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AN INEQUALITY ABOUT q-SERIES

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ABSTRACT. In this paper, we first generalize the traditional notation $(a;q)_n$ to $[g(x);q]_n$ and then obtain an inequality about q-series and some infinite products by means of the new conception. Because many q-series are not summable, our results are useful to study q-series and its application.

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1. INTRODUCTION

q-series, which are also called basic hypergeometric series, play an very important role in many fields. Such as affine root systems, Lie algebras and groups, number theory, orthogonal polynomials, physics (such as representations of quantum groups and Baxter's work on the hard hexagon model). Most of the research work on q-series is to set up identity. But there are also great many q-series whose sums cannot be obtained easily. On these occasions, we must use other methods to study q-series. Using inequalities is one of the choices. In this paper, we obtain an inequality about q-series and some infinite products. Our results are useful for the study of q-series.

2. AN INEQUALITY ABOUT q-SERIES

In this section we will introduce a new concept and obtain an inequality about q-series. First we give some lemmas.

Lemma 2.1. *If $0 < q < 1$, $0 < a < 1$, then for any natural number n, we have*

$$1 < \frac{1-aq^n}{1-q^n} \leq \frac{1-aq}{1-q}.$$

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Proof. Let $g(x) = \frac{1-ax}{1-x}$, $0 \le x < 1$, then $g^{'}(x) = \frac{1-a}{(1-x)^2} > 0$. So $g(x)$ is a strictly increasing function on $[0, 1)$. For any natural number n, we have

$$0 < q^n \le q < 1.$$

So,

$$g(0) < g(q^n) \le g(q).$$

That is

$$1 < \frac{1 - aq^n}{1 - q^n} \le \frac{1 - aq}{1 - q}.$$

□

Lemma 2.2. *If* $0 < a < \frac{1-q}{1+q}$, $0 < q < 1$, *then for any real number* $0 < x \le 1$, *we have*

$$1 + ax\left(ax - \frac{2(1 - aqx)}{1 - q}\right) > 0.$$

Proof. Let

$$g(x) = 1 + ax\left(ax - \frac{2(1 - aqx)}{1 - q}\right) = 1 + \frac{a}{1 - q}[a(1 + q)x^2 - 2x].$$

Under the condition $0 < a < \frac{1-q}{1+q}$, we know $0 < a(1 + q) < 1 - q$, so

$$g^{'}(x) = \frac{2a}{1 - q}[a(1 + q)x - 1] < \frac{2a}{1 - q}[(1 - q)x - 1].$$

Since $0 < 1 - q < 1,\ 0 < x \le 1$, we know $0 < (1 - q)x < 1$. Therefore $g^{'}(x) < 0$ and $g(x)$ is a strictly decreasing function on $(0, 1]$. We have

$$g(x) > g(1) = 1 + a\left(a - \frac{2(1 - aq)}{1 - q}\right) = \frac{1}{1 - q}[(1 + q)a^2 - 2a + (1 - q)].$$

Letting

$$(1 + q)a^2 - 2a + (1 - q) = 0,$$

we have

$$a_1 = \frac{1 - q}{1 + q}, \qquad a_2 = 1.$$

So, when $0 < a < \frac{1-q}{1+q}$,

$$(1 + q)a^2 - 2a + (1 - q) > 0,$$

that is,

$$1 + ax\left(ax - \frac{2(1 - aqx)}{1 - q}\right) > 0.$$

□

Lemma 2.3. *If* $0 < q < \sqrt{2} - 1$, *we have*

$$q < \frac{1 - q}{1 + q}.$$

Proof. Let

$$g(q) = q(1 + q) - (1 - q) = \left(q + 1 + \sqrt{2}\right)\left(q + 1 - \sqrt{2}\right).$$

When $0 < q < \sqrt{2} - 1$, $g(q) < 0$, so we have

$$q < \frac{1 - q}{1 + q}.$$

□

Definition 2.1. Suppose $g(x)$ is a function on $[0,1]$, we denote $[g(x);q]_n$ by

$$[g(x);q]_n = (1-g(q^0))(1-g(q^1))\cdots(1-g(q^{n-1})).$$

We also use the notation $[g(x);q]_\infty$ to express infinite product. That is

$$[g(x);q]_\infty = (1-g(q^0))(1-g(q^1))\cdots(1-g(q^n))\cdots .$$

If $g(x) = ax$, then

$$[g(x);q]_n = (1-a)(1-aq)\cdots(1-aq^{n-1}) = (a;q)_n.$$

So $[g(x);q]_n$ is the expansion of $(a;q)_n$, where $(a;q)_n$ is the q-shifted factorial. Please note that our notation $[g(x);q]_n$ in this paper is different from traditional notation $(a;q)_n$.

