

# A SURPRISING RESULT IN COMPARING ORTHOGONAL AND NONORTHOGONAL LINEAR EXPERIMENTS

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*Key words:* Linear experiment, orthogonal/nonorthogonal experiment, single parameters, comparison of experiments.

*Abstract:* We demonstrate by example that within nonorthogonal linear experiments, a useful condition derived for comparing of the orthogonal ones not only fails but it may also lead to the reverse order.



Comparing Linear Experiments

[Czesław Stępniaik](#)

vol. 10, iss. 2, art. 31, 2009

[Title Page](#)

[Contents](#)

[◀◀](#) [▶▶](#)

[◀](#) [▶](#)

Page 1 of 7

[Go Back](#)

[Full Screen](#)

[Close](#)

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# Contents

- |   |   |
|---|---|
| 1 Preliminaries   | 3 |
| 2 Estimation and Comparison of Linear Experiments for Single Parameters | 5 |



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## Comparing Linear Experiments

Czesław Stępnik

vol. 10, iss. 2, art. 31, 2009

---

Title Page

Contents



Page 2 of 7

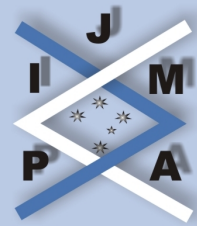
Go Back

Full Screen

Close

journal of **inequalities**  
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# 1. Preliminaries

Any linear experiment is determined by the expectation  $E(\mathbf{y})$  and the variance-covariance matrix  $V(\mathbf{y})$  of the observation vector  $\mathbf{y}$ . In the standard case these two moments have the following representation:

$$(1.1) \quad E(\mathbf{y}) = \mathbf{X}\boldsymbol{\beta} \quad \text{and} \quad V(\mathbf{y}) = \sigma\mathbf{I}_n,$$

where  $\mathbf{X}$  is a known  $n \times p$  design matrix while  $\boldsymbol{\beta} = (\beta_1, \dots, \beta_p)'$  and  $\sigma$  are unknown parameters. To secure the identifiability of the parameters  $\beta_i$ 's we assume that  $\text{rank}(\mathbf{X}) = p$ . Any standard linear experiment, being formally a structure of the form  $(\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \sigma\mathbf{I}_n)$ , will be denoted by  $\mathcal{L}(\mathbf{X})$  and may be identified with its design matrix.

Now let us consider two linear experiments  $\mathcal{L}_1 = \mathcal{L}(\mathbf{X}_1)$  and  $\mathcal{L}_2 = \mathcal{L}(\mathbf{X}_2)$  with design matrices  $\mathbf{X}_1$  and  $\mathbf{X}_2$ , respectively, and with common parameters  $\boldsymbol{\beta}$  and  $\sigma$ . In Stepniak [7], Stepniak and Torgersen [8] and Stepniak, Wang and Wu [9] the experiment  $\mathcal{L}_1$  is said to be at least as good as  $\mathcal{L}_2$  if for any parametric function  $\varphi = \mathbf{c}'\boldsymbol{\beta}$  the variance of its Best Linear Unbiased Estimator (BLUE) in  $\mathcal{L}_1$  is not greater than in  $\mathcal{L}_2$ . It was shown in the above papers that this relation among linear experiments reduces to the Loewner ordering for their information matrices  $\mathbf{M}_1 = \mathbf{X}_1'\mathbf{X}_1$  and  $\mathbf{M}_2 = \mathbf{X}_2'\mathbf{X}_2$ . It appears that this ordering is very strong.

Many authors, among others Kiefer [1], Pukelsheim [4] Liski et al. [3], suggest some weaker criteria, among others of type  $A$ ,  $D$  and  $E$ , based on some scalar functions of the information matrices. In this paper we focus on a reasonable criterion considered by Rao ([5, p. 236]).

Denote by  $\mathcal{C}_p$  the class of all linear experiments with the same parameters  $\boldsymbol{\beta}$  and  $\sigma$ , and by  $\mathcal{O}_p$  its subclass containing orthogonal experiments only. Inspired by Rao we introduce the following definition.

Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 3 of 7

Go Back

Full Screen

Close

**Definition 1.1.** We shall say that an experiment  $\mathcal{L}_1$  belonging to  $\mathcal{C}_p$  is better than  $\mathcal{L}_2$  with respect to the estimation of single parameters (and write:  $\mathcal{L}_1 \succ \mathcal{L}_2$ ) if for any  $\beta_i$ ,  $i = 1, \dots, p$ , its BLUE in  $\mathcal{L}_1$  does not have greater variance than in  $\mathcal{L}_2$  and less for some  $i$ .

One can easily state an algebraic criterion for comparing experiments within  $\mathcal{O}_p$ . The aim of this note is to reveal the fact that this criterion may lead to a reverse order outside this class.



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## Comparing Linear Experiments

Czesław Stępnik

vol. 10, iss. 2, art. 31, 2009

---

Title Page

Contents



Page 4 of 7

Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756

## 2. Estimation and Comparison of Linear Experiments for Single Parameters

In this section we focus on estimation and comparison of linear experiments with respect to the estimation of single parameters  $\beta_i$  for all  $i = 1, \dots, p$ . In this context a simple result provided by Scheffé ([6, Problem 1.5, p. 24]) will be useful. We shall state it in the form of a lemma.

