

HADAMARD PRODUCT VERSIONS OF THE CHEBYSHEV AND KANTOROVICH INEQUALITIES

JAGJIT SINGH MATHARU AND JASPAL SINGH AUJLA

Department of Mathematics

National Institute of Technology

Jalandhar 144011, Punjab, INDIA

EEmail: matharuj@yahoo.com aujla@nitj.ac.in

Received: 10 February, 2009

Accepted: 15 April, 2009

Communicated by: [S. Puntanen](#)

2000 AMS Sub. Class.: Primary 15A48; Secondary 15A18, 15A45.

Key words: Chebyshev inequality, Kantorovich inequality, Hadamard product

Abstract: The purpose of this note is to prove Hadamard product versions of the Chebyshev and the Kantorovich inequalities for positive real numbers. We also prove a generalization of Fiedler's inequality.

Acknowledgements: The authors thank a referee for useful suggestions.

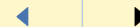
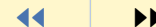
Hadamard Product Versions

Jagjit Singh Matharu and
Jaspal Singh Aujla

vol. 10, iss. 2, art. 51, 2009

[Title Page](#)

[Contents](#)



Page 1 of 13

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

Contents

- | | | |
|---|---|---|
| 1 | Introduction | 3 |
| 2 | The Chebyshev and Kantorovich Inequalities: Matrix Versions | 5 |



Hadamard Product Versions
Jagjit Singh Matharu and
Jaspal Singh Aujla
vol. 10, iss. 2, art. 51, 2009

Title Page

Contents



Page 2 of 13

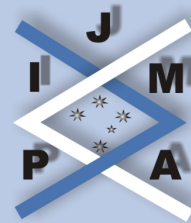
Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



1. Introduction

In what follows, the capital letters A, B, C, \dots denote $m \times m$ complex matrices, whereas the small letters a, b, c, \dots denote real numbers, unless mentioned otherwise. By $X \geq Y$ we mean that $X - Y$ is positive semidefinite ($X > Y$ mean $X - Y$ is positive definite). For $A = (a_{ij})$ and $B = (b_{ij})$, $A \circ B = (a_{ij}b_{ij})$ denotes the Hadamard product of A and B . According to Schur's theorem [4, Page 23] the Hadamard product is monotone in the sense that $A \geq B$, $C \geq D$ implies $A \circ C \geq B \circ D$. The tensor product $A \otimes B$ is the $m^2 \times m^2$ matrix

$$(1.1) \quad \begin{pmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & & \vdots \\ a_{m1}B & \cdots & a_{mm}B \end{pmatrix}.$$

Marcus and Khan in [10] made the simple but important observation that the Hadamard product is a principal submatrix of the tensor product. The inequality

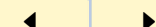
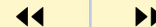
$$(1.2) \quad \left(\sum_{i=1}^n w_i a_i \right) \left(\sum_{i=1}^n w_i b_i \right) \leq \sum_{i=1}^n w_i a_i b_i$$

holds for all $a_1 \geq a_2 \geq \dots \geq a_n \geq 0$, $b_1 \geq b_2 \geq \dots \geq b_n \geq 0$ and weights $w_i \geq 0$, $i = 1, \dots, n$. Hardy, Littlewood and Polya [6, page 43] attribute this inequality to Chebyshev. For $0 < a \leq a_i \leq b$, $w_i \geq 0$, $i = 1, 2, \dots, n$, Kantorovich's inequality states that

$$(1.3) \quad \left(\sum_{i=1}^n w_i a_i \right) \left(\sum_{i=1}^n \frac{w_i}{a_i} \right) \leq \frac{(a+b)^2}{4ab} \left(\sum_{i=1}^n w_i \right)^2.$$

Title Page

Contents



Page 3 of 13

Go Back

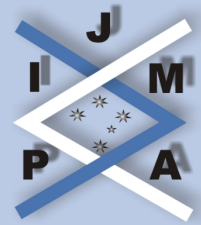
Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

In Section 2, we state and prove matrix versions of inequalities (1.2) and (1.3) involving the Hadamard product. A generalization of Fiedler's inequality is also proved in this section. There are several generalizations of Kantorovich and Fiedler's inequality; see [2, 3, 8, 9].



