

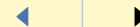
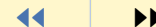
Quasi β -Power Increasing
Sequences

Santosh Kr. Saxena

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ON QUASI β -POWER INCREASING SEQUENCES

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Abstract: In this paper we prove a general theorem on $|\bar{N}, p_n^\alpha; \delta|_k$ summability, which generalizes a theorem of Özarşlan [6] on $|\bar{N}, p_n; \delta|_k$ summability, under weaker conditions and by using quasi β -power increasing sequences instead of almost increasing sequences.

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1. Introduction

A positive sequence (γ_n) is said to be a quasi β -power increasing sequence if there exists a constant $K = K(\beta, \gamma) \geq 1$ such that

$$(1.1) \quad Kn^\beta \gamma_n \geq m^\beta \gamma_m$$

holds for all $n \geq m \geq 1$. It should be noted that every almost increasing sequence is a quasi β -power increasing sequence for any non-negative β , but the converse need not be true as can be seen by taking the example, say $\gamma_n = n^{-\beta}$ for $\beta > 0$. So we are weakening the hypotheses of the theorem of Özarşlan [6], replacing an almost increasing sequence by a quasi β -power increasing sequence.

Let $\sum a_n$ be a given infinite series with partial sums (s_n) and let (p_n) be a sequence with $p_0 > 0$, $p_n \geq 0$ for $n > 0$ and $P_n = \sum_{\nu=0}^n p_\nu$. We define

$$(1.2) \quad p_n^\alpha = \sum_{\nu=0}^n A_{n-\nu}^{\alpha-1} p_\nu, \quad P_n^\alpha = \sum_{\nu=0}^n p_\nu^\alpha, \quad (P_{-i}^\alpha = p_{-i}^\alpha = 0, i \geq 1),$$

where

$$(1.3) \quad A_0^\alpha = 1, \quad A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!}, \quad (\alpha > -1, n = 1, 2, 3, \dots)$$

The sequence-to-sequence transformation

$$(1.4) \quad U_n^\alpha = \frac{1}{P_n^\alpha} \sum_{\nu=0}^n p_\nu^\alpha s_\nu$$

defines the sequence (U_n^α) of the (\bar{N}, p_n^α) mean of the sequence (s_n) , generated by the sequence of coefficients (p_n^α) (see [7]).



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The series $\sum a_n$ is said to be summable $|\bar{N}, p_n^\alpha|_k, k \geq 1$, if (see [2])

$$(1.5) \quad \sum_{n=1}^{\infty} \left(\frac{P_n^\alpha}{p_n^\alpha} \right)^{k-1} |U_n^\alpha - U_{n-1}^\alpha|^k < \infty,$$

and it is said to be summable $|\bar{N}, p_n^\alpha; \delta|_k, k \geq 1$ and $\delta \geq 0$, if (see [7])

$$(1.6) \quad \sum_{n=1}^{\infty} \left(\frac{P_n^\alpha}{p_n^\alpha} \right)^{\delta k + k - 1} |U_n^\alpha - U_{n-1}^\alpha|^k < \infty.$$

In the special case when $\delta = 0, \alpha = 0$ (respectively, $p_n = 1$ for all values of n) $|\bar{N}, p_n^\alpha; \delta|_k$ summability is the same as $|\bar{N}, p_n|_k$ (respectively $|C, 1; \delta|_k$) summability.

Mishra and Srivastava [4] proved the following theorem for $|C, 1|_k$ summability.

Later on Bor [3] generalized the theorem of Mishra and Srivastava [4] for $|\bar{N}, p_n|_k$ summability.

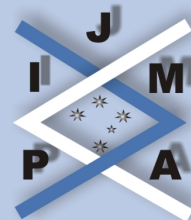
Quite recently Özarlan [6] has generalized the theorem of Bor [3] under weaker conditions. For this, Özarlan [6] used the concept of almost increasing sequences. A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $A c_n \leq b_n \leq B c_n$ (see [1]). Obviously every increasing sequence is an almost increasing sequence but the converse needs not be true as can be seen from the example $b_n = n e^{(-1)^n}$.

