

Journal of Inequalities in Pure and Applied Mathematics

THE DISCRETE VERSION OF OSTROWSKI'S INEQUALITY IN NORMED LINEAR SPACES

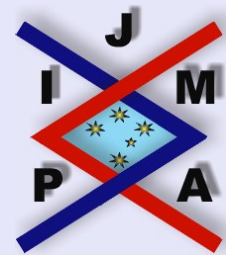
S.S. DRAGOMIR

School of Communications and Informatics
Victoria University of Technology
PO Box 14428
Melbourne City MC
8001 Victoria, Australia

*E*Mail: sever@matilda.vu.edu.au

*U*RL: <http://rgmia.vu.edu.au/SSDragomirWeb.html>

©2000 Victoria University
ISSN (electronic): 1443-5756
042-01



volume 3, issue 1, article 2,
2002.

Received 14 May, 2001;
accepted 02 July, 2001.

Communicated by: R.P. Agarwal

Abstract

Contents



Home Page

Go Back

Close

Quit

Abstract

Discrete versions of Ostrowski's inequality for vectors in normed linear spaces are given.

2000 Mathematics Subject Classification: Primary 26D15; Secondary 26D99.

Key words: Discrete Ostrowski's Inequality.

Contents

1	Introduction	3
2	Some Identities	8
3	Discrete Ostrowski's Inequality	12
4	Weighted Ostrowski Inequality	23
	References	



The Discrete Version of Ostrowski's Inequality in Normed Linear Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 2 of 34

1. Introduction

The following result is known in the literature as Ostrowski's inequality [10].

Theorem 1.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with the property that $|f'(t)| \leq M$ for all $t \in (a, b)$. Then*

$$(1.1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) M$$

for all $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

A simple proof of this fact can be done by using the identity:

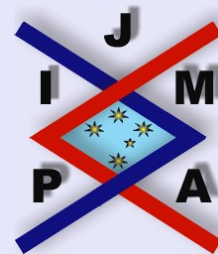
$$(1.2) \quad f(x) = \frac{1}{b-a} \int_a^b f(t) dt + \frac{1}{b-a} \int_a^b p(x, t) f'(t) dt, \quad x \in [a, b],$$

where

$$p(x, t) := \begin{cases} t - a & \text{if } a \leq t \leq x \\ t - b & \text{if } x < t \leq b \end{cases}$$

which also holds for absolutely continuous functions $f : [a, b] \rightarrow \mathbb{R}$.

The following Ostrowski type result for absolutely continuous functions holds (see [6] – [8]).



The Discrete Version of
Ostrowski's Inequality in
Normed Linear Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 3 of 34

Theorem 1.2. Let $f : [a, b] \rightarrow \mathbb{R}$ be absolutely continuous on $[a, b]$. Then, for all $x \in [a, b]$, we have:

$$(1.3) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \begin{cases} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \|f'\|_\infty & \text{if } f' \in L_\infty[a, b]; \\ \frac{1}{(p+1)^{\frac{1}{p}}} \left[\left(\frac{x-a}{b-a} \right)^{p+1} + \left(\frac{b-x}{b-a} \right)^{p+1} \right]^{\frac{1}{p}} (b-a)^{\frac{1}{p}} \|f'\|_q & \text{if } f' \in L_q[a, b], \\ & \frac{1}{p} + \frac{1}{q} = 1, p > 1; \\ \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \|f'\|_1 & \end{cases}$$

where $\|\cdot\|_r$ ($r \in [1, \infty]$) are the usual Lebesgue norms on $L_r[a, b]$, i.e.,

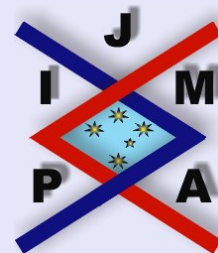
$$\|g\|_\infty := \operatorname{ess\,sup}_{t \in [a, b]} |g(t)|$$

and

$$\|g\|_r := \left(\int_a^b |g(t)|^r dt \right)^{\frac{1}{r}}, \quad r \in [1, \infty).$$

The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{\frac{1}{p}}}$ and $\frac{1}{2}$ respectively are sharp in the sense presented in Theorem 1.1.

The above inequalities can also be obtained from the Fink result in [9] on choosing $n = 1$ and performing some appropriate computations.



The Discrete Version of
Ostrowski's Inequality in
Normed Linear Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 4 of 34

If one drops the condition of absolute continuity and assumes that f is Hölder continuous, then one may state the result (see [5]):

Theorem 1.3. *Let $f : [a, b] \rightarrow \mathbb{R}$ be of $r - H$ -Hölder type, i.e.,*

$$(1.4) \quad |f(x) - f(y)| \leq H |x - y|^r, \text{ for all } x, y \in [a, b],$$

where $r \in (0, 1]$ and $H > 0$ are fixed. Then, for all $x \in [a, b]$, we have the inequality:

$$(1.5) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{H}{r+1} \left[\left(\frac{b-x}{b-a} \right)^{r+1} + \left(\frac{x-a}{b-a} \right)^{r+1} \right] (b-a)^r.$$

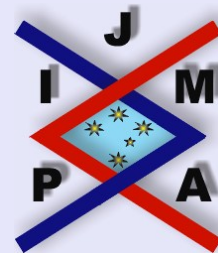
The constant $\frac{1}{r+1}$ is also sharp in the above sense.

Note that if $r = 1$, i.e., f is Lipschitz continuous, then we get the following version of Ostrowski's inequality for Lipschitzian functions (with L instead of H) (see [4])

$$(1.6) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) L.$$

Here the constant $\frac{1}{4}$ is also best.

Moreover, if one drops the condition of the continuity of the function, and assumes that it is of bounded variation, then the following result may be stated (see [2]).



The Discrete Version of
Ostrowski's Inequality in
Normed Linear Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 5 of 34

Theorem 1.4. Assume that $f : [a, b] \rightarrow \mathbb{R}$ is of bounded variation and denote by $\bigvee_a^b(f)$ its total variation. Then

$$(1.7) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b(f)$$

for all $x \in [a, b]$. The constant $\frac{1}{2}$ is the best possible.

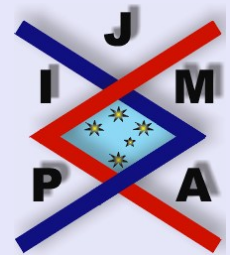
If we assume more about f , i.e., f is monotonically increasing, then the inequality (1.7) may be improved in the following manner [3] (see also [1]).

