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A NEW INEQUALITY SIMILAR TO HILBERT'S INEQUALITY

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Abstract

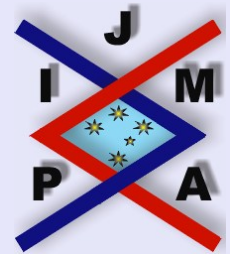
In this paper, we build a new inequality similar to Hilbert's inequality with a best constant factor. As an application, we consider its equivalent form.

2000 Mathematics Subject Classification: 26D15

Key words: Hilbert's inequality, Weight coefficient, Cauchy's inequality.

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1. Introduction

If $0 < \sum_{n=0}^{\infty} a_n^2 < \infty$ and $0 < \sum_{n=0}^{\infty} b_n^2 < \infty$, then the famous Hilbert's inequality (see Hardy et al. [1]) is given by

$$(1.1) \quad \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{m+n+1} < \pi \left(\sum_{n=0}^{\infty} a_n^2 \sum_{n=0}^{\infty} b_n^2 \right)^{\frac{1}{2}},$$

where the constant factor π is the best possible. Recently, Yang and Debnath [2, 3] and Yang [4, 5] gave (1.1) some extensions and improvements, and Kuang and Debnath [6] considered its strengthened versions and generalizations.

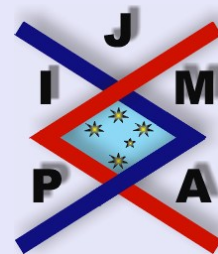
The major objective of this paper is to build a new inequality similar to (1.1), which relates to the double series form as

$$(1.2) \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\ln m + \ln n + 1} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\ln emn}.$$

For this, we must estimate the following weight coefficient

$$(1.3) \quad \omega(n) = \sum_{m=1}^{\infty} \frac{1}{m \ln emn} \left(\frac{\ln \sqrt{en}}{\ln \sqrt{em}} \right)^{\frac{1}{2}} \quad (n \in N),$$

and do some preparatory works.



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2. Some Lemmas

Let f have its first four derivatives on $[1, \infty)$ and $(-1)^n f^{(n)}(x) > 0$ ($n = 0, \dots, 4$), and $f(x), f'(x) \rightarrow 0$ ($x \rightarrow \infty$), then (see [6, (2.1)])

$$(2.1) \quad \sum_{k=1}^{\infty} f(k) < \int_1^{\infty} f(x) dx + \frac{1}{2}f(1) - \frac{1}{12}f'(1).$$

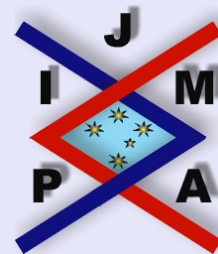
Lemma 2.1. For $n \in N$, define $R(n)$ as

$$(2.2) \quad R(n) = \frac{1}{(2 \ln \sqrt{en})^{\frac{1}{2}}} \int_0^{\frac{1}{2 \ln \sqrt{en}}} \frac{1}{(1+u)u^{\frac{1}{2}}} du - \frac{2}{3 \ln en} - \frac{1}{12(\ln en)^2}.$$

Then we have $R(n) > 0$ ($n \in N$).

Proof. Integrating by parts, we have

$$\begin{aligned} \int_0^{\frac{1}{2 \ln \sqrt{en}}} \frac{1}{(1+u)u^{\frac{1}{2}}} du &= 2 \int_0^{\frac{1}{2 \ln \sqrt{en}}} \frac{1}{(1+u)} du^{\frac{1}{2}} \\ &= (2 \ln \sqrt{en})^{\frac{1}{2}} \frac{1}{\ln en} + 2 \int_0^{\frac{1}{2 \ln \sqrt{en}}} u^{\frac{1}{2}} \frac{1}{(1+u)^2} du \\ &= (2 \ln \sqrt{en})^{\frac{1}{2}} \frac{1}{\ln en} + \frac{4}{3} \int_0^{\frac{1}{2 \ln \sqrt{en}}} \frac{1}{(1+u)^2} du^{3/2} \\ &= (2 \ln \sqrt{en})^{\frac{1}{2}} \frac{1}{\ln en} + \frac{1}{3} (2 \ln \sqrt{en})^{\frac{1}{2}} \frac{1}{(\ln en)^2} \\ &\quad + \frac{8}{3} \int_0^{\frac{1}{2 \ln \sqrt{en}}} u^{3/2} \frac{1}{(1+u)^3} du \end{aligned}$$



