A SURPRISING RESULT IN COMPARING ORTHOGONAL AND NONORTHOGONAL LINEAR EXPERIMENTS

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Abstract: We demonstrate by example that within nonorthogonal linear experiments, a useful condition derived for comparing of the orthogonal ones not only fails but it may also lead to the reverse order.

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1. Preliminaries

Any linear experiment is determined by the expectation $E(\mathbf{y})$ and the variance-covariance matrix $V(\mathbf{y})$ of the observation vector $\mathbf{y}$. In the standard case these two moments have the following representation:

$$E(\mathbf{y}) = \mathbf{X}\boldsymbol{\beta} \quad \text{and} \quad V(\mathbf{y}) = \sigma\mathbf{I}_n, \tag{1.1}$$

where $\mathbf{X}$ is a known $n \times p$ *design matrix* while $\boldsymbol{\beta} = (\beta_1, \dots, \beta_p)'$ and σ are unknown parameters. To secure the identifiability of the parameters β_i's we assume that *rank*$(\mathbf{X}) = p$. Any standard linear experiment, being formally a structure of the form $(\mathbf{y},\mathbf{X}\boldsymbol{\beta},\sigma\mathbf{I}_n)$, will be denoted by $\mathcal{L}(\mathbf{X})$ and may be identified with its design matrix.

Now let us consider two linear experiments $\mathcal{L}_1 = \mathcal{L}(\mathbf{X}_1)$ and $\mathcal{L}_2 = \mathcal{L}(\mathbf{X}_2)$ with design matrices $\mathbf{X}_1$ and $\mathbf{X}_2$, respectively, and with common parameters $\boldsymbol{\beta}$ and σ. In Stępniak [7], Stępniak and Torgersen [8] and Stępniak, Wang and Wu [9] the experiment $\mathcal{L}_1$ is said to be at least as good as $\mathcal{L}_2$ if for any parametric function $\varphi = \mathbf{c}'\boldsymbol{\beta}$ the variance of its Best Linear Unbiased Estimator (BLUE) in $\mathcal{L}_1$ is not greater than in $\mathcal{L}_2$. It was shown in the above papers that this relation among linear experiments reduces to the Loewner ordering for their information matrices $\mathbf{M}_1 = \mathbf{X}_1'\mathbf{X}_1$ and $\mathbf{M}_2 = \mathbf{X}_2'\mathbf{X}_2$. It appears that this ordering is very strong.

Many authors, among others Kiefer [1] , Pukelsheim [4] Liski et al. [3], suggest some weaker criteria, among others of type A, D and E, based on some scalar functions of the information matrices. In this paper we focus on a reasonable criterion considered by Rao ([5, p. 236]).

Denote by $\mathcal{C}_p$ the class of all linear experiments with the same parameters $\boldsymbol{\beta}$ and σ, and by $\mathcal{O}_p$ its subclass containing orthogonal experiments only. Inspired by Rao we introduce the following definition.

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Definition 1.1. *We shall say that an experiment $\mathcal{L}_1$ belonging to $\mathcal{C}_p$ is better than $\mathcal{L}_2$ with respect to the estimation of single parameters (and write:$\mathcal{L}_1 \succ \mathcal{L}_2$) if for any β_i, $i = 1, ..., p$, its BLUE in $\mathcal{L}_1$ does not have greater variance than in $\mathcal{L}_2$ and less for some i.*

One can easily state an algebraic criterion for comparing experiments within $\mathcal{O}_p$. The aim of this note is to reveal the fact that this criterion may lead to a reverse order outside this class.

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2. Estimation and Comparison of Linear Experiments for Single Parameters

In this section we focus on estimation and comparison of linear experiments with respect to the estimation of single parameters β_i for all $i = 1, ..., p$. In this context a simple result provided by Scheffé ([6, Problem 1.5, p. 24]) will be useful. We shall state it in the form of a lemma.

Let $\mathcal{L} = \mathcal{L}(\mathbf{X})$ be a linear experiment of the form (1.1), where $\mathbf{X}$ is an $n \times p$ design matrix of rank p and let $\mathbf{x}_1, ..., \mathbf{x}_p$ be the columns of $\mathbf{X}$. For a given $\mathbf{x}_i$, $i = 1, ..., p$ denote by $\mathbf{P}_i$ the orthogonal projector onto the linear space generated by the remaining columns $\mathbf{x}_j$, $j \neq i$.

Lemma 2.1. *Under the above assumptions each parameter β_i in the experiment (1.1) is unbiasedly estimable and the variance of its BLUE may be presented in the form $\sigma(\mathbf{a}_i'\mathbf{a}_i)^{-1}$, where $\mathbf{a}_i = (\mathbf{I} - \mathbf{P}_i)\mathbf{x}_i$.*

In fact this lemma is a consequence of the well known Lehmann-Scheffé theorem on minimum variance unbiased estimation (cf. Lehmann and Scheffé [2]).

