HADAMARD PRODUCT VERSIONS OF THE CHEBYSHEV AND KANTOROVICH INEQUALITIES

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Abstract: The purpose of this note is to prove Hadamard product versions of the Chebyshev and the Kantorovich inequalities for positive real numbers. We also prove a generalization of Fiedler's inequality.

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Contents

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1. Introduction

In what follows, the capital letters $A, B, C, \ldots$ denote $m \times m$ complex matrices, whereas the small letters $a, b, c, \ldots$ denote real numbers, unless mentioned otherwise. By $X \geq Y$ we mean that $X - Y$ is positive semidefinite ($X > Y$ mean $X - Y$ is positive definite). For $A = (a_{ij})$ and $B = (b_{ij})$, $A \circ B = (a_{ij}b_{ij})$ denotes the Hadamard product of A and B. According to Schur's theorem [4, Page 23] the Hadamard product is monotone in the sense that $A \geq B$, $C \geq D$ implies $A \circ C \geq B \circ D$. The tensor product $A \otimes B$ is the $m^2 \times m^2$ matrix

$$\begin{pmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & & \vdots \\ a_{m1}B & \cdots & a_{mm}B \end{pmatrix}. \tag{1.1}$$

Marcus and Khan in [10] made the simple but important observation that the Hadamard product is a principal submatrix of the tensor product. The inequality

$$\left(\sum_{i=1}^{n} w_i a_i\right)\left(\sum_{i=1}^{n} w_i b_i\right) \leq \sum_{i=1}^{n} w_i a_i b_i \tag{1.2}$$

holds for all $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$, $b_1 \geq b_2 \geq \cdots \geq b_n \geq 0$ and weights $w_i \geq 0$, $i = 1, \ldots, n$. Hardy, Littlewood and Polya [6, page 43] attribute this inequality to Chebyshev. For $0 < a \leq a_i \leq b, w_i \geq 0, i = 1, 2, \ldots, n$, Kantorovich's inequality states that

$$\left(\sum_{i=1}^{n} w_i a_i\right)\left(\sum_{i=1}^{n} \frac{w_i}{a_i}\right) \leq \frac{(a+b)^2}{4ab}\left(\sum_{i=1}^{n} w_i\right)^2. \tag{1.3}$$

Hadamard Product Versions
Jagjit Singh Matharu and Jaspal Singh Aujla
vol. 10, iss. 2, art. 51, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

In Section 2, we state and prove matrix versions of inequalities (1.2) and (1.3) involving the Hadamard product. A generalization of Fiedler's inequality is also proved in this section. There are several generalizations of Kantorovich and Fiedler's inequality; see [2, 3, 8, 9].

journal of inequalities in pure and applied mathematics
issn: 1443-5756

2. The Chebyshev and Kantorovich Inequalities: Matrix Versions

We begin with a Hadamard product version of inequality (1.2).

Theorem 2.1. *Let* $A_1 \geq \cdots \geq A_n \geq 0$ *and* $B_1 \geq \cdots \geq B_n \geq 0$. *Then*

$$\left(\sum_{i=1}^{n} w_i A_i\right) \circ \left(\sum_{i=1}^{n} w_i B_i\right) \leq \left(\sum_{i=1}^{n} w_i\right) \left(\sum_{i=1}^{n} w_i (A_i \circ B_i)\right), \tag{2.1}$$

where $w_i \geq 0,\ i = 1, \ldots, n$, *are weights.*

Proof. We have

$$\begin{aligned}
\left(\sum_{i=1}^{n} w_i\right) \left(\sum_{i=1}^{n} w_i (A_i \circ B_i)\right) - \left(\sum_{i=1}^{n} w_i A_i\right) \circ \left(\sum_{i=1}^{n} w_i B_i\right) \qquad\qquad (2.2)\\
= \sum_{i,j=1}^{n} (w_i w_j (A_j \circ B_j) - w_i w_j (A_i \circ B_j)) \\
= \frac{1}{2} \sum_{i,j=1}^{n} \Big(w_i w_j (A_j \circ B_j) - w_i w_j (A_i \circ B_j) \\
+ w_j w_i (A_i \circ B_i) - w_j w_i (A_j \circ B_i) \Big) \\
= \frac{1}{2} \sum_{i,j=1}^{n} w_i w_j (A_i - A_j) \circ (B_i - B_j).
\end{aligned}$$

Since the Hadamard product of two positive semidefinite matrices is positive semidefinite, therefore the summand in 2.2 is positive semidefinite. □

Hadamard Product Versions
Jagjit Singh Matharu and Jaspal Singh Aujla
vol. 10, iss. 2, art. 51, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Our next result is a Hadamard product version of inequality (1.3) .