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$$> (2 \ln \sqrt{en})^{\frac{1}{2}} \frac{1}{\ln en} + \frac{1}{3} (2 \ln \sqrt{en})^{\frac{1}{2}} \frac{1}{(\ln en)^2}.$$

Hence by (2.2), we have

$$R(n) > \frac{1}{\ln en} + \frac{1}{3(\ln en)^2} - \frac{2}{3 \ln en} - \frac{1}{12(\ln en)^2} = \frac{1}{3 \ln en} + \frac{1}{4(\ln en)^2} > 0.$$

The lemma is thus proved. \square

Lemma 2.2. *If $\omega(n)$ is defined by (1.3), then $\omega(n) < \pi$, for $n \in N$.*

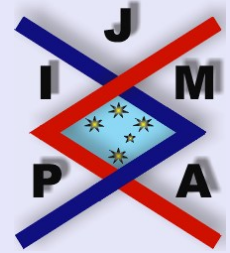
Proof. For fixed $n \in N$, setting

$$f_n(x) = \frac{1}{x \ln enx} \left(\frac{\ln \sqrt{en}}{\ln \sqrt{ex}} \right)^{\frac{1}{2}}, \quad x \in [1, \infty),$$

we find $f_n(1) = \frac{1}{\ln en} (2 \ln \sqrt{en})^{\frac{1}{2}}$, and

$$f'_n(x) = -\frac{1}{x^2 \ln enx} \left(\frac{\ln \sqrt{en}}{\ln \sqrt{ex}} \right)^{\frac{1}{2}} - \frac{1}{x^2 \ln^2 enx} \left(\frac{\ln \sqrt{en}}{\ln \sqrt{ex}} \right)^{\frac{1}{2}} - \frac{1}{2x^2 \ln enx} \cdot \frac{(\ln \sqrt{en})^{\frac{1}{2}}}{(\ln \sqrt{ex})^{\frac{3}{2}}},$$

$$f'_n(1) = -\left(\frac{2}{\ln en} + \frac{1}{\ln^2 en} \right) (2 \ln \sqrt{en})^{\frac{1}{2}}.$$



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Setting $u = \frac{\ln \sqrt{ex}}{\ln \sqrt{en}}$ in the following integral, we obtain

$$\int_1^\infty f_n(x) dx = \int_{\frac{1}{2 \ln \sqrt{en}}}^\infty \frac{1}{1+u} \left(\frac{1}{u}\right)^{\frac{1}{2}} du = \pi - \int_0^{\frac{1}{2 \ln \sqrt{en}}} \frac{1}{1+u} \left(\frac{1}{u}\right)^{\frac{1}{2}} du.$$

Hence by (2.1), (2.2) and Lemma 2.1, we have

$$\begin{aligned} \omega(n) &= \sum_{m=1}^\infty f_n(m) \\ &< \int_1^\infty f_n(x) dx + \frac{1}{2} f_n(1) - \frac{1}{12} f_n'(1) \\ &= \pi - \int_0^{1/(2 \ln \sqrt{en})} \frac{1}{1+u} \left(\frac{1}{u}\right)^{\frac{1}{2}} du + \left(\frac{2}{3 \ln en} + \frac{1}{12 \ln^2 en}\right) (2 \ln \sqrt{en})^{\frac{1}{2}} \\ &= \pi - (2 \ln \sqrt{en})^{\frac{1}{2}} R(n) < \pi. \end{aligned}$$

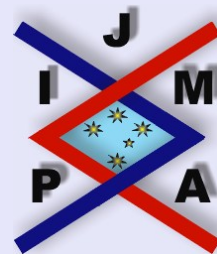
The lemma is proved. □

Lemma 2.3. For $0 < \epsilon < 1$, we have

$$(2.3) \quad \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{1}{mn \ln emn} \left(\frac{1}{\ln \sqrt{em} \ln \sqrt{en}}\right)^{\frac{1+\epsilon}{2}} > \frac{1}{\epsilon} (\pi + o(1)) \quad (\epsilon \rightarrow 0^+).$$

Proof. Setting $u = \frac{\ln \sqrt{ex}}{\ln \sqrt{ey}}$ in the following integral, we find

$$\int_{\sqrt{e}}^\infty \frac{1}{x \ln ex} \left(\frac{1}{\ln \sqrt{ex}}\right)^{\frac{1+\epsilon}{2}} dx$$