Theorem 2.4. *Suppose $0 < a < \frac{1-q}{1+q}$, $0 < q < \sqrt{2}-1$, $0 < z < 1$, then the following inequality holds*

$$\frac{[g_2(x,a);q]_\infty}{(1-z)[g_1(x,q);q]_\infty} \le \sum_{n=0}^{\infty} \frac{(a;q)_n^2}{(q;q)_n^2} z^n \le \frac{[g_1(x,a);q]_\infty}{(1-z)[g_2(x,q);q]_\infty}, \tag{2.1}$$

where

$$\begin{aligned} g_1(x,a) &= -ax(ax-2)z, \\ g_2(x,a) &= -ax\left(ax - \frac{2(1-aqx)}{1-q}\right) z. \end{aligned}$$

When $a = q$, the equality holds.

Proof. Let $f(a,z) = \sum\limits_{n=0}^{\infty} \frac{(a;q)_n^2}{(q;q)_n^2} z^n$, since $1-a = (1-aq^n) - a(1-q^n)$

$$\begin{aligned} f(a,z) &= 1 + \sum_{n=1}^{\infty} \frac{(a;q)_n^2}{(q;q)_n^2} z^n \\ &= 1 + \sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}^2}{(q;q)_n^2} [(1-aq^n) - a(1-q^n)]^2 z^n \\ &= 1 + \sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}^2}{(q;q)_n^2} [(1-aq^n)^2 - 2a(1-q^n)(1-aq^n) + a^2(1-q^n)^2] z^n \\ &= 1 + \sum_{n=1}^{\infty} \frac{(aq;q)_n^2}{(q;q)_n^2} z^n + a^2 \sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}^2}{(q;q)_{n-1}^2} z^n \\ &\quad - 2a \sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}^2}{(q;q)_n^2} (1-q^n)(1-aq^n) z^n \\ &= 1 + \sum_{n=1}^{\infty} \frac{(aq;q)_n^2}{(q;q)_n^2} z^n + a^2 \sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}^2}{(q;q)_{n-1}^2} z^n - 2a \sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}^2}{(q;q)_{n-1}^2} \frac{1-aq^n}{1-q^n} z^n. \end{aligned}$$

From Lemma 2.1, we know $\frac{1-aq^n}{1-q^n} > 1$. So

$$\begin{aligned} f(a,z) &= 1 + \sum_{n=1}^{\infty} \frac{(aq;q)_n^2}{(q;q)_n^2} z^n + a^2 \sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}^2}{(q;q)_{n-1}^2} z^n - 2a \sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}^2}{(q;q)_{n-1}^2} \frac{1-aq^n}{1-q^n} z^n \\ &\leq f(aq,z) + a^2 z f(aq,z) - 2a \sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}^2}{(q;q)_{n-1}^2} z^n \\ &= f(aq,z) + a^2 z f(aq,z) - 2az f(aq,z) \\ &= (1 + a(a-2)z) f(aq,z). \end{aligned}$$

By iterating this functional inequality $n-1$ times we get that

$$f(a,z) \leq [g_1(x,a);q]_n f(aq^n,z), \qquad n = 1,2,\ldots,$$

where $g_1(x,a) = -ax(ax-2)z$. Which on letting $n \to \infty$ and using $q^n \to 0$ gives

$$f(a,z) \leq [g_1(x,a);q]_\infty f(0,z). \tag{2.2}$$

Again from Lemma 2.1, we know $\frac{1-aq^n}{1-q^n} \leq \frac{1-aq}{1-q}$. So

$$\begin{aligned} f(a,z) &= 1 + \sum_{n=1}^{\infty} \frac{(aq;q)_n^2}{(q;q)_n^2} z^n + a^2 \sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}^2}{(q;q)_{n-1}^2} z^n \\ &\qquad - 2a \sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}^2}{(q;q)_{n-1}^2} \frac{1-aq^n}{1-q^n} z^n \\ &\geq f(aq,z) + a^2 z f(aq,z) - 2a \frac{1-aq}{1-q} \sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}^2}{(q;q)_{n-1}^2} z^n \\ &= \left(1 + a^2 z - 2az \frac{1-aq}{1-q}\right) f(aq,z) \\ &= \left(1 + a\left(a - \frac{2(1-aq)}{1-q}\right) z\right) f(aq,z). \end{aligned}$$