Theorem 1.5. Let $f : [a, b] \rightarrow \mathbb{R}$ be monotonic nondecreasing. Then for all $x \in [a, b]$, we have the inequality:

$$(1.8) \quad \begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{1}{b-a} \left\{ [2x - (a+b)] f(x) + \int_a^b \operatorname{sgn}(t-x) f(t) dt \right\} \\ & \leq \frac{1}{b-a} \{ (x-a)[f(x) - f(a)] + (b-x)[f(b) - f(x)] \} \\ & \leq \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] [f(b) - f(a)]. \end{aligned}$$

All the inequalities in (1.8) are sharp and the constant $\frac{1}{2}$ is the best possible.

For other recent results including Ostrowski type inequalities for n -time differentiable functions, visit the RGMIA website at <http://rgmia.vu.edu.au/database.html>.



The Discrete Version of
Ostrowski's Inequality in
Normed Linear Spaces

S.S. Dragomir

Title Page

Contents



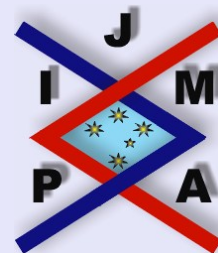
Go Back

Close

Quit

Page 6 of 34

In this paper we point out some discrete Ostrowski type inequalities for vectors in normed linear spaces.



**The Discrete Version of
Ostrowski's Inequality in
Normed Linear Spaces**

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 7 of 34

2. Some Identities

The following lemma holds.

Lemma 2.1. *Let x_i ($i = 1, \dots, n$) be vectors in X . Then we have the representation*

$$(2.1) \quad x_i = \frac{1}{n} \sum_{j=1}^n x_j + \frac{1}{n} \sum_{j=1}^n p(i, j) \Delta x_j, \quad i \in \{1, \dots, n\},$$

where

$$(2.2) \quad p(1, j) = j - n \quad \text{if } 1 \leq j \leq n - 1;$$

$$(2.3) \quad p(n, j) = j \quad \text{if } 1 \leq j \leq n - 1;$$

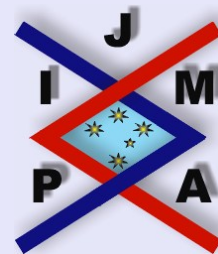
and

$$(2.4) \quad p(i, j) = \begin{cases} j & \text{if } 1 \leq j \leq i - 1, \\ j - n & \text{if } i \leq j \leq n - 1, \end{cases}$$

where $2 \leq i \leq n - 1$ and $1 \leq j \leq n - 1$.

Proof. For $i = 1$, we have to prove that

$$(2.5) \quad x_1 = \frac{1}{n} \sum_{j=1}^n x_j + \frac{1}{n} \sum_{j=1}^n (j - n) \Delta x_j.$$



The Discrete Version of
Ostrowski's Inequality in
Normed Linear Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 8 of 34

Using the summation by parts formula, we have

$$\begin{aligned}
 \sum_{j=1}^n (j-n) \Delta x_j &= (j-n) x_j \Big|_{j=1}^n - \sum_{j=1}^{n-1} \Delta (j-n) x_{j+1} \\
 &= (n-1) x_1 - \sum_{j=1}^{n-1} x_{j+1} \\
 &= n x_1 - \sum_{j=1}^n x_j
 \end{aligned}$$

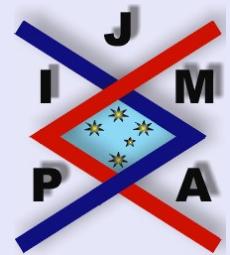
and the formula (2.5) is proved.

For $i = n$, we can prove similarly that

$$(2.6) \quad x_n = \frac{1}{n} \sum_{j=1}^n x_j + \frac{1}{n} \sum_{j=1}^{n-1} j \Delta x_j.$$

Let $2 \leq i \leq n-1$. We have

$$\begin{aligned}
 (2.7) \quad \sum_{j=1}^{n-1} p(i, j) \Delta x_j &= \sum_{j=1}^{i-1} p(i, j) \Delta x_j + \sum_{j=i}^{n-1} p(i, j) \Delta x_j \\
 &= \sum_{j=1}^{i-1} i \Delta x_j + \sum_{j=i}^{n-1} (j-n) \Delta x_j.
 \end{aligned}$$



The Discrete Version of
Ostrowski's Inequality in
Normed Linear Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 9 of 34

Using the summation by parts formula, we have

$$\begin{aligned}
 (2.8) \quad \sum_{j=1}^{i-1} i \Delta x_j &= j x_j \Big|_{j=i}^n - \sum_{j=1}^{i-1} \Delta(i) x_{j+1} \\
 &= i x_i - x_1 - \sum_{j=1}^{i-1} x_{j+1} = (i-1) x_i - \sum_{j=1}^{i-1} x_j
 \end{aligned}$$

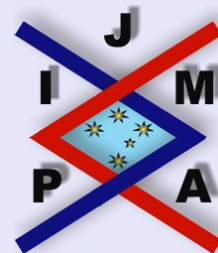
and

$$\begin{aligned}
 (2.9) \quad \sum_{j=i}^{n-1} (j-n) \Delta x_j &= (j-n) x_j \Big|_{j=i}^n - \sum_{j=i}^{n-1} \Delta(j-n) x_{j+1} \\
 &= (n-i) x_i - \sum_{j=i}^{n-1} x_{j+1} \\
 &= (n-i+1) x_i - \sum_{j=i}^n x_j.
 \end{aligned}$$

Using (2.7) – (2.9), we deduce

$$\begin{aligned}
 \sum_{j=1}^{n-1} p(i, j) \Delta x_j &= (i-1) x_i - \sum_{j=1}^{i-1} x_j + (n-i+1) x_i - \sum_{j=i}^n x_j \\
 &= n x_i - \sum_{j=1}^n x_j
 \end{aligned}$$

and the identity (2.1) is proved. \square



**The Discrete Version of
Ostrowski's Inequality in
Normed Linear Spaces**

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 10 of 34

The following corollaries hold.

Corollary 2.2. *We have the identity*

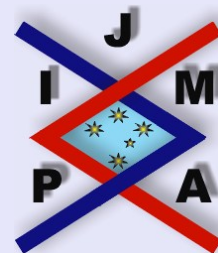
$$(2.10) \quad \frac{x_1 + x_n}{2} = \frac{1}{n} \sum_{j=1}^n x_j + \frac{1}{n} \sum_{j=1}^n \left(j - \frac{n}{2} \right) \Delta x_j.$$

Corollary 2.3. *Let $n = 2m + 1$. Then we have*

$$(2.11) \quad x_{m+1} = \frac{1}{2m+1} \sum_{j=1}^{2m+1} x_j + \frac{1}{2m+1} \sum_{j=1}^{2m} p_m(j) \Delta x_j,$$

where

$$p_m(j) = \begin{cases} j & \text{if } 1 \leq j \leq m, \\ j - 2m - 1 & \text{if } m + 1 \leq j \leq 2m. \end{cases}$$



The Discrete Version of
Ostrowski's Inequality in
Normed Linear Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 11 of 34

3. Discrete Ostrowski's Inequality

The following discrete inequality of Ostrowski type holds.

