ON QUASI β-POWER INCREASING SEQUENCES

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Abstract: In this paper we prove a general theorem on $\left|\bar{N}, p_n^{\alpha}; \delta\right|_k$ summability, which generalizes a theorem of Özarslan [6] on $\left|\bar{N}, p_n; \delta\right|_k$ summability, under weaker conditions and by using quasi β -power increasing sequences instead of almost increasing sequences.

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1. Introduction

A positive sequence (γ_n) is said to be a quasi β-power increasing sequence if there exists a constant $K = K(\beta, \gamma) \geq 1$ such that

$$Kn^{\beta}\gamma_n \geq m^{\beta}\gamma_m \tag{1.1}$$

holds for all $n \geq m \geq 1$. It should be noted that every almost increasing sequence is a quasi β-power increasing sequence for any non-negative β, but the converse need not be true as can be seen by taking the example, say $\gamma_n = n^{-\beta}$ for $\beta > 0$. So we are weakening the hypotheses of the theorem of Özarslan [6], replacing an almost increasing sequence by a quasi β-power increasing sequence.

Let $\sum a_n$ be a given infinite series with partial sums (s_n) and let (p_n) be a sequence with $p_0 > 0$, $p_n \geq 0$ for $n > 0$ and $P_n = \sum_{\nu=0}^{n} p_\nu$. We define

$$p_n^{\alpha} = \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} p_\nu, \qquad P_n^{\alpha} = \sum_{\nu=0}^{n} p_\nu^{\alpha}, \qquad \left(P_{-i}^{\alpha} = p_{-i}^{\alpha} = 0, i \geq 1\right), \tag{1.2}$$

where

$$A_0^{\alpha} = 1, \quad A_n^{\alpha} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!}, \quad (\alpha > -1, n = 1, 2, 3, \ldots) \tag{1.3}$$

The sequence-to-sequence transformation

$$U_n^{\alpha} = \frac{1}{P_n^{\alpha}} \sum_{\nu=0}^{n} p_\nu^{\alpha} s_\nu \tag{1.4}$$

defines the sequence (U_n^{α}) of the $\left(\bar{N}, p_n^{\alpha}\right)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n^{α}) (see [7]).

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The series $\sum a_n$ is said to be summable $\left|\bar{N}, p_n^\alpha\right|_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{k-1} \left|U_n^\alpha - U_{n-1}^\alpha\right|^k < \infty, \tag{1.5}$$

and it is said to be summable $\left|\bar{N}, p_n^\alpha; \delta\right|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [7])

$$\sum_{n=1}^{\infty} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\delta k+k-1} \left|U_n^\alpha - U_{n-1}^\alpha\right|^k < \infty. \tag{1.6}$$

In the special case when $\delta = 0$, $\alpha = 0$ (respectively, $p_n = 1$ for all values of n) $\left|\bar{N}, p_n^\alpha; \delta\right|_k$ summability is the same as $\left|\bar{N}, p_n\right|_k$(respectively $|C, 1; \delta|_k$) summability.

Mishra and Srivastava [4] proved the following theorem for $|C, 1|_k$ summability.

Later on Bor [3] generalized the theorem of Mishra and Srivastava [4] for $\left|\bar{N}, p_n\right|_k$ summability.

Quite recently Özarslan [6] has generalized the theorem of Bor [3] under weaker conditions. For this, Özarslan [6] used the concept of almost increasing sequences. A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). Obviously every increasing sequence is an almost increasing sequence but the converse needs not be true as can be seen from the example $b_n = ne^{(-1)^n}$.

Theorem 1.1. *Let (X_n) be an almost increasing sequence and the sequences (ρ_n) and (λ_n) such that the conditions*

$$|\Delta\lambda_n| \leq \rho_n, \tag{1.7}$$

$$\rho_n \to 0 \quad \textit{as } n \to \infty, \tag{1.8}$$

$$|\lambda_n| X_n = O(1), \quad \textit{as } n \to \infty, \tag{1.9}$$

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$$\sum_{n=1}^{\infty} n\,|\Delta\rho_n|\,X_n < \infty. \tag{1.10}$$

are satisfied. If (p_n) *is a sequence such that the condition*

$$P_n = O\left(np_n\right), \quad \text{as } n \to \infty, \tag{1.11}$$

is satisfied and

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k-1} |s_n|^k = O\left(X_m\right), \quad \text{as } m \to \infty, \tag{1.12}$$

$$\sum_{n=\nu+1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}} = O\left\{\left(\frac{P_\nu}{p_\nu}\right)^{\delta k} \frac{1}{P_\nu}\right\}, \tag{1.13}$$

then the series $\sum a_n\lambda_n$ *is summable* $\left|\bar{N}, p_n; \delta\right|_k$ *for* $k \geq 1$ *and* $0 \leq \delta < \frac{1}{k}$.