Using Lemma 2.2, we know that, for any natural number n

$$\begin{aligned} &1 + aq^n \left(aq^n - \frac{2(1-aq^{n+1})}{1-q}\right) z \\ &= z\left[\frac{1}{z} + aq^n \left(aq^n - \frac{2(1-aq^{n+1})}{1-q}\right)\right] \\ &\geq z\left[1 + aq^n \left((aq^n - \frac{2(1-aq^{n+1})}{1-q}\right)\right] > 0. \end{aligned}$$

Let $g_2(x,a) = -ax\left(ax - \frac{2(1-aqx)}{1-q}\right) z$, and by iterating this functional inequality $n-1$ times we get that

$$f(a,z) \geq [g_2(x,a);q]_n f(aq^n,z), \qquad n = 1,2,\ldots.$$

Which on letting $n \to \infty$ and using $q^n \to 0$ gives

$$f(a,z) \geq [g_2(x,a);q]_\infty f(0,z). \tag{2.3}$$

Combined with (2.2) and (2.3) gives

$$[g_2(x,a);q]_\infty f(0,z) \leq f(a,z) \leq [g_1(x,a);q]_\infty f(0,z). \tag{2.4}$$

Using Lemma 2.3, when $0 < q < \sqrt{2} - 1$, we have $q < \frac{1-q}{1+q}$. So, let $a = q$, also combining (2.2) and (2.3) gives the following inequality

$$\frac{f(q,z)}{[g_1(x,q);q]_\infty} \le f(0,z) \le \frac{f(q,z)}{[g_2(x,q);q]_\infty}.$$

Because of $f(q,z) = \frac{1}{1-z}$, we have

$$\frac{1}{(1-z)[g_1(x,q);q]_\infty} \le f(0,z) \le \frac{1}{(1-z)[g_2(x,q);q]_\infty}. \tag{2.5}$$

(2.4) and (2.5) yield the following inequality

$$\frac{[g_2(x,a);q]_\infty}{(1-z)[g_1(x,q);q]_\infty} \le f(a,z) \le \frac{[g_1(x,a);q]_\infty}{(1-z)[g_2(x,q);q]_\infty}.$$

That is

$$\frac{[g_2(x,a);q]_\infty}{(1-z)[g_1(x,q);q]_\infty} \le \sum_{n=0}^{\infty} \frac{(a;q)_n^2}{(q;q)_n^2} z^n \le \frac{[g_1(x,a);q]_\infty}{(1-z)[g_2(x,q);q]_\infty}.$$

If $a = q$, we have

$$\frac{[g_2(x,a);q]_\infty}{(1-z)[g_1(x,q);q]_\infty} = \sum_{n=0}^{\infty} \frac{(a;q)_n^2}{(q;q)_n^2} z^n = \frac{[g_1(x,a);q]_\infty}{(1-z)[g_2(x,q);q]_\infty} = \frac{1}{1-z}.$$

So the equality holds. We complete our proof. □

Corollary 2.5. *Under the conditions of the theorem, we have*

$$\sum_{n=0}^{\infty} \left[\frac{(a;q)_n}{(q;q)_n} - \frac{(az;q)_\infty}{(qz;q)_\infty} \right]^2 z^n \le \frac{\begin{vmatrix} [g_1(x,a);q]_\infty & [g_2(x,q);q]_\infty \\ (az;q)_\infty^2 & (qz;q)_\infty^2 \end{vmatrix}}{[g_2(x,q);q]_\infty (z;q)_\infty (qz;q)_\infty}. \tag{2.6}$$