Theorem 3.1. *Let $(X, \|\cdot\|)$ be a normed linear space and x_i ($i = 1, \dots, n$) be vectors in X . Then we have the inequality*

$$(3.1) \quad \left\| x_i - \frac{1}{n} \sum_{k=1}^n x_k \right\| \leq \frac{1}{n} \left[\left(i - \frac{n+1}{2} \right)^2 + \frac{n^2-1}{4} \right] \max_{k=1, \dots, n-1} \|\Delta x_k\|,$$

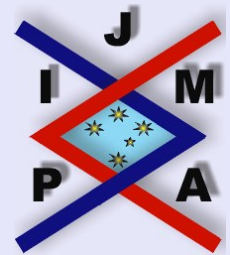
for all $i \in \{1, \dots, n\}$. The constant $c = \frac{1}{4}$ in the right hand side is best in the sense that it cannot be replaced by a smaller one.

Proof. We use the representation (2.1) and the generalised triangle inequality to obtain

$$\begin{aligned} \left\| x_i - \frac{1}{n} \sum_{k=1}^n x_k \right\| &= \frac{1}{n} \left\| \sum_{k=1}^{n-1} p(i, k) \Delta x_k \right\| \\ &\leq \frac{1}{n} \sum_{k=1}^{n-1} |p(i, k)| \|\Delta x_k\| \\ &\leq \max_{k=1, \dots, n-1} \|\Delta x_k\| \times \frac{1}{n} \sum_{k=1}^{n-1} |p(i, k)|. \end{aligned}$$

If $i = 1$, then we have

$$\sum_{k=1}^{n-1} |p(1, k)| = \sum_{k=1}^{n-1} |k - n| = \sum_{k=1}^{n-1} k = \frac{n(n-1)}{2}$$



The Discrete Version of
Ostrowski's Inequality in
Normed Linear Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 12 of 34

and as

$$\left(1 - \frac{n+1}{2}\right)^2 + \frac{n^2-1}{4} = \frac{n(n-1)}{2}, \text{ for } n \geq 1$$

the inequality (3.1) is valid for $i = 1$.

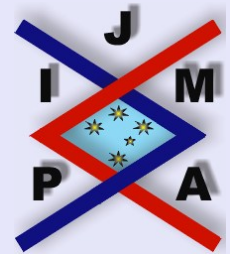
Let $2 \leq i \leq n - 1$. Then

$$\begin{aligned} \sum_{k=1}^{n-1} |p(i, k)| &= \sum_{k=1}^{i-1} |p(i, k)| + \sum_{k=i}^{n-1} |p(i, k)| \\ &= \sum_{k=1}^{i-1} k + \sum_{k=i}^{n-1} (n-k) \\ &= \frac{(i-1)i}{2} + n(n-1-i+1) - \left(\sum_{k=1}^{n-1} k - \sum_{k=1}^{i-1} k \right) \\ &= \frac{(i-1)i}{2} + n(n-i) - \left(\frac{n(n-1)}{2} - \frac{i(i-1)}{2} \right) \\ &= \frac{1}{2} (2i^2 + n^2 - 2ni + n) \\ &= \left(i - \frac{n+1}{2} \right)^2 + \frac{n^2-1}{4} \end{aligned}$$

and the inequality (3.1) is also proved for $i \in \{2, \dots, n-1\}$.

For $i = n$, we have $p(n, k) = k, k = 1, \dots, n-1$ giving

$$\sum_{k=1}^{n-1} |p(n, k)| = \sum_{k=1}^{n-1} k = \frac{n(n-1)}{2}$$



The Discrete Version of
Ostrowski's Inequality in
Normed Linear Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 13 of 34

and as

$$\left(n - \frac{n+1}{2}\right)^2 + \frac{n^2-1}{4} = \frac{n(n-1)}{2}$$

the inequality (3.2) is also valid for $i = n$.

To prove the sharpness of the constant $c = \frac{1}{4}$, assume that (3.1) holds with a constant $c > 0$, i.e.,

$$(3.2) \quad \left\|x_i - \frac{1}{n} \sum_{k=1}^n x_k\right\| \leq \frac{1}{n} \left[\left(i - \frac{n+1}{2}\right)^2 + c(n^2-1) \right] \max_{k=1, \dots, n-1} \|\Delta x_k\|$$

for any x_k ($k = 1, \dots, n$) in X .

Let $x_k = x_1 + (k-1)r$, $k = 1, \dots, n$, $r \in X$, $r \neq 0$, $x_1 \neq 0$ and $i = 1$ in (3.2). Then we get

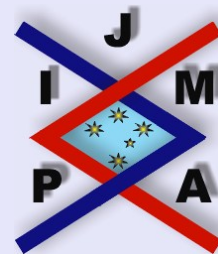
$$(3.3) \quad \left\|x_1 - \frac{1}{n} \sum_{k=1}^n (x_1 + (k-1)r)\right\| \leq \frac{1}{n} \left[\frac{(n-1)^2}{4} + c(n^2-1) \right] \|r\|$$

and as

$$\sum_{k=1}^n (x_1 + (k-1)r) = nx_1 + \frac{n(n-1)}{2}r,$$

then from (3.3) we deduce

$$\left\| \left(\frac{n-1}{2}\right) \cdot r \right\| \leq \frac{1}{n} \left[\frac{(n-1)^2}{4} + c(n^2-1) \right] \|r\|$$



The Discrete Version of
Ostrowski's Inequality in
Normed Linear Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 14 of 34

from where we get

$$\frac{1}{2} \leq \frac{1}{n} \left[\frac{n-1}{4} + c(n+1) \right]$$

i.e.,

$$n+1 \leq 4c(n+1),$$

which implies that $c \geq \frac{1}{4}$, and the theorem is proved. □

Corollary 3.2. *Under the above assumptions and if $n = 2m + 1$, then we have the inequality*

$$(3.4) \quad \left\| x_{m+1} - \frac{1}{2m+1} \sum_{k=1}^{2m+1} x_k \right\| \leq \frac{m(m+1)}{2m+1} \max_{k=1, \dots, 2m} \|\Delta x_k\|.$$

The proof is obvious by the above Theorem 3.1 for $i = m + 1$.

The following corollary also holds.