Hadamard Product Versions
Jagjit Singh Matharu and
Jaspal Singh Aujla
vol. 10, iss. 2, art. 51, 2009

Title Page

Contents



Page 4 of 13

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

2. The Chebyshev and Kantorovich Inequalities: Matrix Versions

We begin with a Hadamard product version of inequality (1.2).

Theorem 2.1. Let $A_1 \geq \dots \geq A_n \geq 0$ and $B_1 \geq \dots \geq B_n \geq 0$. Then

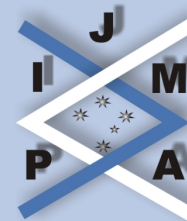
$$(2.1) \quad \left(\sum_{i=1}^n w_i A_i \right) \circ \left(\sum_{i=1}^n w_i B_i \right) \leq \left(\sum_{i=1}^n w_i \right) \left(\sum_{i=1}^n w_i (A_i \circ B_i) \right),$$

where $w_i \geq 0$, $i = 1, \dots, n$, are weights.

Proof. We have

$$(2.2) \quad \begin{aligned} & \left(\sum_{i=1}^n w_i \right) \left(\sum_{i=1}^n w_i (A_i \circ B_i) \right) - \left(\sum_{i=1}^n w_i A_i \right) \circ \left(\sum_{i=1}^n w_i B_i \right) \\ &= \sum_{i,j=1}^n (w_i w_j (A_j \circ B_j) - w_i w_j (A_i \circ B_j)) \\ &= \frac{1}{2} \sum_{i,j=1}^n \left(w_i w_j (A_j \circ B_j) - w_i w_j (A_i \circ B_j) \right. \\ & \quad \left. + w_j w_i (A_i \circ B_i) - w_j w_i (A_j \circ B_i) \right) \\ &= \frac{1}{2} \sum_{i,j=1}^n w_i w_j (A_i - A_j) \circ (B_i - B_j). \end{aligned}$$

Since the Hadamard product of two positive semidefinite matrices is positive semidefinite, therefore the summand in 2.2 is positive semidefinite. \square



Hadamard Product Versions
Jagjit Singh Matharu and
Jaspal Singh Aujla

vol. 10, iss. 2, art. 51, 2009

[Title Page](#)

[Contents](#)



Page 5 of 13

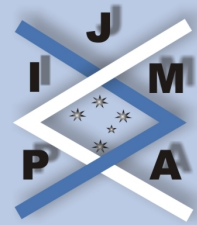
[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



[Title Page](#)

[Contents](#)

[◀◀](#) [▶▶](#)

[◀](#) [▶](#)

Page 6 of 13

[Go Back](#)

[Full Screen](#)

[Close](#)

Our next result is a Hadamard product version of inequality (1.3).

Theorem 2.2. Let A_1, \dots, A_n be such that $0 < aI_m \leq A_i \leq bI_m$, $i = 1, \dots, n$ (here I_m denotes the $m \times m$ identity matrix). Then

$$(2.3) \quad \left(\sum_{i=1}^n W_i^{1/2} A_i W_i^{1/2} \right) \circ \left(\sum_{i=1}^n W_i^{1/2} A_i^{-1} W_i^{1/2} \right) \leq \frac{a^2 + b^2}{2ab} \left(\sum_{i=1}^n W_i \right) \circ \left(\sum_{i=1}^n W_i \right)$$

for all $W_i \geq 0$, $i = 1, \dots, n$.

Proof. We first prove the inequality

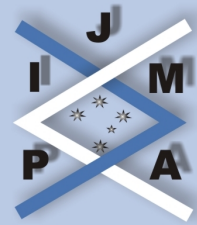
$$(2.4) \quad P^{1/2} A P^{1/2} \circ Q^{1/2} B^{-1} Q^{1/2} + P^{1/2} A^{-1} P^{1/2} \circ Q^{1/2} B Q^{1/2} \leq \frac{a^2 + b^2}{ab} (P \circ Q),$$

when $0 < aI_m \leq A, B \leq bI_m$ and $P, Q \geq 0$. Let $A = UDU^*$ and $B = V\Gamma V^*$ with unitary U and V , and diagonal matrices D and Γ . Then

$$\begin{aligned} A \otimes B^{-1} + A^{-1} \otimes B &= (U \otimes V)(D \otimes \Gamma + \Gamma^{-1} \otimes D)(U \otimes V)^* \\ &\leq (U \otimes V) \left(\frac{a^2 + b^2}{ab} (I_m \otimes I_m) \right) (U \otimes V)^* \\ &= \frac{a^2 + b^2}{ab} (I_m \otimes I_m), \end{aligned}$$

where the inequality follows from (1.3). Thus we have

$$(2.5) \quad P^{1/2} A P^{1/2} \otimes Q^{1/2} B^{-1} Q^{1/2} + P^{1/2} A^{-1} P^{1/2} \otimes Q^{1/2} B Q^{1/2}$$