Proof. Since

$$\begin{aligned}
&\sum_{n=0}^{\infty} \left[\frac{(a;q)_n}{(q;q)_n} - \frac{(az;q)_\infty}{(qz;q)_\infty} \right]^2 z^n \\
&= \sum_{n=0}^{\infty} \left[\frac{(a;q)_n^2}{(q;q)_n^2} - 2\frac{(a;q)_n(az;q)_\infty}{(q;q)_n(qz;q)_\infty} + \frac{(az;q)_\infty^2}{(qz;q)_\infty^2} \right] z^n \\
&= \sum_{n=0}^{\infty} \frac{(a;q)_n^2}{(q;q)_n^2} z^n - 2\frac{(az;q)_\infty}{(qz;q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n + \frac{(az;q)_\infty^2}{(qz;q)_\infty^2} \sum_{n=0}^{\infty} z^n \\
&= \sum_{n=0}^{\infty} \frac{(a;q)_n^2}{(q;q)_n^2} z^n - 2\frac{(az;q)_\infty}{(qz;q)_\infty} \frac{(az;q)_\infty}{(z;q)_\infty} + \frac{(az;q)_\infty^2}{(qz;q)_\infty^2} \frac{1}{1-z} \\
&= \sum_{n=0}^{\infty} \frac{(a;q)_n^2}{(q;q)_n^2} z^n - 2\frac{1}{1-z} \frac{(az;q)_\infty^2}{(qz;q)_\infty^2} + \frac{1}{1-z} \frac{(az;q)_\infty^2}{(qz;q)_\infty^2}
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{(a;q)_n^2}{(q;q)_n^2} z^n - \frac{1}{1-z} \frac{(az;q)_\infty^2}{(qz;q)_\infty^2}$$
$$\leq \frac{[g_1(x,a);q]_\infty}{(1-z)[g_2(x,q);q]_\infty} - \frac{1}{1-z} \frac{(az;q)_\infty^2}{(qz;q)_\infty^2}$$
$$= \frac{\begin{vmatrix} [g_1(x,a);q]_\infty & [g_2(x,q);q]_\infty \\ (az;q)_\infty^2 & (qz;q)_\infty^2 \end{vmatrix}}{[g_2(x,q);q]_\infty (z;q)_\infty (qz;q)_\infty}$$

we gain the inequality we seek. □

Theorem 2.6. *Under the conditions of the theorem, the following inequality holds*

$$\frac{(az;q)_\infty^2}{(qz;q)_\infty^2} \leq \frac{[g_1(x,a);q]_\infty}{[g_2(x,q);q]_\infty}. \tag{2.7}$$

Proof. From the proof of (2.6), we have

$$\frac{[g_1(x,a);q]_\infty}{(1-z)[g_2(x,q);q]_\infty} - \frac{1}{1-z} \frac{(az;q)_\infty^2}{(qz;q)_\infty^2} \geq \sum_{n=0}^{\infty} \left[\frac{(a;q)_n}{(q;q)_n} - \frac{(az;q)_\infty}{(qz;q)_\infty} \right]^2 z^n \geq 0$$

so the inequality (2.7) holds. □

Corollary 2.7. *Suppose* $0 < a < \frac{1-q}{1+q}$, $0 < b < \frac{1-q}{1+q} 0 < q < \sqrt{2} - 1$, $0 < z < 1$, *then the following inequality holds*

$$\sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n^2} z^n \leq \frac{(az,bz;q)_\infty}{(z;q)_\infty^2} + \frac{\begin{vmatrix} [g_1(x,a);q]_\infty & [g_2(x,q);q]_\infty \\ (az;q)_\infty^2 & (qz;q)_\infty^2 \end{vmatrix}^{\frac{1}{2}} \cdot \begin{vmatrix} [g_1(x,b);q]_\infty & [g_2(x,q);q]_\infty \\ (bz;q)_\infty^2 & (qz;q)_\infty^2 \end{vmatrix}^{\frac{1}{2}}}{[g_2(x,q);q]_\infty (z;q)_\infty (qz;q)_\infty}. \tag{2.8}$$