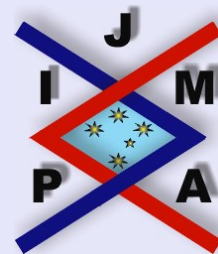
Corollary 3.3. *Under the above assumptions, we have:*

a) *If $n = 2k$, then*

$$(3.5) \quad \left\| \frac{x_1 + x_{2k}}{2} - \frac{1}{2k} \sum_{j=1}^{2k} x_j \right\| \leq \frac{1}{2} (k-1) \max_{j=1, \dots, 2k-1} \|\Delta x_j\|.$$

b) *If $n = 2k + 1$, then*

$$(3.6) \quad \left\| \frac{x_1 + x_{2k+1}}{2} - \frac{1}{2k+1} \sum_{j=1}^{2k+1} x_j \right\| \leq \frac{2k^2 + 2k + 1}{2(2k+1)} \max_{j=1, \dots, 2k} \|\Delta x_j\|.$$



The Discrete Version of
Ostrowski's Inequality in
Normed Linear Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 15 of 34

Proof. The proof is as follows.

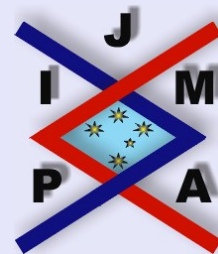
a) If $n = 2k$, then by Corollary 2.2, we have

$$\begin{aligned}
 & \left\| \frac{x_1 + x_{2k}}{2} - \frac{1}{2k} \sum_{j=1}^{2k} x_j \right\| \\
 & \leq \frac{1}{2k} \sum_{j=1}^{2k-1} |j - k| \|\Delta x_j\| \\
 & \leq \frac{1}{2k} \max_{j=1, \dots, 2k-1} \|\Delta x_j\| \sum_{j=1}^{2k-1} |j - k| \\
 & = \frac{1}{2k} \max_{j=1, \dots, 2k-1} \|\Delta x_j\| \left(\sum_{j=1}^k (k - j) + \sum_{j=k+1}^{2k-1} (j - k) \right) \\
 & = \frac{1}{k} \max_{j=1, \dots, 2k-1} \|\Delta x_j\| \frac{(k-1)k}{2} \\
 & = \frac{1}{2} (k-1) \max_{j=1, \dots, 2k-1} \|\Delta x_j\|,
 \end{aligned}$$

and the inequality (3.5) is proved.

b) If $n = 2k + 1$, then by Corollary 2.2, we have

$$\left\| \frac{x_1 + x_{2k+1}}{2} - \frac{1}{2k+1} \sum_{j=1}^{2k+1} x_j \right\| \leq \frac{1}{2k+1} \sum_{j=1}^{2k+1} \left| j - \frac{2k+1}{2} \right| \|\Delta x_j\|$$



The Discrete Version of
Ostrowski's Inequality in
Normed Linear Spaces

S.S. Dragomir

Title Page

Contents

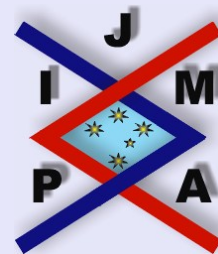


Go Back

Close

Quit

Page 16 of 34



The Discrete Version of
Ostrowski's Inequality in
Normed Linear Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 17 of 34

$$\begin{aligned}
 &\leq \frac{1}{2k+1} \max_{j=1, \dots, 2k} \|\Delta x_j\| \sum_{j=1}^{2k+1} \left| j - k - \frac{1}{2} \right| \\
 &= \frac{1}{2k+1} \max_{j=1, \dots, 2k} \|\Delta x_j\| \left[\sum_{j=1}^k \left(k + \frac{1}{2} - j \right) + \sum_{j=k+1}^{2k+1} \left(j - k - \frac{1}{2} \right) \right] \\
 &= \frac{1}{2k+1} \max_{j=1, \dots, 2k} \|\Delta x_j\| \left[\frac{1}{2}k + \sum_{j=1}^k (k - j) - \frac{1}{2}(k + 1) + \sum_{j=k+1}^{2k+1} (j - k) \right] \\
 &= \frac{1}{2k+1} \max_{j=1, \dots, 2k} \|\Delta x_j\| \left[\frac{k^2 - k + k^2 + 3k + 2 - 1}{2} \right] \\
 &= \max_{j=1, \dots, 2k} \|\Delta x_j\| \frac{2k^2 + 2k + 1}{2(2k + 1)}
 \end{aligned}$$

and the inequality (3.6) is proved. □

The following result including a version of a discrete Ostrowski inequality for l_p -norms of $\{\Delta x_i\}_{i=1, \dots, n-1}$ also holds.

Theorem 3.4. *Let $(X, \|\cdot\|)$ be a normed linear space and x_i ($i = 1, \dots, n$) be vectors in X . Then we have the inequality*

$$(3.7) \quad \left\| x_i - \frac{1}{n} \sum_{k=1}^n x_k \right\| \leq \frac{1}{n} [s_\alpha(i-1) + s_\alpha(n-i)]^{\frac{1}{\alpha}} \left[\sum_{k=1}^{n-1} \|\Delta x_k\|^\beta \right]^{\frac{1}{\beta}}$$

for all $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, where $s_\alpha(\cdot)$ denotes the sum:

$$s_\alpha(m) := \sum_{j=1}^m j^\alpha.$$

When $m = 0$, the sum is assumed to be zero.

Proof. Using representation (2.2) and the generalised triangle inequality, we have:

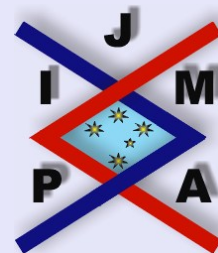
$$\begin{aligned} (3.8) \quad \left\| x_i - \frac{1}{n} \sum_{k=1}^n x_k \right\| &= \frac{1}{n} \left\| \sum_{k=1}^{n-1} p(i, k) \Delta x_k \right\| \\ &\leq \frac{1}{n} \sum_{k=1}^{n-1} |p(i, k)| \|\Delta x_k\| =: B. \end{aligned}$$

Using Hölder's discrete inequality, we have

$$(3.9) \quad B \leq \frac{1}{n} \left(\sum_{k=1}^{n-1} |p(i, k)|^\alpha \right)^{\frac{1}{\alpha}} \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^\beta \right)^{\frac{1}{\beta}}.$$

However,

$$\sum_{k=1}^{n-1} |p(i, k)|^\alpha = \sum_{k=1}^{i-1} |p(i, k)|^\alpha + \sum_{k=i}^{n-1} |p(i, k)|^\alpha$$



The Discrete Version of
Ostrowski's Inequality in
Normed Linear Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 18 of 34

$$\begin{aligned}
&= \sum_{k=1}^{i-1} k^\alpha + \sum_{k=i}^{n-1} (n-k)^\alpha \\
&= 1^\alpha + \dots + (i-1)^\alpha + (n-i)^\alpha + \dots + 1^\alpha \\
&= s_\alpha(i-1) + s_\alpha(n-i)
\end{aligned}$$

and the inequality (3.7) then follows by (3.8) and (3.9). □

The case of $\alpha = \beta = 2$ can be useful in practical applications.