Theorem 1.1. *Let (X_n) be an almost increasing sequence and the sequences (ρ_n) and (λ_n) such that the conditions*

$$(1.7) \quad |\Delta \lambda_n| \leq \rho_n,$$

$$(1.8) \quad \rho_n \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

$$(1.9) \quad |\lambda_n| X_n = O(1), \quad \text{as } n \rightarrow \infty,$$



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$$(1.10) \quad \sum_{n=1}^{\infty} n |\Delta \rho_n| X_n < \infty.$$

are satisfied. If (p_n) is a sequence such that the condition

$$(1.11) \quad P_n = O(np_n), \quad \text{as } n \rightarrow \infty,$$

is satisfied and

$$(1.12) \quad \sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{\delta k - 1} |s_n|^k = O(X_m), \quad \text{as } m \rightarrow \infty,$$

$$(1.13) \quad \sum_{n=\nu+1}^{\infty} \left(\frac{P_n}{p_n} \right)^{\delta k - 1} \frac{1}{P_{n-1}} = O \left\{ \left(\frac{P_\nu}{p_\nu} \right)^{\delta k} \frac{1}{P_\nu} \right\},$$

then the series $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n; \delta|_k$ for $k \geq 1$ and $0 \leq \delta < \frac{1}{k}$.



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2. Main Result

The aim of this paper is to generalize Theorem 1.1 for $|\bar{N}, p_n^\alpha; \delta|_k$ summability under weaker conditions by using quasi β -power increasing sequences instead of almost increasing sequences. Now, we will prove the following theorem.

Theorem 2.1. *Let (X_n) be a quasi β -power increasing sequence for some $0 < \beta < 1$ and the sequences (ρ_n) and (λ_n) such that the conditions (1.7) – (1.10) of Theorem 1.1 are satisfied. If (p_n^α) is a sequence such that*

$$(2.1) \quad P_n^\alpha = O(np_n^\alpha), \quad \text{as } n \rightarrow \infty,$$

is satisfied and

$$(2.2) \quad \sum_{n=1}^m \left(\frac{P_n^\alpha}{p_n^\alpha} \right)^{\delta k - 1} |s_n|^k = O(X_m), \quad \text{as } m \rightarrow \infty,$$

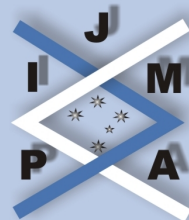
$$(2.3) \quad \sum_{n=\nu+1}^{\infty} \left(\frac{P_n^\alpha}{p_n^\alpha} \right)^{\delta k - 1} \frac{1}{P_{n-1}^\alpha} = O \left\{ \left(\frac{P_\nu^\alpha}{p_\nu^\alpha} \right)^{\delta k} \frac{1}{P_\nu^\alpha} \right\},$$

then the series $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n^\alpha; \delta|_k$ for $k \geq 1$ and $0 \leq \delta < \frac{1}{k}$.

Remark 1. It may be noted that, if we take (X_n) as an almost increasing sequence and $\alpha = 0$ in Theorem 2.1, then we get Theorem 1.1. In this case, conditions (2.1) and (2.2) reduce to conditions (1.11) and (1.12) respectively and condition (2.3) reduces to (1.13). If additionally $\delta = 0$, relation (2.3) reduces to

$$(2.4) \quad \sum_{n=\nu+1}^{\infty} \frac{p_n}{P_n P_{n-1}} = O \left(\frac{1}{P_\nu} \right),$$

which always holds.



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We need the following lemma for the proof of our theorem.