Proof. Noting that

$$\sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n^2} z^n > 0 \qquad \text{and} \qquad \frac{(az,bz;q)_\infty}{(z;q)_\infty^2} > 0,$$

we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n^2} z^n - \frac{(az,bz;q)_\infty}{(z;q)_\infty^2} \\ &\leq \left| \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n^2} z^n - \frac{(az,bz;q)_\infty}{(z;q)_\infty^2} \right| \\ &= \left| \sum_{n=0}^{\infty} \left[\frac{(a;q)_n}{(q;q)_n} - \frac{(az;q)_\infty}{(qz;q)_\infty} \right] \left[\frac{(b;q)_n}{(q;q)_n} - \frac{(bz;q)_\infty}{(qz;q)_\infty} \right] z^n \right| \\ &\leq \sum_{n=0}^{\infty} \left| \frac{(a;q)_n}{(q;q)_n} - \frac{(az;q)_\infty}{(qz;q)_\infty} \right| \cdot \left| \frac{(b;q)_n}{(q;q)_n} - \frac{(bz;q)_\infty}{(qz;q)_\infty} \right| z^n \end{aligned} \tag{2.9}$$

$$= \sum_{n=0}^{\infty} \left| \frac{(a;q)_n}{(q;q)_n} - \frac{(az;q)_\infty}{(qz;q)_\infty} \right| z^{\frac{n}{2}} \cdot \left| \frac{(b;q)_n}{(q;q)_n} - \frac{(bz;q)_\infty}{(qz;q)_\infty} \right| z^{\frac{n}{2}}.$$

Using the Cauchy inequality and (2.6), we have

$$\begin{aligned}
(2.10) \quad & \sum_{n=0}^{\infty} \left| \frac{(a;q)_n}{(q;q)_n} - \frac{(az;q)_\infty}{(qz;q)_\infty} \right| z^{\frac{n}{2}} \cdot \left| \frac{(b;q)_n}{(q;q)_n} - \frac{(bz;q)_\infty}{(qz;q)_\infty} \right| z^{\frac{n}{2}} \\
& \le \left\{ \sum_{n=0}^{\infty} \left[\frac{(a;q)_n}{(q;q)_n} - \frac{(az;q)_\infty}{(qz;q)_\infty} \right]^2 z^n \right\}^{\frac{1}{2}} \cdot \left\{ \sum_{n=0}^{\infty} \left[\frac{(b;q)_n}{(q;q)_n} - \frac{(bz;q)_\infty}{(qz;q)_\infty} \right]^2 z^n \right\}^{\frac{1}{2}} \\
& \le \left\{ \frac{\begin{vmatrix} \frac{[g_1(x,a);q]_\infty}{(az;q)_\infty^2} & \frac{[g_2(x,q);q]_\infty}{(qz;q)_\infty^2} \end{vmatrix}}{[g_2(x,q);q]_\infty (z;q)_\infty (qz;q)_\infty} \right\}^{\frac{1}{2}} \cdot \left\{ \frac{\begin{vmatrix} \frac{[g_1(x,b);q]_\infty}{(bz;q)_\infty^2} & \frac{[g_2(x,q);q]_\infty}{(qz;q)_\infty^2} \end{vmatrix}}{[g_2(x,q);q]_\infty (z;q)_\infty (qz;q)_\infty} \right\}^{\frac{1}{2}} \\
& = \frac{\begin{vmatrix} \frac{[g_1(x,a);q]_\infty}{(az;q)_\infty^2} & \frac{[g_2(x,q);q]_\infty}{(qz;q)_\infty^2} \end{vmatrix}^{\frac{1}{2}} \cdot \begin{vmatrix} \frac{[g_1(x,b);q]_\infty}{(bz;q)_\infty^2} & \frac{[g_2(x,q);q]_\infty}{(qz;q)_\infty^2} \end{vmatrix}^{\frac{1}{2}}}{[g_2(x,q);q]_\infty (z;q)_\infty (qz;q)_\infty}.
\end{aligned}$$

Combining (2.9) and (2.10) gives

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n^2} z^n \le \frac{(az,bz;q)_\infty}{(z;q)_\infty^2} \\
+ \frac{\begin{vmatrix} \frac{[g_1(x,a);q]_\infty}{(az;q)_\infty^2} & \frac{[g_2(x,q);q]_\infty}{(qz;q)_\infty^2} \end{vmatrix}^{\frac{1}{2}} \cdot \begin{vmatrix} \frac{[g_1(x,b);q]_\infty}{(bz;q)_\infty^2} & \frac{[g_2(x,q);q]_\infty}{(qz;q)_\infty^2} \end{vmatrix}^{\frac{1}{2}}}{[g_2(x,q);q]_\infty (z;q)_\infty (qz;q)_\infty}.
\end{aligned}$$

This is the inequality we seek. □

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