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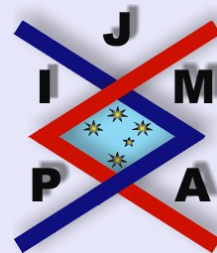
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$$\begin{aligned}
&= \left(\frac{1}{\ln \sqrt{ey}} \right)^{\frac{1+\epsilon}{2}} \int_{\frac{1}{\ln \sqrt{ey}}}^{\infty} \frac{1}{1+u} \left(\frac{1}{u} \right)^{\frac{1+\epsilon}{2}} du \\
&= \left(\frac{1}{\ln \sqrt{ey}} \right)^{\frac{1+\epsilon}{2}} \int_0^{\infty} \frac{1}{1+u} \left(\frac{1}{u} \right)^{\frac{1+\epsilon}{2}} du \\
&\quad - \left(\frac{1}{\ln \sqrt{ey}} \right)^{\frac{1+\epsilon}{2}} \int_0^{\frac{1}{\ln \sqrt{ey}}} \frac{1}{1+u} \left(\frac{1}{u} \right)^{\frac{1+\epsilon}{2}} du \\
&> \left(\frac{1}{\ln \sqrt{ey}} \right)^{\frac{1+\epsilon}{2}} \int_0^{\infty} \frac{1}{1+u} \left(\frac{1}{u} \right)^{\frac{1+\epsilon}{2}} du \\
&\quad - \left(\frac{1}{\ln \sqrt{ey}} \right)^{\frac{1+\epsilon}{2}} \int_0^{\frac{1}{\ln \sqrt{ey}}} \left(\frac{1}{u} \right)^{\frac{1+\epsilon}{2}} du \\
&= \left(\frac{1}{\ln \sqrt{ey}} \right)^{\frac{1+\epsilon}{2}} (\pi + o(1)) - \frac{2}{1-\epsilon} \left(\frac{1}{\ln \sqrt{ey}} \right) (\epsilon \rightarrow 0^+).
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{mn \ln emn} \left(\frac{1}{\ln \sqrt{em} \ln \sqrt{en}} \right)^{\frac{1+\epsilon}{2}} \\
&> \int_{\sqrt{e}}^{\infty} \int_{\sqrt{e}}^{\infty} \frac{1}{xy \ln exy} \left(\frac{1}{\ln \sqrt{ex} \ln \sqrt{ey}} \right)^{\frac{1+\epsilon}{2}} dx dy \\
&= \int_{\sqrt{e}}^{\infty} \frac{1}{y} \left(\frac{1}{\ln \sqrt{ey}} \right)^{\frac{1+\epsilon}{2}} \left[\int_{\sqrt{e}}^{\infty} \frac{1}{x \ln exy} \left(\frac{1}{\ln \sqrt{ex}} \right)^{\frac{1+\epsilon}{2}} dx \right] dy
\end{aligned}$$



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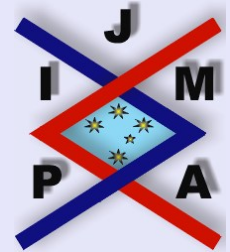
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$$\begin{aligned}
&> (\pi + o(1)) \int_{\sqrt{\epsilon}}^{\infty} \frac{1}{y} \left(\frac{1}{\ln \sqrt{\epsilon} y} \right)^{1+\epsilon} dy - \frac{2}{1-\epsilon} \int_{\sqrt{\epsilon}}^{\infty} \frac{1}{y} \left(\frac{1}{\ln \sqrt{\epsilon} y} \right)^{\frac{1+\epsilon}{2}+1} dy \\
&= (\pi + o(1)) \frac{1}{\epsilon} - \frac{4}{1-\epsilon^2} \\
&= \frac{1}{\epsilon} (\pi + o(1)) \quad (\epsilon \rightarrow 0^+).
\end{aligned}$$

The lemma is proved. □



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3. Main Result and an Application

Theorem 3.1. If $0 < \sum_{n=1}^{\infty} na_n^2 < \infty$ and $0 < \sum_{n=1}^{\infty} nb_n^2 < \infty$, then

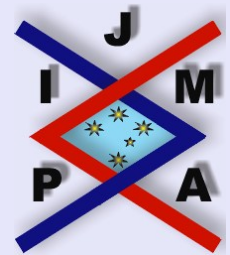
$$(3.1) \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\ln emn} < \pi \left(\sum_{n=1}^{\infty} na_n^2 \sum_{n=1}^{\infty} nb_n^2 \right)^{\frac{1}{2}},$$

where the constant factor π is the best possible.