Corollary 3.5. *With the assumptions of Theorem 3.4, we have*

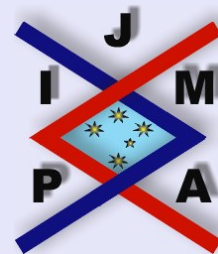
$$\begin{aligned}
(3.10) \quad &\left\| x_i - \frac{1}{n} \sum_{k=1}^n x_k \right\| \\
&\leq \frac{1}{\sqrt{n}} \left[\left(i - \frac{n+1}{2} \right)^2 + \frac{n^2-1}{12} \right]^{\frac{1}{2}} \left[\sum_{k=1}^{n-1} \|\Delta x_k\|^2 \right]^{\frac{1}{2}}.
\end{aligned}$$

Proof. For $\alpha = 2$, we have

$$s_2(i-1) = \sum_{k=1}^{i-1} k^2 = \frac{i(i-1)(2i-1)}{6}$$

and

$$s_2(n-i) = \sum_{k=1}^{n-i} k^2 = \frac{(n-i)(n-i+1)[2(n-i)+1]}{6}.$$



The Discrete Version of Ostrowski's Inequality in Normed Linear Spaces

S.S. Dragomir

Title Page
Contents
◀◀
▶▶
◀
▶
Go Back
Close
Quit
Page 19 of 34

As simple algebra proves that

$$s_2(i-1) + s_2(n-i) = n \left[\left(i - \frac{n+1}{2} \right)^2 + \frac{n^2-1}{12} \right],$$

then, by (3.7) we deduce the desired inequality (3.10). \square

Corollary 3.6. *Under the above assumptions and if $n = 2m + 1$, then we have the inequality:*

$$(3.11) \quad \left\| x_{m+1} - \frac{1}{2m+1} \sum_{k=1}^{2m+1} x_k \right\| \leq \frac{2^{\frac{1}{\alpha}}}{2m+1} [s_\alpha(m)]^{\frac{1}{\alpha}} \left[\sum_{k=1}^{2m} \|\Delta x_k\|^\beta \right]^{\frac{1}{\beta}}$$

for $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

In particular, for $\alpha = \beta = 2$, we have

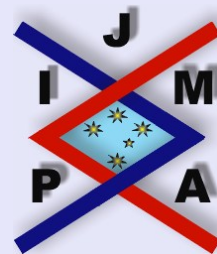
$$(3.12) \quad \left\| x_{m+1} - \frac{1}{2m+1} \sum_{k=1}^{2m+1} x_k \right\| \leq \sqrt{\frac{m(m+1)}{3(2m+1)}} \left[\sum_{k=1}^{2m} \|\Delta x_k\|^2 \right]^{\frac{1}{2}}.$$

The following result providing an upper bound in terms of the l_1 -norm of $(\Delta x_k)_{k=1, n-1}$ also holds.

Theorem 3.7. *Let $(X, \|\cdot\|)$ be a normed linear space and x_i ($i = 1, \dots, n$) be vectors in X . Then we have the inequality*

$$(3.13) \quad \left\| x_i - \frac{1}{n} \sum_{k=1}^n x_k \right\| \leq \frac{1}{n} \left[\frac{1}{2} (n-1) + \left| i - \frac{n+1}{2} \right| \right] \sum_{k=1}^{n-1} \|\Delta x_k\|$$

for all $i \in \{1, \dots, n\}$.



The Discrete Version of
Ostrowski's Inequality in
Normed Linear Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 20 of 34

Proof. As in Theorem 3.4, we have

$$(3.14) \quad \left\| x_i - \frac{1}{n} \sum_{k=1}^n x_k \right\| \leq B,$$

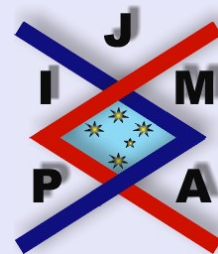
where

$$B := \frac{1}{n} \sum_{k=1}^{n-1} |p(i, k)| \|\Delta x_k\|.$$

It is obvious that

$$\begin{aligned} B &= \frac{1}{n} \left[\sum_{k=1}^{i-1} k \|\Delta x_k\| + \sum_{k=i}^{n-1} (n-k) \|\Delta x_k\| \right] \\ &\leq \frac{1}{n} \left[(i-1) \sum_{k=1}^{i-1} \|\Delta x_k\| + (n-i) \sum_{k=i}^{n-1} \|\Delta x_k\| \right] \\ &= \frac{1}{n} \max \{i-1, n-i\} \left[\sum_{k=1}^{i-1} \|\Delta x_k\| + \sum_{k=i}^{n-1} \|\Delta x_k\| \right] \\ &= \frac{1}{n} \left[\frac{1}{2} (n-1) + \frac{1}{2} |n-i-i+1| \right] \sum_{k=1}^{n-1} \|\Delta x_k\| \\ &= \frac{1}{n} \left[\frac{1}{2} (n-1) + \left| i - \frac{n+1}{2} \right| \right] \sum_{k=1}^{n-1} \|\Delta x_k\| \end{aligned}$$

and the inequality (3.13) is proved. \square



The Discrete Version of
Ostrowski's Inequality in
Normed Linear Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

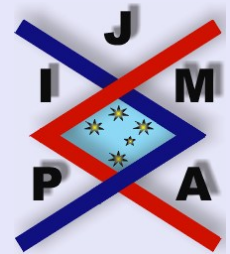
Quit

Page 21 of 34

The following corollary contains the best inequality we can get from (3.13).

Corollary 3.8. *Let $(X, \|\cdot\|)$ be as above and $n = 2m + 1$. Then we have the inequality*

$$(3.15) \quad \left\| x_{m+1} - \frac{1}{2m+1} \sum_{k=1}^{2m+1} x_k \right\| \leq \frac{m}{2m+1} \sum_{k=1}^{2m} \|\Delta x_k\|.$$



The Discrete Version of
Ostrowski's Inequality in
Normed Linear Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 22 of 34

4. Weighted Ostrowski Inequality

We start with the following theorem.

Theorem 4.1. Let $(X, \|\cdot\|)$ be a normed linear space, $x_i \in X$ ($i = 1, \dots, n$) and $p_i \geq 0$ ($i = 1, \dots, n$) with $\sum_{i=1}^n p_i = 1$. Then we have the inequality:

$$(4.1) \quad \left\| x_i - \sum_{j=1}^n p_j x_j \right\| \leq \sum_{j=1}^n p_j |j - i| \cdot \max_{k=1, n-1} \|\Delta x_k\|$$

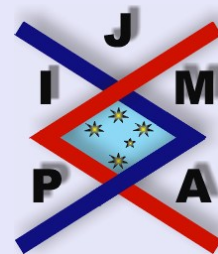
$$\leq \max_{k=1, n-1} \|\Delta x_k\| \times \begin{cases} \frac{n-1}{2} + \left| i - \frac{n+1}{2} \right|, \\ \left(\sum_{j=1}^n |j - i|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^n p_j^q \right)^{\frac{1}{q}} \text{ if } p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \left[\frac{n^2-1}{4} + \left(i - \frac{n+1}{2} \right)^2 \right] \max_{j=1, n} \{p_j\} \end{cases}$$

for all $i \in \{1, \dots, n\}$.