[Title Page](#)

[Contents](#)

[◀◀](#) [▶▶](#)

[◀](#) [▶](#)

Page 7 of 13

[Go Back](#)

[Full Screen](#)

[Close](#)

$$\begin{aligned}
 &= (P^{1/2} \otimes Q^{1/2})(A \otimes B^{-1} + A^{-1} \otimes B)(P^{1/2} \otimes Q^{1/2}) \\
 &\leq \frac{a^2 + b^2}{ab} (P \otimes Q).
 \end{aligned}$$

Since the Hadamard product is a principal submatrix of the tensor product, the inequality (2.4) follows from (2.5). On taking $B = A$ and $Q = P$ in (2.4) we see that (2.3) holds for $n = 1$. Further, by (2.4) we have

$$W_i^{1/2} A_i W_i^{1/2} \circ W_j^{1/2} A_j^{-1} W_j^{1/2} + W_i^{1/2} A_i^{-1} W_i^{1/2} \circ W_j^{1/2} A_j W_j^{1/2} \leq \frac{a^2 + b^2}{ab} (W_i \circ W_j)$$

for $i, j = 1, \dots, n$. Summing over i, j , we have

$$(2.6) \quad 2 \sum_{i,j=1}^n \left[(W_i^{1/2} A_i W_i^{1/2}) \circ (W_j^{1/2} A_j^{-1} W_j^{1/2}) \right] \leq \left(\frac{a^2 + b^2}{ab} \right) \sum_{i,j=1}^n (W_i \circ W_j),$$

which implies that

$$\left(\sum_{i=1}^n W_i^{1/2} A_i W_i^{1/2} \right) \circ \left(\sum_{i=1}^n W_i^{1/2} A_i^{-1} W_i^{1/2} \right) \leq \left(\frac{a^2 + b^2}{2ab} \right) \left(\sum_{i=1}^n W_i \right) \circ \left(\sum_{i=1}^n W_i \right).$$

□

The next corollary follows on taking $W_i = w_i I_m$, $i = 1, \dots, n$.

Corollary 2.3. *Let A_1, \dots, A_n be such that $0 < aI_m \leq A_i \leq bI_m$, and $w_i \geq 0$, $i = 1, \dots, n$ be weights. Then*

$$\left(\sum_{i=1}^n w_i A_i \right) \circ \left(\sum_{i=1}^n w_i A_i^{-1} \right) \leq \left(\frac{a^2 + b^2}{2ab} \right) \left(\sum_{i=1}^n w_i \right)^2 I_m.$$



Title Page

Contents



Page 8 of 13

Go Back

Full Screen

Close

Remark 1. The case $n = 1$ of Corollary 2.3 is proved in [7]. The example

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad a = \frac{3 - \sqrt{5}}{2}, \quad b = \frac{3 + \sqrt{5}}{2}$$

shows that the inequality

$$A \circ A^{-1} \leq \frac{(a + b)^2}{4ab} I_2$$

need not be true.

For our next result we need the following lemma.