Now let us consider the class $\mathcal{O}_p$ of all orthogonal experiments, i.e. satisfying the condition $\mathbf{x}_i'\mathbf{x}_j = 0$ for $i \neq j$, with the same parameters $\boldsymbol{\beta}$ and σ. Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be matrices with columns $\mathbf{x}_{1,1}, ..., \mathbf{x}_{1,p}$ and $\mathbf{x}_{2,1}, ..., \mathbf{x}_{2,p}$, respectively. The following theorem is a direct consequence of Lemma 2.1.

Theorem 2.2. *For any orthogonal experiments $\mathcal{L}_1 = \mathcal{L}(\mathbf{X}_1)$ and $\mathcal{L}_2 = \mathcal{L}(\mathbf{X}_2)$ belonging to the class $\mathcal{O}_p$ the first one is better than the second one for estimation of single parameters, i.e. $\mathcal{L}_1 \succ \mathcal{L}_2$, if and only if,*

$$\mathbf{x}_{1,i}'\mathbf{x}_{1,i} \geq \mathbf{x}_{2,i}'\mathbf{x}_{2,i} \quad \text{for } i = 1, ..., p \text{ with strict inequality for some } i. \tag{2.1}$$

Now we shall demonstrate by example that the ordering rule (2.1) may lead to unexpected results outside the class $\mathcal{O}_p$.

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Example 2.1. Let $\mathbf{x}$ be an arbitrary n-column such that $\mathbf{x}'\mathbf{1}_n \neq \mathbf{0}$ and $\mathbf{x} \neq \lambda\mathbf{1}_n$ for any scalar λ. Consider two linear experiments $\mathcal{L}_1 = \mathcal{L}([\mathbf{1}_n, \mathbf{x}])$ and $\mathcal{L}_2 = \mathcal{L}([\mathbf{1}_n, (\mathbf{I}_n - \mathbf{P})\mathbf{x}])$ where $\mathbf{P} = \frac{1}{n}\mathbf{1}_n\mathbf{1}_n'$ is the orthogonal projector onto the one-dimensional linear space generated by $\mathbf{1}_n$. Since $\mathbf{x}'(\mathbf{I} - \mathbf{P})\mathbf{x} < \mathbf{x}'\mathbf{x}$, the condition (2.1) holds for $\mathbf{X}_1 = [\mathbf{1}_n, \mathbf{x}]$ and $\mathbf{X}_2 = [\mathbf{1}_n, (\mathbf{I}_n - \mathbf{P})\mathbf{x}]$. This may suggest that the experiment $\mathcal{L}_1$ is at least as good as $\mathcal{L}_2$ for estimation of the single parameters β_1 and β_2, i.e. that $\mathcal{L}(\mathbf{X}_1) \succ \mathcal{L}(\mathbf{X}_2)$.However, by Lemma 2.1, the variances of the BLUE's for β_2 in these two experiments are the same, while for β_1 the corresponding variance in $\mathcal{L}(\mathbf{X}_2)$ is less than in $\mathcal{L}(\mathbf{X}_1)$.

Conclusion. *In this example the condition (2.1) is met while* $\mathcal{L}(\mathbf{X}_2) \succ \mathcal{L}(\mathbf{X}_1)$.

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References

[1] J. KIEFER, Optimum Experimental Designs, *J. Roy. Statist. Soc. B*, **21** (1959), 272–304.

[2] E.L. LEHMANN AND H. SCHEFFÉ, Completeness, similar regions, and unbiased estimation - Part I, *Sankhyā*, **10** (1950), 305–340.

[3] E.P. LISKI, K.R. MANDAL, K.R. SHAH AND B.K. SINHA, *Topics in Optimal Designs*, Lecture Notes in Statistics, Springer-Verlag, New York- (2002)

[4] F. PUKELSHEIM, *Optimal Designs of Experiments*, Wiley, New York, 1993.

[5] C.R. RAO, *Linear Statistical Inference and its Applications*, 2nd. Ed., Wiley, New York, 1973.

[6] H. SCHEFFÉ, *The Analysis of Variance*, Wiley, New York, 1959.

[7] C. STĘPNIAK, Optimal allocation of units in experimental designs with hierarchical and cross classification, *Ann. Inst. Statist. Math. A*, **35** (1983), 461–473.

[8] C. STĘPNIAK AND E. TORGERSEN, Comparison of linear models with partially known covariances with respect to unbiased estimation, *Scand. J. Statist.*, **8** (1981), 183–184.

[9] C. STĘPNIAK, S.G. WANG AND C.F.J. WU, Comparison of linear experiments with known covariances, *Ann. Statist.*, **12** (1984), 358–365.

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