Lemma 2.2 ([5]). Under the conditions on (X_n) , (β_n) and (λ_n) as taken in the statement of the theorem, the following conditions hold

$$(2.5) \quad n\rho_n X_n = O(1), \quad \text{as } n \rightarrow \infty,$$

$$(2.6) \quad \sum_{n=1}^{\infty} \rho_n X_n < \infty.$$

Proof of Theorem 2.1. Let (T_n^α) be the (\bar{N}, p_n^α) mean of the series $\sum a_n \lambda_n$. Then by definition, we have

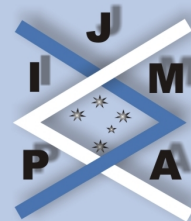
$$T_n^\alpha = \frac{1}{P_n^\alpha} \sum_{\nu=0}^n p_\nu^\alpha \sum_{w=0}^{\nu} a_w \lambda_w = \frac{1}{P_n^\alpha} \sum_{\nu=0}^n (P_n^\alpha - P_{\nu-1}^\alpha) a_\nu \lambda_\nu.$$

Then, for $n \geq 1$, we get

$$T_n^\alpha - T_{n-1}^\alpha = \frac{p_n^\alpha}{P_n^\alpha P_{n-1}^\alpha} \sum_{\nu=1}^n P_{\nu-1}^\alpha a_\nu \lambda_\nu.$$

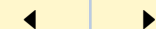
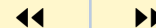
Applying Abel's transformation, we have

$$\begin{aligned} T_n^\alpha - T_{n-1}^\alpha &= \frac{p_n^\alpha}{P_n^\alpha P_{n-1}^\alpha} \sum_{\nu=1}^{n-1} \Delta (P_{\nu-1}^\alpha \lambda_\nu) s_\nu + \frac{p_n^\alpha}{P_n^\alpha} s_n \lambda_n \\ &= -\frac{p_n^\alpha}{P_n^\alpha P_{n-1}^\alpha} \sum_{\nu=1}^{n-1} p_\nu^\alpha s_\nu \lambda_\nu + \frac{p_n^\alpha}{P_n^\alpha P_{n-1}^\alpha} \sum_{\nu=1}^{n-1} P_\nu^\alpha s_\nu \Delta \lambda_\nu + \frac{p_n^\alpha}{P_n^\alpha} s_n \lambda_n \\ &= T_{n,1}^\alpha + T_{n,2}^\alpha + T_{n,3}^\alpha, \quad \text{say.} \end{aligned}$$



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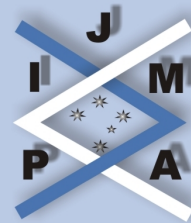
$$|T_{n,1}^\alpha + T_{n,2}^\alpha + T_{n,3}^\alpha|^k \leq 3^k \left(|T_{n,1}^\alpha|^k + |T_{n,2}^\alpha|^k + |T_{n,3}^\alpha|^k \right),$$

to complete the proof of Theorem 2.1, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n^\alpha}{p_n^\alpha} \right)^{\delta k + k - 1} |T_{n,w}^\alpha|^k < \infty, \quad \text{for } w = 1, 2, 3.$$

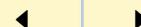
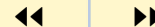
Now, when $k > 1$, applying Hölder's inequality with indices k and k' , where $\frac{1}{k} + \frac{1}{k'} = 1$, and using $|\lambda_n| = O\left(\frac{1}{X_n}\right) = O(1)$, by (1.9), we have

