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SOME REMARKS ON A PAPER BY A. MCD. MERCER

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Abstract

In this note we give a necessary and sufficient condition in order that an inequality established by A. McD. Mercer to be true for every convex sequence.

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Contents

1. Introduction

In [1] A. McD. Mercer proved the following result:

If the sequence $\{u_k\}$ is convex then

$$\sum_{k=0}^{n}\left[\frac{1}{n+1}-\frac{1}{2^n}\binom{n}{k}\right]u_k \geq 0. \tag{1.1}$$

In [2] this inequality was generalized to the following:

Suppose that the polynomial

$$\sum_{k=0}^{n} a_k x^k \tag{1.2}$$

has $x = 1$ as a double root and the coefficients c_k, $k = 0, 1, \ldots, n-2$ of the polynomial

$$\frac{\sum_{k=0}^{n} a_k x^k}{(x-1)^2} = \sum_{k=0}^{n-2} c_k x^k \tag{1.3}$$

are positive. Then

$$\sum_{k=0}^{n} a_k u_k \geq 0 \tag{1.4}$$

if the sequence $\{u_k\}$ is convex.

The aim of this note is to show that the inequality (1.4) holds for every convex sequence $\{u_k\}$ if and only if the polynomial given by (1.2) has $x = 1$ as a double root and the coefficients c_k $(k = 0, 1, \ldots, n-2)$ of the polynomial given by (1.3) are positive.

2. A Result of Tiberiu Popoviciu

Let n be a fixed natural number and

$$x_0 < x_1 < \cdots < x_n \tag{2.1}$$

$n+1$ distinct points on the real axis. We denote by S the linear subspace of the real functions defined on the set of the points (2.1). If $a_0, a_1, \dots, a_n$ are $n+1$ fixed real numbers we define the linear functional A, $A : S \to \mathbb{R}$ by

$$A(f) = \sum_{k=0}^{n} a_k f(x_k). \tag{2.2}$$

T. Popoviciu ([3]) proved the following results:

Theorem 2.1.

(a) The functional A is zero for every polynomial of degree at the most one if and only if there exist the constants $\alpha_0, \alpha_1, \dots, \alpha_{n-2}$ independent of the function f, such that the following equality holds:

$$A(f) = \sum_{k=0}^{n-2} \alpha_k [x_k, x_{k+1}, x_{k+2}; f], \tag{2.3}$$

where $[x_k, x_{k+1}, x_{k+2}; f]$ is divided difference of the function f.

(b) If there exists an index k $(0 \le k \le n-2)$ such that $\alpha_k \neq 0$, then

$$A(f) \ge 0, \tag{2.4}$$

for every convex function f if and only if

$$\alpha_i \ge 0, \quad i = 0, 1, \dots, n-2. \tag{2.5}$$

3. Main Result

Theorem 3.1. *Let* $a_0, a_1, \dots, a_n$ *be* $n+1$ *fixed real numbers such that* $\sum_{k=0}^{n} a_k^2 > 0$. *The inequality*

$$\sum_{k=0}^{n} a_k u_k \geq 0 \tag{3.1}$$

holds for every convex sequence $\{u_k\}$ *if and only if the polynomial given by (1.2) has* $x = 1$ *as a double root and all coefficients* c_k *of the polynomial given by (1.3) are positive.*

Proof. The sufficiency of the theorem was proved by A. Mc D. Mercer in [2].

We suppose that the inequality (3.1) is valid for every convex sequence. The sequences $\{1\}$, $\{-1\}$, $\{k\}$ and $\{-k\}$ are convex sequences. By (3.1) we get

$$\sum_{k=0}^{n} a_k = 0$$

(3.2)

$$\sum_{k=1}^{n} k a_k = 0.$$

We denote by f, $f : [0,1] \to \mathbb{R}$, the polygonal line having its vertices $\left(\frac{k}{n}, u_k\right)$, $k = 0, 1, \dots, n$.

The sequence $\{u_k\}$ is convex if and only if the function f is convex.

Let us denote by

$$A(f)=\sum_{k=0}^{n} a_k f\left(\frac{k}{n}\right).$$

The inequality (3.1) holds for every convex sequence $\{u_k\}$ if and only if

$$A(f)\geq 0 \tag{3.3}$$

for every function f which is convex on the set $\{0,\frac{1}{n},\dots,\frac{n}{n}\}$.

By (3.2) we have

$$A(P)=0$$

for every polynomial P having its degree at the most one. Using Popoviciu's Theorem 2.1, it follows that there exist the constants $\alpha_0,\alpha_1,\dots,\alpha_{n-2}$, independent of the function f such that

$$A(f)=\sum_{k=0}^{n-2}\alpha_k\left[\frac{k}{n},\frac{k+1}{n},\frac{k+2}{n};f\right], \tag{3.4}$$

for every function f defined of the set $\{0,\frac{1}{n},\dots,\frac{n}{n}\}$.

By the equality

$$\sum_{k=0}^{n}\alpha_k\left[\frac{k}{n},\frac{k+1}{n},\frac{k+2}{n};f\right]=\sum_{k=0}^{n} a_k f\left(\frac{k}{n}\right),$$

we get $\alpha_k=\frac{2}{n^2}c_k$, $k=0,1,\dots,n-2$.

Because $x = 1$ is a double root for the polynomial given by (1.2) we have

$$\sum_{k=0}^{n} c_k \neq 0.$$

Using again Popoviciu's Theorem (b), $A(f) \geq 0$ if and only if $c_k \geq 0$, $k = 0, \dots, n-2$, and our theorem is proved. $\square$

4. Another Proof of (1.1)

Let us consider the Bernstein operator B_n,

$$B_n(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \tag{4.1}$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $k = 0, 1, \dots, n$.

It is well known that for every convex function f, B_n is a convex function too. For such a function, we have, by Jensen's inequality,

$$\int_0^1 B_n(f)(x)dx \geq B_n(f)\left(\frac{1}{2}\right). \tag{4.2}$$

On the other hand we have

$$\int_0^1 p_{n,k}(x)dx = \frac{1}{n+1}, \tag{4.3}$$

$$p_{n,k}\left(\frac{1}{2}\right) = \binom{n}{k}\frac{1}{2^n}, \quad k = 0, 1, \dots, n.$$

Now, the inequality (1.1) follows by (4.2) and (4.3).

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