COEFFICIENT BOUNDS FOR MEROMORPHIC STARLIKE AND CONVEX FUNCTIONS

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Abstract: In this paper, some subclasses of meromorphic univalent functions in the unit disk Δ are extended. Let $U(p)$ denote the class of normalized univalent meromorphic functions f in Δ with a simple pole at $z = p > 0$. Let ϕ be a function with positive real part on Δ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps Δ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. The class $\sum^*(p, w_0, \phi)$ consists of functions $f \in U(p)$ satisfying

$$-\left(\frac{zf'(z)}{f(z)-w_0} + \frac{p}{z-p} - \frac{pz}{1-pz}\right) \prec \phi(z).$$

The class $\sum(p, \phi)$ consists of functions $f \in U(p)$ satisfying

$$-\left(1 + z\frac{f''(z)}{f'(z)} + \frac{2p}{z-p} - \frac{2pz}{1-pz}\right) \prec \phi(z).$$

The bounds for w_0 and some initial coefficients of f in $\sum^*(p, w_0, \phi)$ and $\sum(p, \phi)$ are obtained.

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1. Introduction

Let $U(p)$ denote the class of univalent meromorphic functions f in the unit disk Δ with a simple pole at $z = p > 0$ and with the normalization $f(0) = 0$ and $f'(0) = 1$. Let $U^*(p, w_0)$ be the subclass of $U(p)$ such that $f(z) \in U^*(p, w_0)$ if and only if there is a ρ, $0 < \rho < 1$, with the property that

$$\Re \frac{zf'(z)}{f(z) - w_0} < 0$$

for $\rho < |z| < 1$. The functions in $U^*(p, w_0)$ map $|z| < r < \rho$ (for some ρ, $p < \rho < 1$) onto the complement of a set which is starlike with respect to w_0. Further the functions in $U^*(p, w_0)$ all omit the value w_0. This class of starlike meromorphic functions is developed from Robertson's concept of star center points [11]. Let $\mathcal{P}$ denote the class of functions $P(z)$ which are meromorphic in Δ and satisfy $P(0) = 1$ and $\Re\{P(z)\} \geq 0$ for all $z \in \Delta$.

For $f(z) \in U^*(p, w_0)$, there is a function $P(z) \in \mathcal{P}$ such that

$$z\frac{f'(z)}{f(z) - w_0} + \frac{p}{z - p} - \frac{pz}{1 - pz} = -P(z) \tag{1.1}$$

for all $z \in \Delta$. Let $\sum^*(p, w_0)$ denote the class of functions $f(z)$ which satisfy (1.1) and the condition $f(0) = 0$, $f'(0) = 1$. Then $U^*(p, w_0)$ is a subset of $\sum^*(p, w_0)$. Miller [9] proved that $U^*(p, w_0) = \sum^*(p, w_0)$ for $p \leq 2 - \sqrt{3}$.

Let $K(p)$ denote the class of functions which belong to $U(p)$ and map $|z| < r < \rho$ (for some $p < \rho < 1$) onto the complement of a convex set. If $f \in K(p)$, then there is a $p < \rho < 1$, such that for each z, $\rho < |z| < 1$

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \leq 0.$$

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If $f \in K(p)$, then for each z in Δ,

$$\Re\left\{1 + z\frac{f''(z)}{f'(z)} + \frac{2p}{z-p} - \frac{2pz}{1-pz}\right\} \leq 0. \tag{1.2}$$

Let $\sum(p)$ denote the class of functions f which satisfy (1.2) and the conditions $f(0) = 0$ and $f'(0) = 1$. The class $K(p)$ is contained in $\sum(p)$. Royster [12] showed that for $0 < p \leq 2 - \sqrt{3}$, if $f \in \sum(p)$ and is meromorphic, then $f \in K(p)$. Also, for each function $f \in \sum(p)$, there is a function $P \in \mathcal{P}$ such that

$$1 + z\frac{f''(z)}{f'(z)} + \frac{2p}{z-p} - \frac{2pz}{1-pz} = -P(z).$$

The class $U(p)$ and related classes have been studied in [3], [4], [5] and [6].