Theorem 2.2. *Let* $A_1, \ldots, A_n$ *be such that* $0 < aI_m \le A_i \le bI_m,\ i = 1, \ldots, n$ *(here* I_m *denotes the* $m \times m$ *identity matrix). Then*

$$\left(\sum_{i=1}^{n} W_i^{1/2} A_i W_i^{1/2}\right) \circ \left(\sum_{i=1}^{n} W_i^{1/2} A_i^{-1} W_i^{1/2}\right) \le \frac{a^2+b^2}{2ab}\left(\sum_{i=1}^{n} W_i\right) \circ \left(\sum_{i=1}^{n} W_i\right) \tag{2.3}$$

for all $W_i \ge 0,\ i = 1, \ldots, n.$

Proof. We first prove the inequality

$$P^{1/2}AP^{1/2} \circ Q^{1/2}B^{-1}Q^{1/2} + P^{1/2}A^{-1}P^{1/2} \circ Q^{1/2}BQ^{1/2} \le \frac{a^2+b^2}{ab}(P \circ Q), \tag{2.4}$$

when $0 < aI_m \le A, B \le bI_m$ and $P, Q \ge 0$. Let $A = UDU^*$ and $B = V\Gamma V^*$ with unitary U and V, and diagonal matrices D and Γ. Then

$$\begin{aligned} A \otimes B^{-1} + A^{-1} \otimes B &= (U \otimes V)(D \otimes \Gamma + \Gamma^{-1} \otimes D)(U \otimes V)^* \\ &\le (U \otimes V)\left(\frac{a^2+b^2}{ab}(I_m \otimes I_m)\right)(U \otimes V)^* \\ &= \frac{a^2+b^2}{ab}(I_m \otimes I_m), \end{aligned}$$

where the inequality follows from (1.3). Thus we have

$$P^{1/2}AP^{1/2} \otimes Q^{1/2}B^{-1}Q^{1/2} + P^{1/2}A^{-1}P^{1/2} \otimes Q^{1/2}BQ^{1/2} \tag{2.5}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

$$= (P^{1/2} \otimes Q^{1/2})(A \otimes B^{-1} + A^{-1} \otimes B)(P^{1/2} \otimes Q^{1/2})$$
$$\leq \frac{a^2+b^2}{ab}(P \otimes Q).$$

Since the Hadamard product is a principal submatrix of the tensor product, the inequality (2.4) follows from (2.5). On taking $B = A$ and $Q = P$ in (2.4) we see that (2.3) holds for $n = 1$. Further, by (2.4) we have

$$W_i^{1/2}A_iW_i^{1/2} \circ W_j^{1/2}A_j^{-1}W_j^{1/2} + W_i^{1/2}A_i^{-1}W_i^{1/2} \circ W_j^{1/2}A_jW_j^{1/2} \leq \frac{a^2+b^2}{ab}(W_i \circ W_j)$$

for $i, j = 1, \ldots, n$. Summing over i, j, we have

$$2\sum_{i,j=1}^{n}\left[\left(W_i^{1/2}A_iW_i^{1/2}\right) \circ \left(W_j^{1/2}A_j^{-1}W_j^{1/2}\right)\right] \leq \left(\frac{a^2+b^2}{ab}\right)\sum_{i,j=1}^{n}(W_i \circ W_j), \tag{2.6}$$

which implies that

$$\left(\sum_{i=1}^{n} W_i^{1/2}A_iW_i^{1/2}\right) \circ \left(\sum_{i=1}^{n} W_i^{1/2}A_i^{-1}W_i^{1/2}\right) \leq \left(\frac{a^2+b^2}{2ab}\right)\left(\sum_{i=1}^{n} W_i\right) \circ \left(\sum_{i=1}^{n} W_i\right).$$

□

The next corollary follows on taking $W_i = w_iI_m$, $i = 1, \ldots, n$.