Proof. By Cauchy's inequality and (1.3), we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\ln emn} \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\frac{a_m}{(\ln emn)^{\frac{1}{2}}} \left(\frac{\ln \sqrt{em}}{\ln \sqrt{en}} \right)^{\frac{1}{4}} \left(\frac{m}{n} \right)^{\frac{1}{2}} \right] \\ & \quad \times \left[\frac{b_n}{(\ln emn)^{\frac{1}{2}}} \left(\frac{\ln \sqrt{en}}{\ln \sqrt{em}} \right)^{\frac{1}{4}} \left(\frac{n}{m} \right)^{\frac{1}{2}} \right] \\ &\leq \left[\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m^2}{\ln emn} \left(\frac{\ln \sqrt{em}}{\ln \sqrt{en}} \right)^{\frac{1}{2}} \left(\frac{m}{n} \right) \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{b_n^2}{\ln emn} \left(\frac{\ln \sqrt{en}}{\ln \sqrt{em}} \right)^{\frac{1}{2}} \left(\frac{n}{m} \right) \right]^{\frac{1}{2}} \\ &= \left(\sum_{m=1}^{\infty} \omega(m) ma_m^2 \sum_{n=1}^{\infty} \omega(n) nb_n^2 \right)^{\frac{1}{2}}. \end{aligned}$$

By Lemma 2.2, we have (3.1).



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For $0 < \epsilon < 1$, setting a'_n as:

$$a'_n = \frac{1}{n(\ln \sqrt{en})^{\frac{1+\epsilon}{2}}}, \quad n \in N,$$

then we have

$$\begin{aligned} \sum_{n=1}^{\infty} na_n'^2 &= \frac{1}{(\ln \sqrt{e})^{1+\epsilon}} + \frac{1}{2(\ln 2\sqrt{e})^{1+\epsilon}} + \sum_{n=3}^{\infty} \frac{1}{n(\ln \sqrt{en})^{1+\epsilon}} \\ &< \frac{1}{(\ln \sqrt{e})^{1+\epsilon}} + \frac{1}{2(\ln 2\sqrt{e})^{1+\epsilon}} + \int_{\sqrt{e}}^{\infty} \frac{1}{x(\ln \sqrt{ex})^{1+\epsilon}} dx \\ (3.2) \quad &= \frac{1}{(\ln \sqrt{e})^{1+\epsilon}} + \frac{1}{2(\ln 2\sqrt{e})^{1+\epsilon}} + \frac{1}{\epsilon} = \frac{1}{\epsilon}(1 + o(1)) \quad (\epsilon \rightarrow 0^+). \end{aligned}$$

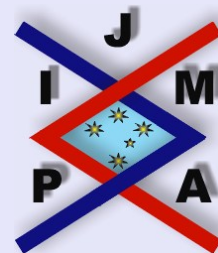
If the constant factor π in (3.1) is not the best possible, then there exists a positive number $K < \pi$, such that (3.1) is valid if we change π to K . In particular, we have

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a'_m a'_n}{\ln emn} < K \sum_{n=1}^{\infty} na_n'^2.$$

By (2.3) and (3.2), we have

$$(\pi + o(1)) < \epsilon \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a'_m a'_n}{\ln emn} < K(1 + o(1)) \quad (\epsilon \rightarrow 0^+),$$

and $\pi \leq K$. This contradicts that $K < \pi$. Hence the constant factor π in (3.1) is the best possible. The theorem is proved. \square



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Remark 3.1. Inequality (3.1) is more similar to the following Mulholland's inequality for $p = q = 2$ (see [7]):

$$(3.3) \quad \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \frac{a_m b_n}{mn \ln emn} < \frac{\pi}{\sin(\frac{\pi}{p})} \left(\sum_{n=2}^{\infty} n^{-1} a_n^p \right)^{\frac{1}{p}} \left(\sum_{n=2}^{\infty} n^{-1} b_n^q \right)^{\frac{1}{q}}.$$

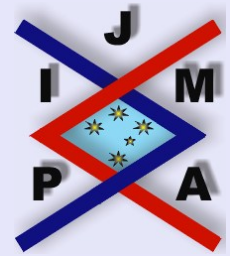
Theorem 3.2. If $0 < \sum_{n=1}^{\infty} n a_n^2 < \infty$, then we have

$$(3.4) \quad \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{m=1}^{\infty} \frac{a_m}{\ln emn} \right)^2 < \pi^2 \sum_{n=1}^{\infty} n a_n^2,$$

where the constant factor π^2 is the best possible. Inequalities (3.1) and (3.4) are equivalent.