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2. Main Result

The aim of this paper is to generalize Theorem 1.1 for $\left|\bar{N}, p_n^\alpha; \delta\right|_k$ summability under weaker conditions by using quasi β-power increasing sequences instead of almost increasing sequences. Now, we will prove the following theorem.

Theorem 2.1. *Let* (X_n) *be a quasi* β*-power increasing sequence for some* $0 < \beta < 1$ *and the sequences* (ρ_n) *and* (λ_n) *such that the conditions (1.7) – (1.10) of Theorem 1.1 are satisfied. If* (p_n^α) *is a sequence such that*

$$P_n^\alpha = O\left(np_n^\alpha\right), \quad \text{as } n \to \infty, \tag{2.1}$$

is satisfied and

$$\sum_{n=1}^{m} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\delta k-1} |s_n|^k = O\left(X_m\right), \quad \text{as } m \to \infty, \tag{2.2}$$

$$\sum_{n=\nu+1}^{\infty} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\delta k-1} \frac{1}{P_{n-1}^\alpha} = O\left\{\left(\frac{P_\nu^\alpha}{p_\nu^\alpha}\right)^{\delta k} \frac{1}{P_\nu^\alpha}\right\}, \tag{2.3}$$

then the series $\sum a_n\lambda_n$ *is summable* $\left|\bar{N}, p_n^\alpha; \delta\right|_k$ *for* $k \geq 1$ *and* $0 \leq \delta < \frac{1}{k}$.

Remark 1. It may be noted that, if we take (X_n) as an almost increasing sequence and $\alpha = 0$ in Theorem 2.1, then we get Theorem 1.1. In this case, conditions (2.1) and (2.2) reduce to conditions (1.11) and (1.12) respectively and condition (2.3) reduces to (1.13). If additionally $\delta = 0$, relation (2.3) reduces to

$$\sum_{n=\nu+1}^{\infty} \frac{p_n}{P_nP_{n-1}} = O\left(\frac{1}{P_\nu}\right), \tag{2.4}$$

which always holds.

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We need the following lemma for the proof of our theorem.

Lemma 2.2 ([5]). *Under the conditions on (X_n), (β_n) and (λ_n) as taken in the statement of the theorem, the following conditions hold*

$$n\rho_n X_n = O(1), \quad \text{as } n \to \infty, \tag{2.5}$$

$$\sum_{n=1}^{\infty} \rho_n X_n < \infty. \tag{2.6}$$

Proof of Theorem 2.1. Let (T_n^α) be the $\left(\bar{N}, p_n^\alpha\right)$ mean of the series $\sum a_n\lambda_n$. Then by definition, we have

$$T_n^\alpha = \frac{1}{P_n^\alpha}\sum_{\nu=0}^{n} p_\nu^\alpha \sum_{w=0}^{\nu} a_w\lambda_w = \frac{1}{P_n^\alpha}\sum_{\nu=0}^{n}\left(P_n^\alpha - P_{\nu-1}^\alpha\right)a_\nu\lambda_\nu.$$

Then, for $n \geq 1$, we get

$$T_n^\alpha - T_{n-1}^\alpha = \frac{p_n^\alpha}{P_n^\alpha P_{n-1}^\alpha}\sum_{\nu=1}^{n} P_{\nu-1}^\alpha a_\nu\lambda_\nu.$$

Applying Abel's transformation, we have

$$\begin{aligned} T_n^\alpha - T_{n-1}^\alpha &= \frac{p_n^\alpha}{P_n^\alpha P_{n-1}^\alpha}\sum_{\nu=1}^{n-1}\Delta\left(P_{\nu-1}^\alpha\lambda_\nu\right)s_\nu + \frac{p_n^\alpha}{P_n^\alpha}s_n\lambda_n \\ &= -\frac{p_n^\alpha}{P_n^\alpha P_{n-1}^\alpha}\sum_{\nu=1}^{n-1} p_\nu^\alpha s_\nu\lambda_\nu + \frac{p_n^\alpha}{P_n^\alpha P_{n-1}^\alpha}\sum_{\nu=1}^{n-1} P_\nu^\alpha s_\nu\Delta\lambda_\nu + \frac{p_n^\alpha}{P_n^\alpha}s_n\lambda_n \\ &= T_{n,1}^\alpha + T_{n,2}^\alpha + T_{n,3}^\alpha, \quad \text{say.} \end{aligned}$$