Let $\mathcal{A}$ be the class of all analytic functions of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ in Δ. Several subclasses of univalent functions are characterized by the quantities $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ lying often in a region in the right-half plane. Ma and Minda [7] gave a unified presentation of various subclasses of convex and starlike functions. For an analytic function ϕ with positive real part on Δ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk Δ onto a region starlike (univalent) with respect to 1 which is symmetric with respect to the real axis, they considered the class $S^*(\phi)$ consisting of functions $f \in \mathcal{A}$ for which $zf'(z)/f(z) \prec \phi(z) \quad (z \in \Delta)$. They also investigated a corresponding class $C(\phi)$ of functions $f \in \mathcal{A}$ satisfying $1 + zf''(z)/f'(z) \prec \phi(z) \quad (z \in \Delta)$. For related results, see [1, 2, 8, 13]. In the following definition, we consider the corresponding extension for meromorphic univalent functions.

Definition 1.1. *Let ϕ be a function with positive real part on Δ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps Δ onto a region starlike with respect to 1 which is symmetric*

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with respect to the real axis. The class $\sum^*(p, w_0, \phi)$ *consists of functions* $f \in U(p)$ *satisfying*

$$-\left(\frac{zf'(z)}{f(z)-w_0} + \frac{p}{z-p} - \frac{pz}{1-pz}\right) \prec \phi(z) \quad (z \in \Delta).$$

The class $\sum(p, \phi)$ *consists of functions* $f \in U(p)$ *satisfying*

$$-\left(1 + z\frac{f''(z)}{f'(z)} + \frac{2p}{z-p} - \frac{2pz}{1-pz}\right) \prec \phi(z) \quad (z \in \Delta).$$

In this paper, the bounds on $|w_0|$ will be determined. Also the bounds for some coefficients of f in $\sum^*(p, w_0, \phi)$ and $\sum(p, \phi)$ will be obtained.

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2. Coefficients Bound Problem

To prove our main result, we need the following:

Lemma 2.1 ([7]). *If* $p_1(z) = 1 + c_1z + c_2z^2 + \cdots$ *is a function with positive real part in* Δ*, then*

$$|c_2 - vc_1^2| \le \begin{cases} -4v+2 & \text{if} \quad v \le 0, \\ 2 & \text{if} \quad 0 \le v \le 1, \\ 4v-2 & \text{if} \quad v \ge 1. \end{cases}$$

When $v < 0$ *or* $v > 1$*, equality holds if and only if* $p_1(z)$ *is* $(1+z)/(1-z)$ *or one of its rotations. If* $0 < v < 1$*, then equality holds if and only if* $p_1(z)$ *is* $(1+z^2)/(1-z^2)$ *or one of its rotations. If* $v = 0$*, the equality holds if and only if*

$$p_1(z) = \left(\frac{1}{2} + \frac{1}{2}\lambda\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\lambda\right)\frac{1-z}{1+z} \quad (0 \le \lambda \le 1)$$

or one of its rotations. If $v = 1$*, the equality holds if and only if* p_1 *is the reciprocal of one of the functions such that equality holds in the case of* $v = 0$*.*

Theorem 2.2. *Let* $\phi(z) = 1 + B_1z + B_2z^2 + \cdots$ *and* $f(z) = z + a_2z^2 + \cdots$ *in* $|z| < p$*. If* $f \in \sum^*(p, w_0, \phi)$*, then*

$$w_0 = \frac{2p}{pB_1c_1 - 2p^2 - 2}$$

and

$$\frac{p}{p^2 + B_1p + 1} \le |w_0| \le \frac{p}{p^2 - B_1p + 1}. \tag{2.1}$$

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Also, we have

$$\left|a_2 + \frac{w_0}{2}\left(p^2 + \frac{1}{p^2} + \frac{1}{w_0^2}\right)\right| \leq \begin{cases} \frac{|w_0||B_2|}{2} & \text{if } |B_2| \geq B_1, \\ \frac{|w_0|B_1}{2} & \text{if } |B_2| \leq B_1. \end{cases} \tag{2.2}$$