Corollary 2.3. *Let $A_1, \ldots, A_n$ be such that $0 < aI_m \leq A_i \leq bI_m$, and $w_i \geq 0,\ i = 1, \ldots, n$ be weights. Then*

$$\left(\sum_{i=1}^{n} w_iA_i\right) \circ \left(\sum_{i=1}^{n} w_iA_i^{-1}\right) \leq \left(\frac{a^2+b^2}{2ab}\right)\left(\sum_{i=1}^{n} w_i\right)^2 I_m.$$

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Hadamard Product Versions
Jagjit Singh Matharu and
Jaspal Singh Aujla
vol. 10, iss. 2, art. 51, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Remark 1. The case $n = 1$ of Corollary 2.3 is proved in [7]. The example

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad a = \frac{3-\sqrt{5}}{2}, \quad b = \frac{3+\sqrt{5}}{2}$$

shows that the inequality

$$A \circ A^{-1} \le \frac{(a+b)^2}{4ab} I_2$$

need not be true.

For our next result we need the following lemma.

Lemma 2.4. *Let* $0 \le r \le 1$. *Then* $A^r + A^{-r} \le A + A^{-1}$ *for all* $A > 0$.

Proof. Suppose that $A = U\Gamma U^*$ with unitary U and diagonal matrix Γ. Then

$$\begin{aligned} A^r + A^{-r} &= U(\Gamma^r + \Gamma^{-r})U^* \\ &\le U(\Gamma + \Gamma^{-1})U^* = A + A^{-1} \end{aligned}$$

since $x^r + x^{-r} \le x + x^{-1}$ for any positive real number x and $0 \le r \le 1$. □

Theorem 2.5. *Let* $0 \le \alpha < \beta$. *Then*

$$\begin{aligned} &\left(\sum_{i=1}^n W_i^{1/2} A_i^{\alpha} W_i^{1/2}\right) \circ \left(\sum_{i=1}^n W_i^{1/2} A_i^{-\alpha} W_i^{1/2}\right) \\ &\qquad \le \left(\sum_{i=1}^n W_i^{1/2} A_i^{\beta} W_i^{1/2}\right) \circ \left(\sum_{i=1}^n W_i^{1/2} A_i^{-\beta} W_i^{1/2}\right) \end{aligned}$$

for all $A_i > 0$ *and* $W_i \ge 0$, $i = 1, \dots, n$.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Proof. We first prove the inequality

$$\left(W_i^{1/2}A_i^{\alpha}W_i^{1/2}\right)\circ\left(W_j^{1/2}A_j^{-\alpha}W_j^{1/2}\right) + \left(W_i^{1/2}A_i^{-\alpha}W_i^{1/2}\right)\circ\left(W_j^{1/2}A_j^{\alpha}W_j^{1/2}\right) \le \left(W_i^{1/2}A_i^{\beta}W_i^{1/2}\right)\circ\left(W_j^{1/2}A_j^{-\beta}W_j^{1/2}\right) + \left(W_i^{1/2}A_i^{-\beta}W_i^{1/2}\right)\circ\left(W_j^{1/2}A_j^{\beta}W_j^{1/2}\right) \tag{2.7}$$

for $0 \le \alpha < \beta$. Let $0 \le r \le 1$. Then

$$\begin{aligned}
&\left(W_i^{1/2}A_i^{r}W_i^{1/2}\right)\otimes\left(W_j^{1/2}A_j^{-r}W_j^{1/2}\right) + \left(W_i^{1/2}A_i^{-r}W_i^{1/2}\right)\otimes\left(W_j^{1/2}A_j^{r}W_j^{1/2}\right)\\
&= \left(W_i^{1/2}\otimes W_j^{1/2}\right)\left(A_i^{r}\otimes A_j^{-r} + A_i^{-r}\otimes A_j^{r}\right)\left(W_i^{1/2}\otimes W_j^{1/2}\right)\\
&= \left(W_i^{1/2}\otimes W_j^{1/2}\right)\left(\left(A_i\otimes A_j^{-1}\right)^{r} + \left(A_i\otimes A_j^{-1}\right)^{-r}\right)\left(W_i^{1/2}\otimes W_j^{1/2}\right)\\
&\le \left(W_i^{1/2}\otimes W_j^{1/2}\right)\left(\left(A_i\otimes A_j^{-1}\right) + \left(A_i\otimes A_j^{-1}\right)^{-1}\right)\left(W_i^{1/2}\otimes W_j^{1/2}\right)
\end{aligned}$$

where the inequality follows from Lemma 2.4. Taking $r = \alpha/\beta$ and replacing A_i by A_i^{β} and A_j by A_j^{β}, we have