$$\begin{aligned} \sum_{n=2}^{m+1} \left(\frac{P_n^\alpha}{p_n^\alpha} \right)^{\delta k + k - 1} |T_{n,1}^\alpha|^k &= \sum_{n=2}^{m+1} \left(\frac{P_n^\alpha}{p_n^\alpha} \right)^{\delta k + k - 1} \left| \frac{p_n^\alpha}{P_n^\alpha P_{n-1}^\alpha} \sum_{\nu=1}^{n-1} p_\nu^\alpha s_\nu \lambda_\nu \right|^k \\ &\leq \sum_{n=2}^{m+1} \left(\frac{P_n^\alpha}{p_n^\alpha} \right)^{\delta k - 1} \frac{1}{P_{n-1}^\alpha} \sum_{\nu=1}^{n-1} p_\nu^\alpha |s_\nu|^k |\lambda_\nu|^k \left(\frac{1}{P_{n-1}^\alpha} \sum_{\nu=1}^{n-1} p_\nu^\alpha \right)^{k-1} \\ &= O(1) \sum_{\nu=1}^m p_\nu^\alpha |s_\nu|^k |\lambda_\nu|^k \sum_{n=\nu+1}^{m+1} \left(\frac{P_n^\alpha}{p_n^\alpha} \right)^{\delta k - 1} \left(\frac{1}{P_{n-1}^\alpha} \right) \\ &= O(1) \sum_{\nu=1}^m p_\nu^\alpha |s_\nu|^k |\lambda_\nu|^k \left(\frac{P_\nu^\alpha}{p_\nu^\alpha} \right)^{\delta k} \frac{1}{P_\nu^\alpha} \\ &= O(1) \sum_{\nu=1}^m \left(\frac{P_\nu^\alpha}{p_\nu^\alpha} \right)^{\delta k - 1} |s_\nu|^k |\lambda_\nu|^k \\ &= O(1) \sum_{\nu=1}^m \left(\frac{P_\nu^\alpha}{p_\nu^\alpha} \right)^{\delta k - 1} |s_\nu|^k |\lambda_\nu| |\lambda_\nu|^{k-1} \end{aligned}$$



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$$\begin{aligned}
 &= O(1) \sum_{\nu=1}^m \left(\frac{P_{\nu}^{\alpha}}{p_{\nu}^{\alpha}} \right)^{\delta k-1} |s_{\nu}|^k |\lambda_{\nu}| \\
 &= O(1) \sum_{\nu=1}^{m-1} \Delta |\lambda_{\nu}| \sum_{u=1}^{\nu} \left(\frac{P_u^{\alpha}}{p_u^{\alpha}} \right)^{\delta k-1} |s_u|^k + O(1) |\lambda_m| \sum_{\nu=1}^m \left(\frac{P_{\nu}^{\alpha}}{p_{\nu}^{\alpha}} \right)^{\delta k-1} |s_{\nu}|^k \\
 &= O(1) \sum_{\nu=1}^{m-1} |\Delta \lambda_{\nu}| X_{\nu} + O(1) |\lambda_m| X_m \\
 &= O(1) \sum_{\nu=1}^{m-1} \rho_{\nu} X_{\nu} + O(1) |\lambda_m| X_m \\
 &= O(1), \quad \text{as } m \rightarrow \infty,
 \end{aligned}$$

by virtue of the hypotheses of Theorem 2.1 and Lemma 2.2. Since $\nu \rho_{\nu} = O\left(\frac{1}{X_{\nu}}\right) = O(1)$, by (2.5), using the fact that $|\Delta \lambda_n| \leq \rho_n$ by (1.7) and $P_n^{\alpha} = O(np_n^{\alpha})$ by (2.1) and after applying the Hölder's inequality again, we obtain

$$\begin{aligned}
 &\sum_{n=2}^{m+1} \left(\frac{P_n^{\alpha}}{p_n^{\alpha}} \right)^{\delta k+k-1} |T_{n,2}^{\alpha}|^k \\
 &\leq \sum_{n=2}^{m+1} \left(\frac{P_n^{\alpha}}{p_n^{\alpha}} \right)^{\delta k-1} \left(\frac{1}{P_{n-1}^{\alpha}} \right)^k \left\{ \sum_{\nu=1}^{n-1} P_{\nu}^{\alpha} |\Delta \lambda_{\nu}| |s_{\nu}| \right\}^k \\
 &\leq \sum_{n=2}^{m+1} \left(\frac{P_n^{\alpha}}{p_n^{\alpha}} \right)^{\delta k-1} \frac{1}{P_{n-1}^{\alpha}} \left\{ \sum_{\nu=1}^{n-1} p_{\nu}^{\alpha} (\nu \rho_{\nu})^k |s_{\nu}|^k \right\} \left\{ \frac{1}{P_{n-1}^{\alpha}} \sum_{\nu=1}^{n-1} p_{\nu}^{\alpha} \right\}^{k-1} \\
 &= O(1) \sum_{\nu=1}^m p_{\nu}^{\alpha} (\nu \rho_{\nu})^k |s_{\nu}|^k \sum_{n=\nu+1}^{m+1} \left(\frac{P_n^{\alpha}}{p_n^{\alpha}} \right)^{\delta k-1} \frac{1}{P_{n-1}^{\alpha}}
 \end{aligned}$$