Proof. Let h be defined by

$$h(z) = -\left[\frac{zf'(z)}{f(z) - w_0} + \frac{p}{z-p} - \frac{pz}{1-pz}\right] = 1 + b_1 z + b_2 z^2 + \cdots .$$

Then it follows that

$$b_1 = p + \frac{1}{p} + \frac{1}{w_0}, \quad \text{and} \tag{2.3}$$

$$b_2 = p^2 + \frac{1}{p^2} + \frac{1}{w_0^2} + \frac{2a_2}{w_0}. \tag{2.4}$$

Since ϕ is univalent and $h \prec \phi$, the function

$$p_1(z) = \frac{1 + \phi^{-1}(h(z))}{1 - \phi^{-1}(h(z))} = 1 + c_1 z + c_2 z^2 + \cdots$$

is analytic and has a positive real part in Δ. Also, we have

$$h(z) = \phi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right) \tag{2.5}$$

and from this equation (2.5), we obtain

$$b_1 = \frac{1}{2} B_1 c_1 \tag{2.6}$$

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and

$$b_2 = \frac{1}{2}B_1\left(c_2 - \frac{1}{2}c_1^2\right) + \frac{1}{4}B_2 c_1^2. \tag{2.7}$$

From (2.3), (2.4), (2.6) and (2.7), we get

$$w_0 = \frac{2p}{pB_1c_1 - 2p^2 - 2} \tag{2.8}$$

and

$$a_2 = \frac{w_0}{8}(2B_1c_2 - B_1c_1^2 + B_2c_1^2) - \frac{p^2w_0}{2} - \frac{w_0}{2p^2} - \frac{1}{2w_0}. \tag{2.9}$$

From (2.3) and (2.6), we obtain

$$p + \frac{1}{p} + \frac{1}{w_0} = \frac{1}{2}B_1c_1$$

and, since $|c_1| \leq 2$ for a function with positive real part, we have

$$\left|p + \frac{1}{p} - \frac{1}{|w_0|}\right| \leq \left|p + \frac{1}{p} + \frac{1}{w_0}\right| \leq \frac{1}{2}B_1|c_1| \leq B_1$$

or

$$-B_1 \leq p + \frac{1}{p} - \frac{1}{|w_0|} \leq B_1.$$

Rewriting the inequality, we obtain

$$\frac{p}{p^2 + B_1p + 1} \leq |w_0| \leq \frac{p}{p^2 - B_1p + 1}.$$

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From (2.9), we obtain

$$\left|a_2 + \frac{w_0}{2}\left(p^2 + \frac{1}{p^2} + \frac{1}{w_0^2}\right)\right| = \left|\frac{w_0}{2}\left(\frac{1}{2}B_1\left(c_2 - \frac{1}{2}c_1^2\right) + \frac{1}{4}B_2 c_1^2\right)\right|$$
$$= \frac{|w_0|B_1}{4}\left|c_2 - \left(\frac{B_1 - B_2}{2B_1}\right)c_1^2\right|.$$

The result now follows from Lemma 2.1. □

The classes $\sum^*(p, w_0, \phi)$ and $\sum(p, \phi)$ are indeed a more general class of functions, as can be seen in the following corollaries.