$$\begin{aligned}
&\left(W_i^{1/2}A_i^{\alpha}W_i^{1/2}\right)\otimes\left(W_j^{1/2}A_j^{-\alpha}W_j^{1/2}\right) + \left(W_i^{1/2}A_i^{-\alpha}W_i^{1/2}\right)\otimes\left(W_j^{1/2}A_j^{\alpha}W_j^{1/2}\right)\\
&\le \left(W_i^{1/2}A_i^{\beta}W_i^{1/2}\right)\otimes\left(W_j^{1/2}A_j^{-\beta}W_j^{1/2}\right) + \left(W_i^{1/2}A_i^{-\beta}W_i^{1/2}\right)\otimes\left(W_j^{1/2}A_j^{\beta}W_j^{1/2}\right).
\end{aligned}$$

Again using the fact that the Hadamard product is a principal submatrix of the tensor

journal of inequalities in pure and applied mathematics
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product, the preceding inequality implies (2.7). Summing over i, j in (2.7), we have

$$\left(\sum_{i=1}^{n} W_i^{1/2} A_i^{\alpha} W_i^{1/2}\right) \circ \left(\sum_{i=1}^{n} W_i^{1/2} A_i^{-\alpha} W_i^{1/2}\right) \leq \left(\sum_{i=1}^{n} W_i^{1/2} A_i^{\beta} W_i^{1/2}\right) \circ \left(\sum_{i=1}^{n} W_i^{1/2} A_i^{-\beta} W_i^{1/2}\right).$$

□

Corollary 2.6. *Let* $0 \leq \alpha < \beta$. *Then*

$$\left(\sum_{i=1}^{n} A_i^{\alpha}\right) \circ \left(\sum_{j=1}^{n} A_j^{-\alpha}\right) \leq \left(\sum_{i=1}^{n} A_i^{\beta}\right) \circ \left(\sum_{j=1}^{n} A_j^{-\beta}\right)$$

for all $A_i > 0$, $i = 1, \ldots, n$.

Proof. Taking $W_i = I_m$ in Theorem 2.5 we get the desired result. □

Corollary 2.7. *Let* $0 \leq \beta$. *Then*

$$I_m \leq \left(\sum_{i=1}^{n} W_i^{1/2} A_i^{\beta} W_i^{1/2}\right) \circ \left(\sum_{i=1}^{n} W_i^{1/2} A_i^{-\beta} W_i^{1/2}\right)$$

for all $A_i > 0$ *and* $W_i \geq 0$, $i = 1, \ldots, n$, *where* $\sum_{i=1}^{n} W_i = I_m$.

Proof. Taking $\alpha = 0$ in Theorem 2.5 gives the desired inequality. □

Remark 2. Corollary 2.7 is another generalization of Fiedler's inequality [5]

$$A \circ A^{-1} \geq I_m.$$

journal of inequalities in pure and applied mathematics
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Next we prove a convexity theorem involving the Hadamard product.

Theorem 2.8. *The function*

$$f(t) = A^{1+t} \circ B^{1-t} + A^{1-t} \circ B^{1+t}$$

is convex on the interval $[-1, 1]$ *and attains its minimum at* $t = 0$ *for all* $A, B > 0$.

Proof. Since f is continuous we need to prove only that f is mid-point convex. Note that for $A, B > 0$ and $s,\ t$ in $[-1, 1]$ the matrices

$$\begin{pmatrix} A^{1+s+t} & A^{1+s} \\ A^{1+s} & A^{1+(s-t)} \end{pmatrix}, \qquad \begin{pmatrix} A^{1-(s+t)} & A^{1-s} \\ A^{1-s} & A^{1-(s-t)} \end{pmatrix},$$

$$\begin{pmatrix} B^{1+s+t} & B^{1+s} \\ B^{1+s} & B^{1+(s-t)} \end{pmatrix}, \qquad \begin{pmatrix} B^{1-(s+t)} & B^{1-s} \\ B^{1-s} & B^{1-(s-t)} \end{pmatrix}$$

are positive semidefinite. Hence the matrix

$$X = \begin{pmatrix} A^{1+s+t} \circ B^{1-(s+t)} + A^{1-(s+t)} \circ B^{1+s+t} & A^{1+s} \circ B^{1-s} + A^{1-s} \circ B^{1+s} \\ A^{1+s} \circ B^{1-s} + A^{1-s} \circ B^{1+s} & A^{1+(s-t)} \circ B^{1-(s-t)} + A^{1-(s-t)} \circ B^{1+(s-t)} \end{pmatrix}$$