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Since

$$\left|T^{\alpha}_{n,1}+T^{\alpha}_{n,2}+T^{\alpha}_{n,3}\right|^{k} \leq 3^{k}\left(\left|T^{\alpha}_{n,1}\right|^{k}+\left|T^{\alpha}_{n,2}\right|^{k}+\left|T^{\alpha}_{n,3}\right|^{k}\right),$$

to complete the proof of Theorem 2.1, it is sufficient to show that

$$\sum_{n=1}^{\infty}\left(\frac{P^{\alpha}_{n}}{p^{\alpha}_{n}}\right)^{\delta k+k-1}\left|T^{\alpha}_{n,w}\right|^{k}<\infty, \quad \text{for } w=1,2,3.$$

Now, when $k>1$, applying Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'}=1$, and using $|\lambda_n| = O\left(\frac{1}{X_n}\right) = O(1)$, by (1.9), we have

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P^{\alpha}_{n}}{p^{\alpha}_{n}}\right)^{\delta k+k-1}\left|T^{\alpha}_{n,1}\right|^{k} &= \sum_{n=2}^{m+1}\left(\frac{P^{\alpha}_{n}}{p^{\alpha}_{n}}\right)^{\delta k+k-1}\left|\frac{p^{\alpha}_{n}}{P^{\alpha}_{n}P^{\alpha}_{n-1}}\sum_{\nu=1}^{n-1}p^{\alpha}_{\nu}s_{\nu}\lambda_{\nu}\right|^{k} \\
&\leq \sum_{n=2}^{m+1}\left(\frac{P^{\alpha}_{n}}{p^{\alpha}_{n}}\right)^{\delta k-1}\frac{1}{P^{\alpha}_{n-1}}\sum_{\nu=1}^{n-1}p^{\alpha}_{\nu}|s_{\nu}|^{k}|\lambda_{\nu}|^{k}\left(\frac{1}{P^{\alpha}_{n-1}}\sum_{\nu=1}^{n-1}p^{\alpha}_{\nu}\right)^{k-1} \\
&= O(1)\sum_{\nu=1}^{m}p^{\alpha}_{\nu}|s_{\nu}|^{k}|\lambda_{\nu}|^{k}\sum_{n=\nu+1}^{m+1}\left(\frac{P^{\alpha}_{n}}{p^{\alpha}_{n}}\right)^{\delta k-1}\left(\frac{1}{P^{\alpha}_{n-1}}\right) \\
&= O(1)\sum_{\nu=1}^{m}p^{\alpha}_{\nu}|s_{\nu}|^{k}|\lambda_{\nu}|^{k}\left(\frac{P^{\alpha}_{\nu}}{p^{\alpha}_{\nu}}\right)^{\delta k}\frac{1}{P^{\alpha}_{\nu}} \\
&= O(1)\sum_{\nu=1}^{m}\left(\frac{P^{\alpha}_{\nu}}{p^{\alpha}_{\nu}}\right)^{\delta k-1}|s_{\nu}|^{k}|\lambda_{\nu}|^{k} \\
&= O(1)\sum_{\nu=1}^{m}\left(\frac{P^{\alpha}_{\nu}}{p^{\alpha}_{\nu}}\right)^{\delta k-1}|s_{\nu}|^{k}|\lambda_{\nu}|\,|\lambda_{\nu}|^{k-1}
\end{aligned}$$

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$$
\begin{aligned}
&= O\left(1\right)\sum_{\nu=1}^{m}\left(\frac{P_{\nu}^{\alpha}}{p_{\nu}^{\alpha}}\right)^{\delta k-1}\left|s_{\nu}\right|^{k}\left|\lambda_{\nu}\right| \\
&= O\left(1\right)\sum_{\nu=1}^{m-1}\Delta\left|\lambda_{\nu}\right|\sum_{u=1}^{\nu}\left(\frac{P_{u}^{\alpha}}{p_{u}^{\alpha}}\right)^{\delta k-1}\left|s_{u}\right|^{k}+O\left(1\right)\left|\lambda_{m}\right|\sum_{\nu=1}^{m}\left(\frac{P_{\nu}^{\alpha}}{p_{\nu}^{\alpha}}\right)^{\delta k-1}\left|s_{\nu}\right|^{k} \\
&= O(1)\sum_{\nu=1}^{m-1}\left|\Delta\lambda_{\nu}\right|X_{\nu}+O\left(1\right)\left|\lambda_{m}\right|X_{m} \\
&= O\left(1\right)\sum_{\nu=1}^{m-1}\rho_{\nu}X_{\nu}+O\left(1\right)\left|\lambda_{m}\right|X_{m} \\
&= O\left(1\right), \quad \text{as } m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 2.1 and Lemma 2.2. Since $\nu\rho_{\nu}=O\left(\frac{1}{X_{\nu}}\right)=O\left(1\right)$, by (2.5), using the fact that $\left|\Delta\lambda_{n}\right|\leq\rho_{n}$ by (1.7) and $P_{n}^{\alpha}=O\left(np_{n}^{\alpha}\right)$ by (2.1) and after applying the Hölder's inequality again, we obtain