Proof. Using the properties of the norm, we have

$$(4.2) \quad \sum_{j=1}^n p_j \|x_i - x_j\| \geq \left\| \sum_{j=1}^n p_j (x_i - x_j) \right\|$$

$$= \left\| x_i \sum_{j=1}^n p_j - \sum_{j=1}^n p_j x_j \right\| = \left\| x_i - \sum_{j=1}^n p_j x_j \right\|,$$



The Discrete Version of
Ostrowski's Inequality in
Normed Linear Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 23 of 34

for all $i \in \{1, \dots, n\}$.

On the other hand,

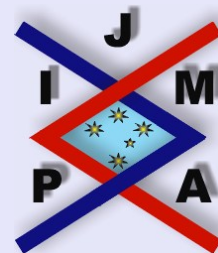
$$\begin{aligned}
 (4.3) \quad & \sum_{j=1}^n p_j \|x_i - x_j\| \\
 &= \sum_{j=1}^{i-1} p_j \|x_i - x_j\| + \sum_{j=i+1}^n p_j \|x_i - x_j\| \\
 &= \sum_{j=1}^{i-1} p_j \left\| \sum_{k=j}^{i-1} (x_{k+1} - x_k) \right\| + \sum_{j=i+1}^n p_j \left\| \sum_{l=i}^{j-1} (x_{l+1} - x_l) \right\| \\
 &\leq \sum_{j=1}^{i-1} p_j \left(\sum_{k=j}^{i-1} \|\Delta x_k\| \right) + \sum_{j=i+1}^n p_j \left(\sum_{l=i}^{j-1} \|\Delta x_l\| \right) \\
 &=: A.
 \end{aligned}$$

Now, as

$$\sum_{k=j}^{i-1} \|\Delta x_k\| \leq (i-j) \max_{k=j, i-1} \|\Delta x_k\| \quad (\text{where } j \leq i-1)$$

and

$$\sum_{l=i}^{s-1} \|\Delta x_l\| \leq (s-i) \max_{l=i, n-1} \|\Delta x_l\| \quad (\text{where } i \leq s-1),$$



**The Discrete Version of
Ostrowski's Inequality in
Normed Linear Spaces**

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 24 of 34

then we deduce that

$$\begin{aligned}
 A &\leq \sum_{j=1}^{i-1} p_j (i-j) \cdot \max_{k=j, i-1} \|\Delta x_k\| + \sum_{j=i+1}^n p_j (j-i) \cdot \max_{l=i, n-1} \|\Delta x_l\| \\
 &\leq \max_{k=1, n-1} \|\Delta x_k\| \left[\sum_{j=1}^{i-1} p_j (i-j) + \sum_{j=i+1}^n p_j (j-i) \right] \\
 &= \max_{k=1, n-1} \|\Delta x_k\| \cdot \sum_{j=1}^n p_j |i-j|
 \end{aligned}$$

and the first inequality in (4.1) is proved.

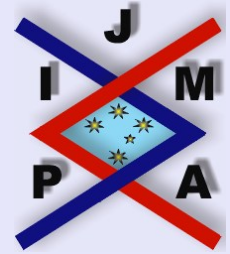
Now, we observe that

$$\begin{aligned}
 \sum_{j=1}^n p_j |i-j| &\leq \max_{j=1, n} |i-j| \sum_{j=1}^n p_j \\
 &= \max_{j=1, n} |i-j| \\
 &= \max \{i-1, n-i\} \\
 &= \frac{n-1}{2} + \left| i - \frac{n+1}{2} \right|,
 \end{aligned}$$

which proves the first part of the second inequality in (4.1).

By Hölder's discrete inequality, we also have

$$\sum_{j=1}^n p_j |i-j| \leq \left(\sum_{j=1}^n p_j^q \right)^{\frac{1}{q}} \left(\sum_{j=1}^n |i-j|^p \right)^{\frac{1}{p}},$$



The Discrete Version of
Ostrowski's Inequality in
Normed Linear Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 25 of 34

where $p > q$ and $\frac{1}{p} + \frac{1}{q} = 1$, and the second part of the second inequality in (4.1) holds.

Finally, we also have

$$\sum_{j=1}^n p_j |i - j| \leq \max_{j=1, n} |p_j| \sum_{j=1}^n |i - j|.$$

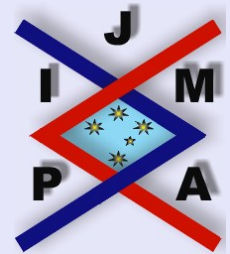
Now, let us observe that

$$\begin{aligned} \sum_{j=1}^n |i - j| &= \sum_{j=1}^i |i - j| + \sum_{j=i+1}^n |i - j| \\ &= \sum_{j=1}^i (i - j) + \sum_{j=i+1}^n (j - i) \\ &= i^2 - \frac{i(i+1)}{2} + \sum_{j=1}^n j - \sum_{j=1}^i j - i(n-i) \\ &= \frac{n^2 - 1}{4} + \left(i - \frac{n+1}{2}\right)^2 \end{aligned}$$

and the last part of the second inequality in (4.1) is proved. \square

Remark 4.1. In some practical applications the case $p = q = 2$ in the second part of the second inequality may be useful. As

$$\sum_{j=1}^n (j - i)^2 = \sum_{j=1}^n j^2 - 2i \sum_{j=1}^n j + ni^2 = n \left[\frac{n^2 - 1}{12} + \left(i - \frac{n+1}{2}\right)^2 \right],$$



The Discrete Version of
Ostrowski's Inequality in
Normed Linear Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 26 of 34

then we may state the inequality

$$(4.4) \quad \left\| x_i - \sum_{j=1}^n p_j x_j \right\| \leq \sqrt{n} \left[\frac{n^2 - 1}{12} + \left(i - \frac{n+1}{2} \right)^2 \right]^{\frac{1}{2}} \left(\sum_{j=1}^n p_j^2 \right)^{\frac{1}{2}} \max_{k=1, n-1} \|\Delta x_k\|$$

for all $i \in \{1, \dots, n\}$.

The following particular case was proved in a different manner in Theorem 3.1.