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$$\begin{aligned}
 &= O(1) \sum_{\nu=1}^m \left(\frac{P_{\nu}^{\alpha}}{p_{\nu}^{\alpha}} \right)^{\delta k-1} (\nu \rho_{\nu})^k |s_{\nu}|^k \\
 &= O(1) \sum_{\nu=1}^{m-1} \Delta(\nu \rho_{\nu}) \sum_{w=1}^{\nu} \left(\frac{P_w^{\alpha}}{p_w^{\alpha}} \right)^{\delta k-1} |s_w|^k + O(1) m \rho_m \sum_{\nu=1}^m \left(\frac{P_{\nu}^{\alpha}}{p_{\nu}^{\alpha}} \right)^{\delta k-1} |s_{\nu}|^k \\
 &= O(1) \sum_{\nu=1}^{m-1} |\Delta(\nu \rho_{\nu})| X_{\nu} + O(1) m \rho_m X_m \\
 &= O(1) \sum_{\nu=1}^{m-1} \nu |\Delta \rho_{\nu}| X_{\nu} + O(1) \sum_{\nu=1}^{m-1} \rho_{\nu+1} X_{\nu+1} + O(1) m \rho_m X_m \\
 &= O(1), \quad \text{as } m \rightarrow \infty,
 \end{aligned}$$

by the virtue of the hypotheses of Theorem 2.1 and Lemma 2.2. Finally, using the fact that $P_n^{\alpha} = O(np_n^{\alpha})$, by (2.1) as in $T_{n,1}^{\alpha}$, we have

$$\begin{aligned}
 \sum_{n=1}^m \left(\frac{P_n^{\alpha}}{p_n^{\alpha}} \right)^{\delta k+k-1} |T_{n,3}^{\alpha}|^k &= O(1) \sum_{n=1}^m \left(\frac{P_n^{\alpha}}{p_n^{\alpha}} \right)^{\delta k-1} |s_n|^k |\lambda_n| \\
 &= O(1), \quad \text{as } m \rightarrow \infty.
 \end{aligned}$$

Therefore, we get

$$\sum_{n=1}^{\infty} \left(\frac{P_n^{\alpha}}{p_n^{\alpha}} \right)^{\delta k+k-1} |T_{n,w}^{\alpha}|^k = O(1), \quad \text{as } m \rightarrow \infty, \quad \text{for } w = 1, 2, 3.$$

This completes the proof of Theorem 2.1. □

If we take $p_n = 1$ and $\alpha = 0$ for all values of n in Theorem 2.1, then we obtain a result concerning the $|C, 1, \delta|_k$ summability.



References

- [1] S. ALJANČIĆ AND D. ARANDELOVIĆ, O-regular varying functions, *Publ. Inst. Math.*, **22**(36) (1977), 5–22.
- [2] H. BOR, A note on some absolute summability methods, *J. Nigerian Math. Soc.*, **6** (1987), 41–46.
- [3] H. BOR, A note on absolute summability factors, *Internet J. Math. and Math. Sci.*, **17**(3) (1994), 479–482.
- [4] K.N. MISHRA AND R.S.L. SRIVASTAVA, On absolute Cesàro summability factors of infinite series, *Portugal. Math.*, **42**(1) (1985), 53–61.
- [5] L. LEINDLER, A new application of quasi power increasing sequences, *Publ. Math. (Debrecen)*, **58** (2001), 791–796.
- [6] H.S. ÖZARSLAN, On almost increasing sequences and its applications, *Internat. J. Math. and Math. Sci.*, **25**(5) (2001), 293–298.
- [7] S.K. SAXENA AND S.K. SAXENA, A note on $|\bar{N}, p_n^\alpha; \delta|_k$ summability factors, *Soochow J. Math.*, **33**(4) (2007), 829–834.

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