is positive semidefinite. Similarly, the matrix

$$Y = \begin{pmatrix} A^{1+(s-t)} \circ B^{1-(s-t)} + A^{1-(s-t)} \circ B^{1+(s-t)} & A^{1+s} \circ B^{1-s} + A^{1-s} \circ B^{1+s} \\ A^{1+s} \circ B^{1-s} + A^{1-s} \circ B^{1+s} & A^{1+(s+t)} \circ B^{1-(s+t)} + A^{1-(s+t)} \circ B^{1+s+t} \end{pmatrix}$$

is positive semidefinite. Hence

$$X + Y = \begin{pmatrix} f(s+t) + f(s-t) & 2f(s) \\ 2f(s) & f(s+t) + f(s-t) \end{pmatrix} \tag{2.8}$$

is positive semidefinite, which implies that

$$f(s) \leq \frac{1}{2}[f(s+t) + f(s-t)].$$

Hadamard Product Versions
Jagjit Singh Matharu and Jaspal Singh Aujla
vol. 10, iss. 2, art. 51, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

This proves the convexity of f. Further, note that $f(t) = f(-t)$. This together with the convexity of f implies that f attains its minimum at 0. □

Corollary 2.9. *The function*

$$g(t) = A^t \circ B^{1-t} + A^{1-t} \circ B^t$$

is decreasing on $[0, 1/2]$, *increasing on* $[1/2, 1]$, *and attains its minimum at* $t = \frac{1}{2}$ *for all* $A, B > 0$.

Proof. The proof follows on replacing A, B by $A^{1/2}, B^{1/2}$ and t by $\frac{1+t}{2}$ in Theorem 2.8. □

A norm $|||\cdot|||$ on $m \times m$ complex matrices is called unitarily invariant if $|||UXV||| = |||X|||$ for all unitary matrices U, V. If A is positive semidefinite and X is any matrix, then

$$|||A \circ X||| \leq \max\ a_{ii}|||X|||$$

for all unitarily invariant norms $|||\cdot|||$ [1]. Thus the proof of the following corollary follows from Corollary 2.9 using the fact that $g(1/2) \leq g(t) \leq g(1) = g(0)$.

Corollary 2.10. *Let* $0 \leq t \leq 1$. *Then,*

$$2|||A^{1/2} \circ B^{1/2}||| \leq |||A^t \circ B^{1-t} + A^{1-t} \circ B^t||| \leq |||A + B|||$$

for all unitarily invariant norms $|||\cdot|||$ *and all* $A, B > 0$.

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Hadamard Product Versions
Jagjit Singh Matharu and Jaspal Singh Aujla
vol. 10, iss. 2, art. 51, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

References

[1] T. ANDO, R.A. HORN AND C.R. JOHNSON, The singular values of the Hadamard product: A basic inequality, *Linear Multilinear Algebra*, **21** (1987), 345–365.

[2] J.K. BAKSALARY AND S. PUNTANEN, Generalized matrix versions of the Cauchy-Schwarz and Kantorovich inequalities, *Aequationes Math.*, **41** (1991), 103–110.

[3] R.B. BAPAT AND M.K. KWONG, A generalisation of $A \circ A^{-1} \geq I$, *Linear Algebra Appl.*, **93** (1987), 107–112.

[4] R. BHATIA, *Matrix Analysis*, Springer Verlag, New York, 1997.

[5] M. FIEDLER, Über eine Ungleichung für positiv definite Matrizen, *Math. Nachrichten*, **23** (1961), 197–199.

[6] G.H. HARDY, J.E. LITTLEWOOD AND G. POLYA, *Inequalities*, Cambridge University Press, Cambridge, 1959.

[7] J. MIĆIĆ, J. PECARIC AND Y. SEO, Complementary inequalities to inequalities of Jensen and Ando based on the Mond-Pečarić method, *Linear Algebra Appl.*, **318** (2000), 87–107.

[8] A.W. MARSHALL AND I. OLKIN, Matrix versions of the Cauchy and Kantorovich inequalities, *Aequationes Math.*, **40** (1990), 89–93.

[9] M. SINGH, J.S. AUJLA AND H.L. VASUDEVA, Inequalities for Hadamard product and unitarily invariant norms of matrices, *Linear Multilinear Algebra*, **48** (2000), 247–262.

[10] M. MARCUS AND N.A. KHAN, A note on Hadamard product, *Canad. Math. Bull.*, **2** (1950), 81–83.

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