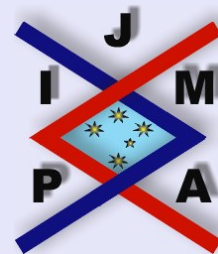
Corollary 4.2. *If x_i ($i = 1, \dots, n$) are vectors in the normed linear space $(X, \|\cdot\|)$, then we have*

$$(4.5) \quad \left\| x_i - \frac{1}{n} \sum_{j=1}^n x_j \right\| \leq \frac{1}{n} \left[\frac{n^2 - 1}{4} + \left(i - \frac{n+1}{2} \right)^2 \right] \max_{k=1, n-1} \|\Delta x_k\|.$$

The following result also holds.

Theorem 4.3. *Let $(X, \|\cdot\|)$ be a normed linear space, $x_i \in X$ ($i = 1, \dots, n$) and $p_i \geq 0$ ($i = 1, \dots, n$) with $\sum_{i=1}^n p_i = 1$. Then, for $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, we have the inequality:*

$$(4.6) \quad \left\| x_i - \sum_{j=1}^n p_j x_j \right\| \leq \sum_{j=1}^n |i - j|^{\frac{1}{\beta}} p_j \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^\alpha \right)^{\frac{1}{\alpha}}$$



The Discrete Version of
Ostrowski's Inequality in
Normed Linear Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 27 of 34

$$\leq \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^\alpha \right)^{\frac{1}{\alpha}} \times \begin{cases} \left[\frac{1}{2}(n-1) + \left| i - \frac{n+1}{2} \right| \right]^{\frac{1}{\beta}}, \\ \left(\sum_{j=1}^n |i-j|^{\frac{\delta}{\beta}} \right)^{\frac{1}{\delta}} \left(\sum_{j=1}^n p_j^\gamma \right)^{\frac{1}{\gamma}} & \text{if } \gamma > 1, \frac{1}{\gamma} + \frac{1}{\delta} = 1, \\ \sum_{j=1}^n |i-j|^{\frac{1}{\beta}} \max_{j=1, n} \{p_j\} \end{cases}$$

for all $i \in \{1, \dots, n\}$.

Proof. Using Hölder's discrete inequality, we may write that

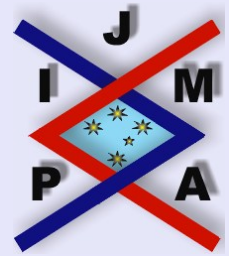
$$\sum_{k=j}^{i-1} \|\Delta x_k\| \leq (i-j)^{\frac{1}{\beta}} \left(\sum_{k=j}^{i-1} \|\Delta x_k\|^\alpha \right)^{\frac{1}{\alpha}}$$

and

$$\sum_{l=i}^{s-1} \|\Delta x_l\| \leq (s-i)^{\frac{1}{\beta}} \left(\sum_{l=i}^{s-1} \|\Delta x_l\|^\alpha \right)^{\frac{1}{\alpha}},$$

which implies for A , as defined in the proof of Theorem 4.1, that

$$\begin{aligned} A &\leq \sum_{j=1}^{i-1} (i-j)^{\frac{1}{\beta}} \left(\sum_{k=j}^{i-1} \|\Delta x_k\|^\alpha \right)^{\frac{1}{\alpha}} p_j + \sum_{s=i+1}^n (s-i)^{\frac{1}{\beta}} \left(\sum_{l=i}^{s-1} \|\Delta x_l\|^\alpha \right)^{\frac{1}{\alpha}} p_s \\ &\leq \left(\sum_{k=1}^{i-1} \|\Delta x_k\|^\alpha \right)^{\frac{1}{\alpha}} \sum_{j=1}^{i-1} (i-j)^{\frac{1}{\beta}} p_j + \left(\sum_{l=i}^{n-1} \|\Delta x_l\|^\alpha \right)^{\frac{1}{\alpha}} \sum_{s=i+1}^n (s-i)^{\frac{1}{\beta}} p_s \end{aligned}$$



The Discrete Version of
Ostrowski's Inequality in
Normed Linear Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 28 of 34

$$\begin{aligned} &\leq \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^\alpha \right)^{\frac{1}{\alpha}} \left[\sum_{j=1}^{i-1} (i-j)^{\frac{1}{\beta}} p_j + \sum_{s=i+1}^n (s-i)^{\frac{1}{\beta}} p_s \right] \\ &= \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^\alpha \right)^{\frac{1}{\alpha}} \sum_{j=1}^n |i-j|^{\frac{1}{\beta}} p_j, \end{aligned}$$

which proves the first inequality in (4.6).

Now it is obvious that

$$\begin{aligned} \sum_{j=1}^n |i-j|^{\frac{1}{\beta}} p_j &\leq \max_{j=1,n} |i-j|^{\frac{1}{\beta}} \sum_{j=1}^n p_j \\ &= \max \left\{ (i-1)^{\frac{1}{\beta}}, (n-i)^{\frac{1}{\beta}} \right\} \\ &= \left[\frac{1}{2}(n-1) + \left| i - \frac{n+1}{2} \right| \right]^{\frac{1}{\beta}}, \end{aligned}$$

proving the first part of the second inequality in (4.6).

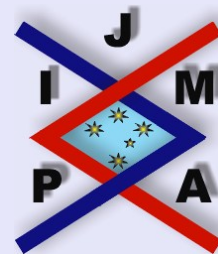
For $\gamma, \delta > 1$ with $\frac{1}{\gamma} + \frac{1}{\delta} = 1$, we have

$$\sum_{j=1}^n |i-j|^{\frac{1}{\beta}} p_j \leq \left(\sum_{j=1}^n p_j^\gamma \right)^{\frac{1}{\gamma}} \left(\sum_{j=1}^n |i-j|^{\frac{\delta}{\beta}} \right)^{\frac{1}{\delta}}$$

obtaining the second part of the second inequality in (4.6).

Finally, we observe that

$$\sum_{j=1}^n |i-j|^{\frac{1}{\beta}} p_j \leq \max_{j=1,n} \{p_j\} \sum_{j=1}^n |i-j|^{\frac{1}{\beta}},$$



**The Discrete Version of
Ostrowski's Inequality in
Normed Linear Spaces**

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 29 of 34

and the theorem is proved. \square

Corollary 4.4. *If x_i ($i = 1, \dots, n$) are vectors in the normed space $(X, \|\cdot\|)$, then for all $i \in \{1, \dots, n\}$ we have:*

$$(4.7) \quad \left\| x_i - \frac{1}{n} \sum_{j=1}^n x_j \right\| \leq \frac{1}{n} \sum_{j=1}^n |i-j|^{\frac{1}{\beta}} \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^\alpha \right)^{\frac{1}{\alpha}}, \quad \alpha > 1, \quad \frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

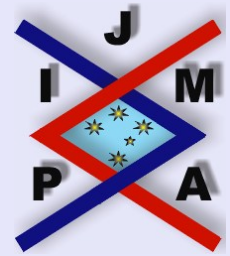
Finally, we may state the following result as well.