$$
\begin{aligned}
&\sum_{n=2}^{m+1}\left(\frac{P_{n}^{\alpha}}{p_{n}^{\alpha}}\right)^{\delta k+k-1}\left|T_{n,2}^{\alpha}\right|^{k} \\
&\leq\sum_{n=2}^{m+1}\left(\frac{P_{n}^{\alpha}}{p_{n}^{\alpha}}\right)^{\delta k-1}\left(\frac{1}{P_{n-1}^{\alpha}}\right)^{k}\left\{\sum_{\nu=1}^{n-1}P_{\nu}^{\alpha}\left|\Delta\lambda_{\nu}\right|\left|s_{\nu}\right|\right\}^{k} \\
&\leq\sum_{n=2}^{m+1}\left(\frac{P_{n}^{\alpha}}{p_{n}^{\alpha}}\right)^{\delta k-1}\frac{1}{P_{n-1}^{\alpha}}\left\{\sum_{\nu=1}^{n-1}p_{\nu}^{\alpha}\left(\nu\rho_{\nu}\right)^{k}\left|s_{\nu}\right|^{k}\right\}\left\{\frac{1}{P_{n-1}^{\alpha}}\sum_{\nu=1}^{n-1}p_{\nu}^{\alpha}\right\}^{k-1} \\
&= O\left(1\right)\sum_{\nu=1}^{m}p_{\nu}^{\alpha}\left(\nu\rho_{\nu}\right)^{k}\left|s_{\nu}\right|^{k}\sum_{n=\nu+1}^{m+1}\left(\frac{P_{n}^{\alpha}}{p_{n}^{\alpha}}\right)^{\delta k-1}\frac{1}{P_{n-1}^{\alpha}}
\end{aligned}
$$

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$$
\begin{aligned}
&= O(1)\sum_{\nu=1}^{m}\left(\frac{P_\nu^\alpha}{p_\nu^\alpha}\right)^{\delta k-1}(\nu\rho_\nu)^k|s_\nu|^k \\
&= O(1)\sum_{\nu=1}^{m-1}\Delta(\nu\rho_\nu)\sum_{w=1}^{\nu}\left(\frac{P_w^\alpha}{p_w^\alpha}\right)^{\delta k-1}|s_w|^k + O(1)\,m\rho_m\sum_{\nu=1}^{m}\left(\frac{P_\nu^\alpha}{p_\nu^\alpha}\right)^{\delta k-1}|s_\nu|^k \\
&= O(1)\sum_{\nu=1}^{m-1}|\Delta(\nu\rho_\nu)|\,X_\nu + O(1)\,m\rho_m X_m \\
&= O(1)\sum_{\nu=1}^{m-1}\nu\,|\Delta\rho_\nu|\,X_\nu + O(1)\sum_{\nu=1}^{m-1}\rho_{\nu+1}X_{\nu+1} + O(1)\,m\rho_m X_m \\
&= O(1), \quad \text{as } m\to\infty,
\end{aligned}
$$

by the virtue of the hypotheses of Theorem 2.1 and Lemma 2.2. Finally, using the fact that $P_n^\alpha = O(np_n^\alpha)$, by (2.1) as in $T_{n,1}^\alpha$, we have

$$
\begin{aligned}
\sum_{n=1}^{m}\left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\delta k+k-1}\left|T_{n,3}^\alpha\right|^k &= O(1)\sum_{n=1}^{m}\left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\delta k-1}|s_n|^k|\lambda_n| \\
&= O(1), \quad \text{as } m\to\infty.
\end{aligned}
$$

Therefore, we get

$$
\sum_{n=1}^{\infty}\left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\delta k+k-1}\left|T_{n,w}^\alpha\right|^k = O(1), \quad \text{as } m\to\infty, \quad \text{for } w=1,2,3.
$$

This completes the proof of Theorem 2.1. □

If we take $p_n = 1$ and $\alpha = 0$ for all values of n in Theorem 2.1, then we obtain a result concerning the $|C,1,\delta|_k$ summability.

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