Let  $\mathcal{L} = \mathcal{L}(\mathbf{X})$  be a linear experiment of the form (1.1), where  $\mathbf{X}$  is an  $n \times p$  design matrix of rank  $p$  and let  $\mathbf{x}_1, \dots, \mathbf{x}_p$  be the columns of  $\mathbf{X}$ . For a given  $\mathbf{x}_i$ ,  $i = 1, \dots, p$  denote by  $\mathbf{P}_i$  the orthogonal projector onto the linear space generated by the remaining columns  $\mathbf{x}_j$ ,  $j \neq i$ .

**Lemma 2.1.** *Under the above assumptions each parameter  $\beta_i$  in the experiment (1.1) is unbiasedly estimable and the variance of its BLUE may be presented in the form  $\sigma(\mathbf{a}'_i \mathbf{a}_i)^{-1}$ , where  $\mathbf{a}_i = (\mathbf{I} - \mathbf{P}_i)\mathbf{x}_i$ .*

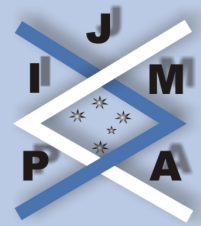
In fact this lemma is a consequence of the well known Lehmann-Scheffé theorem on minimum variance unbiased estimation (cf. Lehmann and Scheffé [2]).

Now let us consider the class  $\mathcal{O}_p$  of all orthogonal experiments, i.e. satisfying the condition  $\mathbf{x}'_i \mathbf{x}_j = 0$  for  $i \neq j$ , with the same parameters  $\beta$  and  $\sigma$ . Let  $\mathbf{X}_1$  and  $\mathbf{X}_2$  be matrices with columns  $\mathbf{x}_{1,1}, \dots, \mathbf{x}_{1,p}$  and  $\mathbf{x}_{2,1}, \dots, \mathbf{x}_{2,p}$ , respectively. The following theorem is a direct consequence of Lemma 2.1.

**Theorem 2.2.** *For any orthogonal experiments  $\mathcal{L}_1 = \mathcal{L}(\mathbf{X}_1)$  and  $\mathcal{L}_2 = \mathcal{L}(\mathbf{X}_2)$  belonging to the class  $\mathcal{O}_p$  the first one is better than the second one for estimation of single parameters, i.e.  $\mathcal{L}_1 \succ \mathcal{L}_2$ , if and only if,*

$$(2.1) \quad \mathbf{x}'_{1,i} \mathbf{x}_{1,i} \geq \mathbf{x}'_{2,i} \mathbf{x}_{2,i} \text{ for } i = 1, \dots, p \text{ with strict inequality for some } i.$$

Now we shall demonstrate by example that the ordering rule (2.1) may lead to unexpected results outside the class  $\mathcal{O}_p$ .



Title Page

Contents

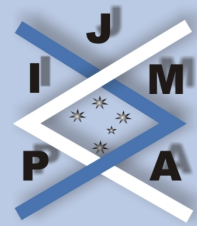


Page 5 of 7

Go Back

Full Screen

Close



*Example 2.1.* Let  $\mathbf{x}$  be an arbitrary  $n$ -column such that  $\mathbf{x}'\mathbf{1}_n \neq 0$  and  $\mathbf{x} \neq \lambda\mathbf{1}_n$  for any scalar  $\lambda$ . Consider two linear experiments  $\mathcal{L}_1 = \mathcal{L}([\mathbf{1}_n, \mathbf{x}])$  and  $\mathcal{L}_2 = \mathcal{L}([\mathbf{1}_n, (\mathbf{I}_n - \mathbf{P})\mathbf{x}])$  where  $\mathbf{P} = \frac{1}{n}\mathbf{1}_n\mathbf{1}_n'$  is the orthogonal projector onto the one-dimensional linear space generated by  $\mathbf{1}_n$ . Since  $\mathbf{x}'(\mathbf{I} - \mathbf{P})\mathbf{x} < \mathbf{x}'\mathbf{x}$ , the condition (2.1) holds for  $\mathbf{X}_1 = [\mathbf{1}_n, \mathbf{x}]$  and  $\mathbf{X}_2 = [\mathbf{1}_n, (\mathbf{I}_n - \mathbf{P})\mathbf{x}]$ . This may suggest that the experiment  $\mathcal{L}_1$  is at least as good as  $\mathcal{L}_2$  for estimation of the single parameters  $\beta_1$  and  $\beta_2$ , i.e. that  $\mathcal{L}(\mathbf{X}_1) \succ \mathcal{L}(\mathbf{X}_2)$ . However, by Lemma 2.1, the variances of the BLUE's for  $\beta_2$  in these two experiments are the same, while for  $\beta_1$  the corresponding variance in  $\mathcal{L}(\mathbf{X}_2)$  is less than in  $\mathcal{L}(\mathbf{X}_1)$ .

**Conclusion.** *In this example the condition (2.1) is met while  $\mathcal{L}(\mathbf{X}_2) \succ \mathcal{L}(\mathbf{X}_1)$ .*

---

## Comparing Linear Experiments

Czesław Stępnik

vol. 10, iss. 2, art. 31, 2009

---

Title Page

Contents



Page 6 of 7

Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756

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Comparing Linear Experiments

Czesław Stepniak

vol. 10, iss. 2, art. 31, 2009

---

[Title Page](#)

[Contents](#)



Page 7 of 7

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756