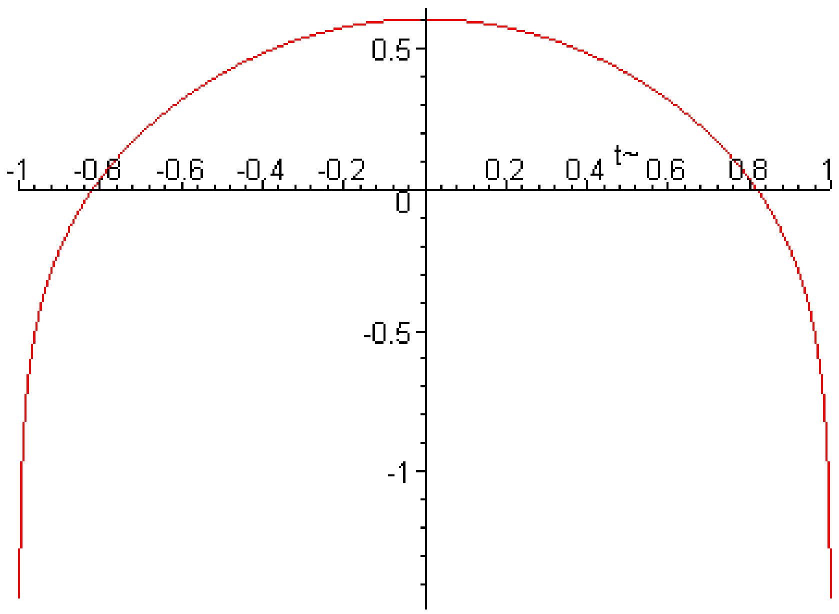

Figure 5.4:

If we choose the value of $n = 100$, then the error $Er(f; a, b, t)$ for the function $f(x) = \sin x, x \in [-1, 1]$ has the variation described in the Figure 5.5 below.

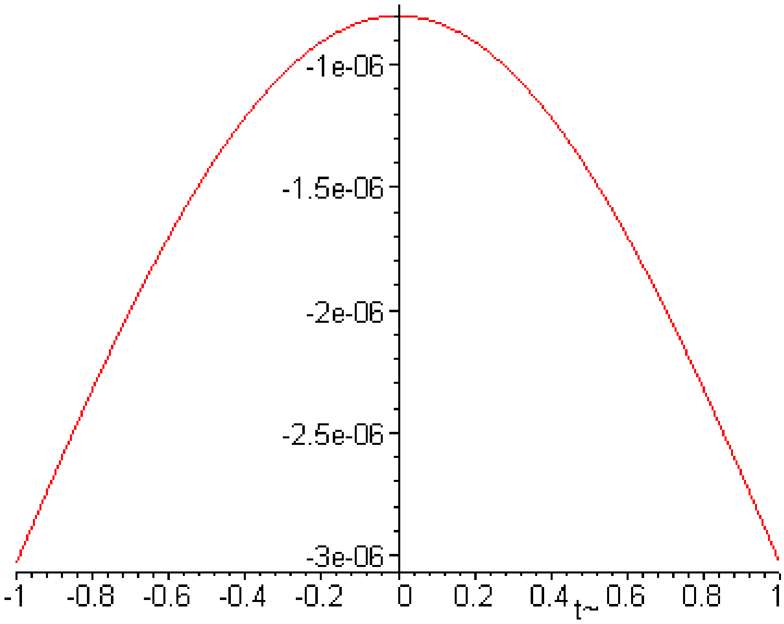

Figure 5.5:

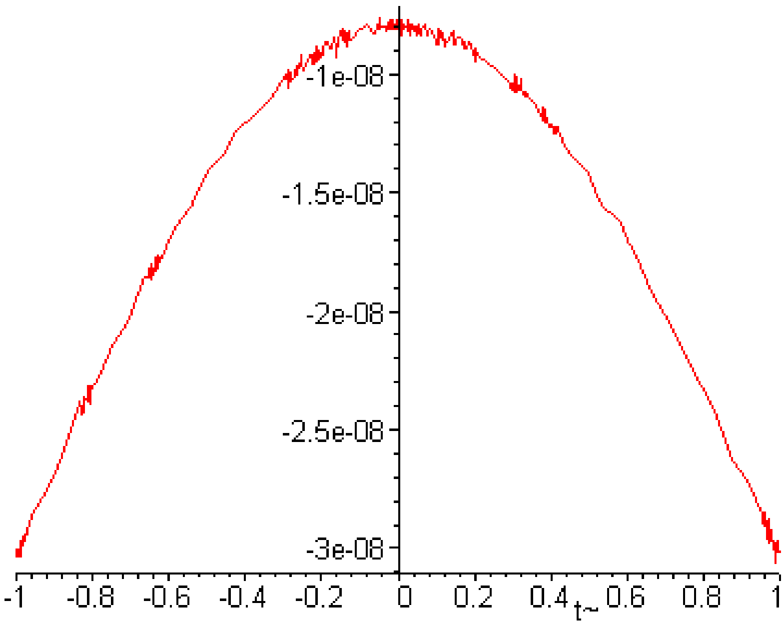

Figure 5.6:

Moreover, for n =100,000, the behaviour of $Er\,(f;a,b,t)$ is plotted in Figure 5.6.

Finally, if we choose the function $f:[-1,1]\to\mathbb{R},\ f(x)=\sin\left(x^2\right)$, the Maple 6 is unable to produce an exact value of the finite Hilbert transform. If we use our formula

$$E_n(f;a,b,t):=\frac{f\,(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)+M_n\,(f;t)\,,t\in[a,b]$$

for n =1,000, then we can produce the plot in Figure 5.7.

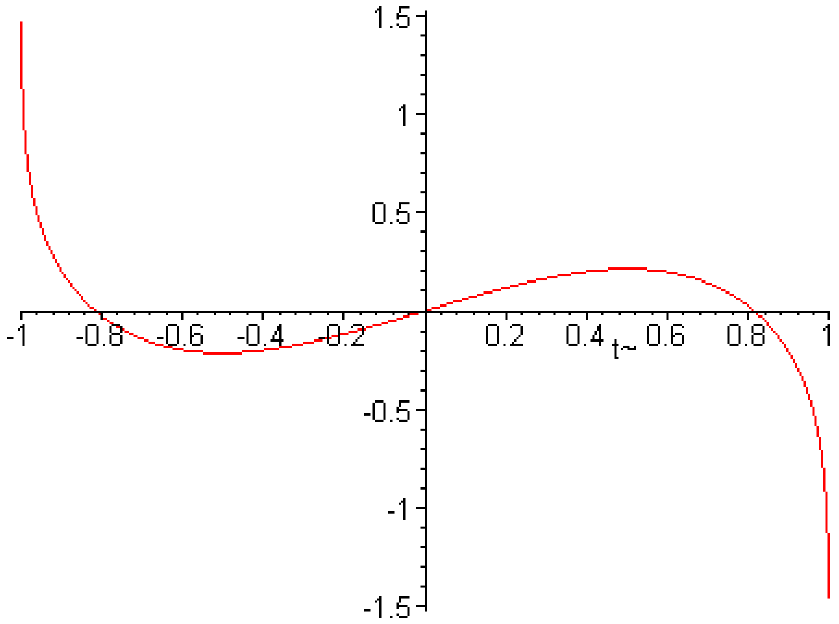

Figure 5.7:

Taking into account the bound (3.12) we know that the accuracy of the plot in Figure 5.7 is at least of order 10^{-5}.

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