Lemma 2.4. *Let $0 \leq r \leq 1$. Then $A^r + A^{-r} \leq A + A^{-1}$ for all $A > 0$.*

Proof. Suppose that $A = U\Gamma U^*$ with unitary U and diagonal matrix Γ . Then

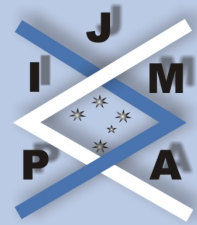
$$\begin{aligned} A^r + A^{-r} &= U(\Gamma^r + \Gamma^{-r})U^* \\ &\leq U(\Gamma + \Gamma^{-1})U^* = A + A^{-1} \end{aligned}$$

since $x^r + x^{-r} \leq x + x^{-1}$ for any positive real number x and $0 \leq r \leq 1$. □

Theorem 2.5. *Let $0 \leq \alpha < \beta$. Then*

$$\begin{aligned} &\left(\sum_{i=1}^n W_i^{1/2} A_i^\alpha W_i^{1/2} \right) \circ \left(\sum_{i=1}^n W_i^{1/2} A_i^{-\alpha} W_i^{1/2} \right) \\ &\leq \left(\sum_{i=1}^n W_i^{1/2} A_i^\beta W_i^{1/2} \right) \circ \left(\sum_{i=1}^n W_i^{1/2} A_i^{-\beta} W_i^{1/2} \right) \end{aligned}$$

for all $A_i > 0$ and $W_i \geq 0$, $i = 1, \dots, n$.



Proof. We first prove the inequality

$$\begin{aligned}
 (2.7) \quad & \left(W_i^{1/2} A_i^\alpha W_i^{1/2}\right) \circ \left(W_j^{1/2} A_j^{-\alpha} W_j^{1/2}\right) \\
 & + \left(W_i^{1/2} A_i^{-\alpha} W_i^{1/2}\right) \circ \left(W_j^{1/2} A_j^\alpha W_j^{1/2}\right) \\
 & \leq \left(W_i^{1/2} A_i^\beta W_i^{1/2}\right) \circ \left(W_j^{1/2} A_j^{-\beta} W_j^{1/2}\right) \\
 & + \left(W_i^{1/2} A_i^{-\beta} W_i^{1/2}\right) \circ \left(W_j^{1/2} A_j^\beta W_j^{1/2}\right)
 \end{aligned}$$

for $0 \leq \alpha < \beta$. Let $0 \leq r \leq 1$. Then

$$\begin{aligned}
 & \left(W_i^{1/2} A_i^r W_i^{1/2}\right) \otimes \left(W_j^{1/2} A_j^{-r} W_j^{1/2}\right) + \left(W_i^{1/2} A_i^{-r} W_i^{1/2}\right) \otimes \left(W_j^{1/2} A_j^r W_j^{1/2}\right) \\
 & = \left(W_i^{1/2} \otimes W_j^{1/2}\right) \left(A_i^r \otimes A_j^{-r} + A_i^{-r} \otimes A_j^r\right) \left(W_i^{1/2} \otimes W_j^{1/2}\right) \\
 & = \left(W_i^{1/2} \otimes W_j^{1/2}\right) \left(\left(A_i \otimes A_j^{-1}\right)^r + \left(A_i \otimes A_j^{-1}\right)^{-r}\right) \left(W_i^{1/2} \otimes W_j^{1/2}\right) \\
 & \leq \left(W_i^{1/2} \otimes W_j^{1/2}\right) \left(\left(A_i \otimes A_j^{-1}\right) + \left(A_i \otimes A_j^{-1}\right)^{-1}\right) \left(W_i^{1/2} \otimes W_j^{1/2}\right)
 \end{aligned}$$

where the inequality follows from Lemma 2.4. Taking $r = \alpha/\beta$ and replacing A_i by A_i^β and A_j by A_j^β , we have

$$\begin{aligned}
 & \left(W_i^{1/2} A_i^\alpha W_i^{1/2}\right) \otimes \left(W_j^{1/2} A_j^{-\alpha} W_j^{1/2}\right) + \left(W_i^{1/2} A_i^{-\alpha} W_i^{1/2}\right) \otimes \left(W_j^{1/2} A_j^\alpha W_j^{1/2}\right) \\
 & \leq \left(W_i^{1/2} A_i^\beta W_i^{1/2}\right) \otimes \left(W_j^{1/2} A_j^{-\beta} W_j^{1/2}\right) + \left(W_i^{1/2} A_i^{-\beta} W_i^{1/2}\right) \otimes \left(W_j^{1/2} A_j^\beta W_j^{1/2}\right).
 \end{aligned}$$

Again using the fact that the Hadamard product is a principal submatrix of the tensor