Corollary 2.3 ([10, inequality 4, p. 447]). *If $f(z) \in \sum^*(p, w_0)$, then*

$$\frac{p}{(1+p)^2} \leq |w_0| \leq \frac{p}{(1-p)^2}.$$

Proof. Let $B_1 = 2$ in (2.1) of Theorem 2.2. □

Corollary 2.4 ([10, Theorem 1, p. 447]). *Let $f \in \sum^*(p, w_0)$ and $f(z) = z + a_2 z^2 + \cdots$ in $|z| < p$. Then the second coefficient a_2 is given by*

$$a_2 = \frac{1}{2}w_0\left(b_2 - p^2 - \frac{1}{p^2} - \frac{1}{w_0^2}\right),$$

where the region of variability for a_2 is contained in the disk

$$\left|a_2 + \frac{1}{2}w_0\left(p^2 + \frac{1}{p^2} + \frac{1}{w_0^2}\right)\right| \leq |w_0|.$$

Proof. Let $B_1 = 2$ in (2.2) of Theorem 2.2. □

The next theorem is for convex meromorphic functions.

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Theorem 2.5. *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ *and* $f(z) = z + a_2 z^2 + \cdots$ *in* $|z| < p$. *If* $f \in \sum(p, \phi)$, *then*

$$\frac{2p^2 - B_1 p + 2}{2p} \leq |a_2| \leq \frac{2p^2 + B_1 p + 2}{2p}.$$

Also

$$\left| a_3 - \frac{1}{3}\left(p^2 + \frac{1}{p^2}\right) - \frac{2}{3}a_2^2 - \mu\left(a_2 - p - \frac{1}{p}\right)^2 \right|$$
$$\leq \begin{cases} \frac{|2B_2 + 3\mu B_1^2|}{12} & \text{if } \left|\frac{2B_2}{B_1} + 3\mu B_1\right| \geq 2, \\ \frac{B_1}{6} & \text{if } \left|\frac{2B_2}{B_1} + 3\mu B_1\right| \leq 2. \end{cases}$$

Proof. Let h now be defined by

$$h(z) = -\left[1 + \frac{zf''(z)}{f'(z)} + \frac{2p}{z-p} - \frac{2pz}{1-pz}\right] = 1 + b_1 z + b_2 z^2 + \cdots$$

and p_1 be defined as in the proof of Theorem 2.2. A computation shows that

$$b_1 = 2\left(p + \frac{1}{p} - a_2\right), \quad \text{and} \tag{2.10}$$

$$b_2 = 2\left(p^2 + \frac{1}{p^2} + 2a_2^2 - 3a_3\right). \tag{2.11}$$

From (2.6) and (2.10), we have

$$a_2 = p + \frac{1}{p} - \frac{B_1 c_1}{4}. \tag{2.12}$$

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From (2.7) and (2.11), we have

$$a_3 = \frac{1}{24}\left(8p^2 + \frac{8}{p^2} + 16a_2^2 - 2B_1c_2 + B_1c_1^2 - B_2c_1^2\right). \tag{2.13}$$

From (2.12), we have

$$2p + \frac{2}{p} - 2a_2 = \frac{1}{2}B_1c_1$$

or

$$\left|2p + \frac{2}{p} - 2|a_2|\right| \leq |2p + \frac{2}{p} - 2a_2| \leq \frac{1}{2}B_1|c_1| \leq B_1.$$

Thus we have

$$-B_1 \leq 2p + (2/p) - 2|a_2| \leq B_1$$

or

$$\frac{2p^2 - B_1p + 2}{2p} \leq |a_2| \leq \frac{2p^2 + B_1p + 2}{2p}.$$

From (2.12) and (2.13), we obtain

$$\begin{aligned}
&\left|a_3 - \frac{1}{3}\left(p^2 + \frac{1}{p^2}\right) - \frac{2}{3}a_2^2 - \mu\left(a_2 - p - \frac{1}{p}\right)^2\right| \\
&= \left|\frac{1}{24}\left(-2B_1c_2 + B_1c_1^2 - B_2c_1^2\right) - \mu\left(\frac{B_1^2c_1^2}{16}\right)\right| \\
&= \frac{B_1}{12}\left|c_2 - \left(\frac{1}{2} - \frac{B_2}{2B_1} - \frac{3\mu B_1}{4}\right)c_1^2\right|.
\end{aligned}$$

The result now follows from Lemma 2.1. □

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