Proof. Since $\sum_{n=1}^{\infty} n a_n^2 > 0$, there exists $k_0 \geq 1$, such that for any $k > k_0$, we have $\sum_{n=1}^k n a_n^2 > 0$, and $b_n(k) = \frac{1}{n} \sum_{m=1}^k \frac{|a_m|}{\ln emn} > 0$ ($n \in N$). By (3.1), we have

$$(3.5) \quad \begin{aligned} 0 &< \left[\sum_{n=1}^k n b_n^2(k) \right]^2 \\ &= \left[\sum_{n=1}^k \frac{1}{n} \left(\sum_{m=1}^k \frac{|a_m|}{\ln emn} \right) \right]^2 \\ &= \left[\sum_{n=1}^k \sum_{m=1}^k \frac{|a_m| b_n(k)}{\ln emn} \right]^2 < \pi^2 \sum_{n=1}^k n a_n^2 \sum_{n=1}^k n b_n^2(k). \end{aligned}$$



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Thus we find

$$(3.6) \quad 0 < \sum_{n=1}^k \frac{1}{n} \left(\sum_{m=1}^k \frac{|a_m|}{\ln emn} \right)^2 = \sum_{n=1}^k nb_n^2(k) < \pi^2 \sum_{n=1}^k na_n^2.$$

It follows that $0 < \sum_{n=1}^{\infty} nb_n^2(\infty) \leq \pi^2 \sum_{n=1}^{\infty} na_n^2 < \infty$. Hence by (3.1), for $k \rightarrow \infty$, neither (3.5) nor (3.6) takes equality, and we have

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{m=1}^{\infty} \frac{a_m}{\ln emn} \right)^2 \leq \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{m=1}^{\infty} \frac{|a_m|}{\ln emn} \right)^2 < \pi^2 \sum_{n=1}^{\infty} na_n^2.$$

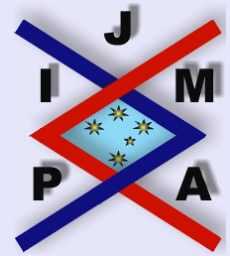
Inequality (3.4) is valid.

On the other hand, if (3.4) holds, by Cauchy's inequality, we have

$$(3.7) \quad \begin{aligned} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\ln emn} &= \sum_{n=1}^{\infty} \left(\frac{1}{n^{\frac{1}{2}}} \sum_{m=1}^{\infty} \frac{a_m}{\ln emn} \right) \left(n^{\frac{1}{2}} b_n \right) \\ &\leq \left[\sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{m=1}^{\infty} \frac{a_m}{\ln emn} \right)^2 \sum_{n=1}^{\infty} nb_n^2 \right]^{\frac{1}{2}}. \end{aligned}$$

By (3.4), we have (3.1).

Hence inequalities (3.1) and (3.4) are equivalent. If the constant factor π^2 in (3.4) is not the best possible, we may show that the constant factor π in (3.1) is not the best possible, by using (3.7). This is a contradiction. The theorem is proved. \square



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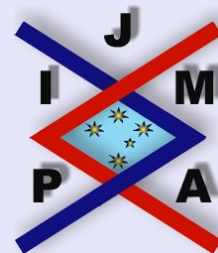
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Remark 3.2. Inequality (3.4) is similar to the following equivalent form of (1.1) (see [2]):

$$(3.8) \quad \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} \frac{a_m}{m+n+1} \right)^2 < \pi^2 \sum_{n=0}^{\infty} a_n^2.$$

Since inequalities (3.1) and (3.4) are similar to (1.1) and its equivalent form with the best constant factors, we have provided some new results.



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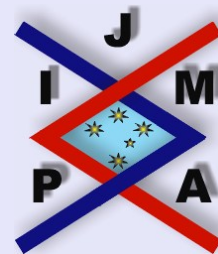
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