Title Page

Contents

◀◀ ▶▶

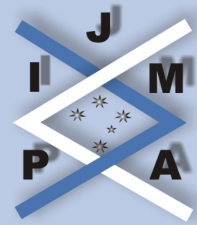
◀ ▶

Page 9 of 13

Go Back

Full Screen

Close



[Title Page](#)

[Contents](#)

◀◀ ▶▶

◀ ▶

Page 10 of 13

[Go Back](#)

[Full Screen](#)

[Close](#)

product, the preceding inequality implies (2.7). Summing over i, j in (2.7), we have

$$\left(\sum_{i=1}^n W_i^{1/2} A_i^\alpha W_i^{1/2} \right) \circ \left(\sum_{i=1}^n W_i^{1/2} A_i^{-\alpha} W_i^{1/2} \right) \leq \left(\sum_{i=1}^n W_i^{1/2} A_i^\beta W_i^{1/2} \right) \circ \left(\sum_{i=1}^n W_i^{1/2} A_i^{-\beta} W_i^{1/2} \right).$$

□

Corollary 2.6. Let $0 \leq \alpha < \beta$. Then

$$\left(\sum_{i=1}^n A_i^\alpha \right) \circ \left(\sum_{j=1}^n A_j^{-\alpha} \right) \leq \left(\sum_{i=1}^n A_i^\beta \right) \circ \left(\sum_{j=1}^n A_j^{-\beta} \right)$$

for all $A_i > 0, i = 1, \dots, n$.

Proof. Taking $W_i = I_m$ in Theorem 2.5 we get the desired result.

□

Corollary 2.7. Let $0 \leq \beta$. Then

$$I_m \leq \left(\sum_{i=1}^n W_i^{1/2} A_i^\beta W_i^{1/2} \right) \circ \left(\sum_{i=1}^n W_i^{1/2} A_i^{-\beta} W_i^{1/2} \right)$$

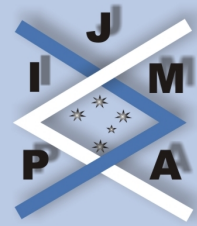
for all $A_i > 0$ and $W_i \geq 0, i = 1, \dots, n$, where $\sum_{i=1}^n W_i = I_m$.

Proof. Taking $\alpha = 0$ in Theorem 2.5 gives the desired inequality.

□

Remark 2. Corollary 2.7 is another generalization of Fiedler's inequality [5]

$$A \circ A^{-1} \geq I_m.$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 11 of 13

Go Back

Full Screen

Close

Next we prove a convexity theorem involving the Hadamard product.

Theorem 2.8. *The function*

$$f(t) = A^{1+t} \circ B^{1-t} + A^{1-t} \circ B^{1+t}$$

is convex on the interval $[-1, 1]$ and attains its minimum at $t = 0$ for all $A, B > 0$.

Proof. Since f is continuous we need to prove only that f is mid-point convex. Note that for $A, B > 0$ and s, t in $[-1, 1]$ the matrices

$$\begin{pmatrix} A^{1+s+t} & A^{1+s} \\ A^{1+s} & A^{1+(s-t)} \end{pmatrix}, \quad \begin{pmatrix} A^{1-(s+t)} & A^{1-s} \\ A^{1-s} & A^{1-(s-t)} \end{pmatrix},$$

$$\begin{pmatrix} B^{1+s+t} & B^{1+s} \\ B^{1+s} & B^{1+(s-t)} \end{pmatrix}, \quad \begin{pmatrix} B^{1-(s+t)} & B^{1-s} \\ B^{1-s} & B^{1-(s-t)} \end{pmatrix}$$

are positive semidefinite. Hence the matrix

$$X = \begin{pmatrix} A^{1+s+t} \circ B^{1-(s+t)} + A^{1-(s+t)} \circ B^{1+s+t} & A^{1+s} \circ B^{1-s} + A^{1-s} \circ B^{1+s} \\ A^{1+s} \circ B^{1-s} + A^{1-s} \circ B^{1+s} & A^{1+(s-t)} \circ B^{1-(s-t)} + A^{1-(s-t)} \circ B^{1+(s-t)} \end{pmatrix}$$