Theorem 4.5. *Let X , x_i and p_i ($i = 1, \dots, n$) be as in Theorem 4.3. Then we have the inequality:*

$$(4.8) \quad \left\| x_i - \sum_{j=1}^n p_j x_j \right\| \leq \begin{cases} \max \{P_{i-1}, 1 - P_i\} \sum_{k=1}^{n-1} \|\Delta x_k\| \\ (1 - p_i) \max \left\{ \sum_{k=1}^{i-1} \|\Delta x_k\|, \sum_{k=i}^{n-1} \|\Delta x_k\| \right\} \end{cases} \leq (1 - p_i) \sum_{j=1}^{n-1} \|\Delta x_k\|$$

for all $i \in \{1, \dots, n\}$, where

$$P_m := \sum_{i=1}^m p_i, \quad m = 1, \dots, n$$



The Discrete Version of
Ostrowski's Inequality in
Normed Linear Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 30 of 34

and $P_0 := 0$.

Proof. It is obvious that

$$\sum_{k=j}^{i-1} \|\Delta x_k\| \leq \sum_{k=1}^{i-1} \|\Delta x_k\|$$

and

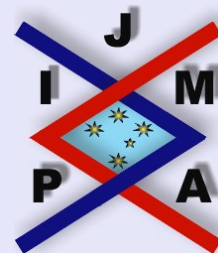
$$\sum_{l=i}^{s-1} \|\Delta x_l\| \leq \sum_{l=i}^{n-1} \|\Delta x_l\|,$$

Then, for A as defined in the proof of Theorem 4.1, we have that

$$\begin{aligned} A &\leq \sum_{k=1}^{i-1} \|\Delta x_k\| \sum_{j=1}^{i-1} p_j + \sum_{l=i}^{n-1} \|\Delta x_l\| \sum_{j=i+1}^n p_j \\ &=: B \\ &\leq \max \{P_{i-1}, 1 - P_i\} \left[\sum_{j=1}^{i-1} \|\Delta x_j\| + \sum_{j=i+1}^{n-1} \|\Delta x_j\| \right] \\ &= \max \{P_{i-1}, 1 - P_i\} \sum_{k=1}^{n-1} \|\Delta x_k\|. \end{aligned}$$

Also, we observe that

$$B \leq \max \left\{ \sum_{j=1}^{i-1} \|\Delta x_j\|, \sum_{j=i+1}^{n-1} \|\Delta x_j\| \right\} (P_{i-1} + 1 - P_i)$$



The Discrete Version of
Ostrowski's Inequality in
Normed Linear Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 31 of 34

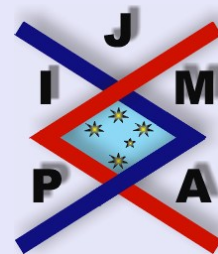
$$= (1 - p_i) \max \left\{ \sum_{k=1}^{i-1} \|\Delta x_k\|, \sum_{k=i}^{n-1} \|\Delta x_k\| \right\}$$

and the theorem is thus proved. □

Corollary 4.6. *Let X and x_i ($i = 1, \dots, n$) be as in Corollary 4.4. Then*

$$(4.9) \quad \left\| x_i - \frac{1}{n} \sum_{j=1}^n x_j \right\| \leq \begin{cases} \frac{1}{n} \left[\frac{1}{2} (n-1) + \left| i - \frac{n+1}{2} \right| \right] \sum_{k=1}^{n-1} \|\Delta x_k\|, \\ \frac{n-1}{n} \max \left\{ \sum_{k=1}^{i-1} \|\Delta x_k\|, \sum_{k=i}^{n-1} \|\Delta x_k\| \right\} \end{cases}$$

for all $i \in \{1, \dots, n\}$.



The Discrete Version of
Ostrowski's Inequality in
Normed Linear Spaces

S.S. Dragomir

Title Page

Contents



Go Back

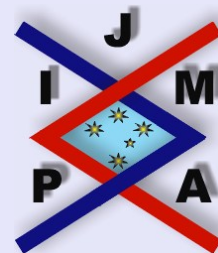
Close

Quit

Page 32 of 34

References

- [1] P. CERONE AND S.S. DRAGOMIR, Midpoint type rules from an inequalities point of view, in *Analytic-Computational Methods in Applied Mathematics*, G.A. Anastassiou (Ed), CRC Press, New York, 2000, 135–200.
- [2] S.S. DRAGOMIR, On the Ostrowski's inequality for mappings of bounded variation and applications, *Math. Ineq. & Appl.*, **4**(1) (2001), 59–66. Preprint on line: *RGMIA Res. Rep. Coll.*, **2**(1) (1999), Article 7, <http://rgmia.vu.edu.au/v2n1.html>
- [3] S.S. DRAGOMIR, Ostrowski's inequality for monotonous mappings and applications, *J. KSIAM*, **3**(1) (1999), 127–135.
- [4] S.S. DRAGOMIR, The Ostrowski's integral inequality for Lipschitzian mappings and applications, *Comp. and Math. with Appl.*, **38** (1999), 33–37.
- [5] S.S. DRAGOMIR, P. CERONE, J. ROUMELIOTIS AND S. WANG, A weighted version of Ostrowski inequality for mappings of Hölder type and applications in numerical analysis, *Bull. Math. Soc. Sci. Math. Roumanie*, **42(90)**(4) (1992), 301–314.
- [6] S.S. DRAGOMIR AND S. WANG, A new inequality of Ostrowski's type in L_1 -norm and applications to some special means and to some numerical quadrature rules, *Tamkang J. of Math.*, **28** (1997), 239–244.
- [7] S.S. DRAGOMIR AND S. WANG, A new inequality of Ostrowski's type in L_p -norm, *Indian J. of Math.*, **40**(3) (1998), 245–304.



The Discrete Version of
Ostrowski's Inequality in
Normed Linear Spaces

S.S. Dragomir

Title Page

Contents



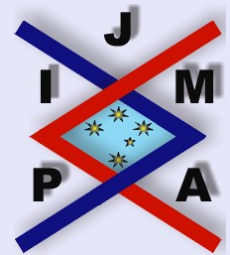
Go Back

Close

Quit

Page 33 of 34

- [8] S.S. DRAGOMIR AND S. WANG, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules, *Appl. Math. Lett.*, **11** (1998), 105–109.
- [9] A.M. FINK, Bounds on the derivation of a function from its averages, *Czech. Math. J.*, **42(117)** (1992), 289–310.
- [10] A. OSTROWSKI, Uber die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert, *Comment. Math. Hel*, **10** (1938), 226–227.



The Discrete Version of
Ostrowski's Inequality in
Normed Linear Spaces

S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 34 of 34