is positive semidefinite. Similarly, the matrix

$$Y = \begin{pmatrix} A^{1+(s-t)} \circ B^{1-(s-t)} + A^{1-(s-t)} \circ B^{1+(s-t)} & A^{1+s} \circ B^{1-s} + A^{1-s} \circ B^{1+s} \\ A^{1+s} \circ B^{1-s} + A^{1-s} \circ B^{1+s} & A^{1+(s+t)} \circ B^{1-(s+t)} + A^{1-(s+t)} \circ B^{1+s+t} \end{pmatrix}$$

is positive semidefinite. Hence

$$(2.8) \quad X + Y = \begin{pmatrix} f(s+t) + f(s-t) & 2f(s) \\ 2f(s) & f(s+t) + f(s-t) \end{pmatrix}$$

is positive semidefinite, which implies that

$$f(s) \leq \frac{1}{2}[f(s+t) + f(s-t)].$$



This proves the convexity of f . Further, note that $f(t) = f(-t)$. This together with the convexity of f implies that f attains its minimum at 0. \square

Corollary 2.9. *The function*

$$g(t) = A^t \circ B^{1-t} + A^{1-t} \circ B^t$$

is decreasing on $[0, 1/2]$, increasing on $[1/2, 1]$, and attains its minimum at $t = \frac{1}{2}$ for all $A, B > 0$.

Proof. The proof follows on replacing A, B by $A^{1/2}, B^{1/2}$ and t by $\frac{1+t}{2}$ in Theorem 2.8. \square

A norm $|||\cdot|||$ on $m \times m$ complex matrices is called unitarily invariant if $|||UXV||| = |||X|||$ for all unitary matrices U, V . If A is positive semidefinite and X is any matrix, then

$$|||A \circ X||| \leq \max a_{ii} |||X|||$$

for all unitarily invariant norms $|||\cdot|||$ [1]. Thus the proof of the following corollary follows from Corollary 2.9 using the fact that $g(1/2) \leq g(t) \leq g(1) = g(0)$.

Corollary 2.10. *Let $0 \leq t \leq 1$. Then,*

$$2|||A^{1/2} \circ B^{1/2}||| \leq |||A^t \circ B^{1-t} + A^{1-t} \circ B^t||| \leq |||A + B|||$$

for all unitarily invariant norms $|||\cdot|||$ and all $A, B > 0$.

Title Page

Contents



Page 12 of 13

Go Back

Full Screen

Close

References

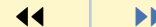
- [1] T. ANDO, R.A. HORN AND C.R. JOHNSON, The singular values of the Hadamard product: A basic inequality, *Linear Multilinear Algebra*, **21** (1987), 345–365.
- [2] J.K. BAKSALARY AND S. PUNTANEN, Generalized matrix versions of the Cauchy-Schwarz and Kantorovich inequalities, *Aequationes Math.*, **41** (1991), 103–110.
- [3] R.B. BAPAT AND M.K. KWONG, A generalisation of $A \circ A^{-1} \geq I$, *Linear Algebra Appl.*, **93** (1987), 107–112.
- [4] R. BHATIA, *Matrix Analysis*, Springer Verlag, New York, 1997.
- [5] M. FIEDLER, Über eine Ungleichung für positiv definite Matrizen, *Math. Nachrichten*, **23** (1961), 197–199.
- [6] G.H. HARDY, J.E. LITTLEWOOD AND G. POLYA, *Inequalities*, Cambridge University Press, Cambridge, 1959.
- [7] J. MIĆIĆ, J. PECARIC AND Y. SEO, Complementary inequalities to inequalities of Jensen and Ando based on the Mond-Pečarić method, *Linear Algebra Appl.*, **318** (2000), 87–107.
- [8] A.W. MARSHALL AND I. OLKIN, Matrix versions of the Cauchy and Kantorovich inequalities, *Aequationes Math.*, **40** (1990), 89–93.
- [9] M. SINGH, J.S. AUJLA AND H.L. VASUDEVA, Inequalities for Hadamard product and unitarily invariant norms of matrices, *Linear Multilinear Algebra*, **48** (2000), 247–262.
- [10] M. MARCUS AND N.A. KHAN, A note on Hadamard product, *Canad. Math. Bull.*, **2** (1950), 81–83.



Hadamard Product Versions
Jagjit Singh Matharu and
Jaspal Singh Aujla
vol. 10, iss. 2, art. 51, 2009

Title Page

Contents



Page 13 of 13

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756