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NEW OSTROWSKI TYPE INEQUALITIES INVOLVING THE PRODUCT OF TWO FUNCTIONS

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ABSTRACT. In this paper we establish new Ostrowski type inequalities involving product of two functions. The analysis used in the proofs is elementary and based on the use of the integral identity recently established by Dedić , Pečarić and Ujević.

Key words and phrases: Ostrowski type inequalities, Product of two functions, Integral identity, Harmonic sequence.

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1. INTRODUCTION

In 1938, Ostrowski [7, p. 468] proved the following inequality:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)M, \tag{1.1}$$

for all $x \in [a,b]$, where $f:[a,b]\to\mathbb{R}$ is a differentiable function such that $|f'(x)| \le M$ for all $x \in [a,b]$.

In 1992, Fink [4] and earlier in 1976, Milovanović and Pečarić [6] have obtained some interesting generalizations of (1.1) in the form

$$\left|\frac{1}{n}\left(f(x)+\sum_{k=1}^{n-1}F_k(x)\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le C(n,p,x)\left\|f^{(n)}\right\|_\infty, \tag{1.2}$$

where

$$F_k(x)=\frac{n-k}{k!}\cdot\frac{f^{(k-1)}(a)(x-a)^k-f^{(k-1)}(b)(x-b)^k}{b-a},$$

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as usual $\frac{1}{p}+\frac{1}{p'}=1$ with $p'=1$ for $p=\infty$, $p'=\infty$ for $p=1$ and

$$\|f\|_p=\left(\int_a^b |f(t)|^p\,dt\right)^{\frac{1}{p}}.$$

In fact, Milovanović and Pečarić [6] (see also [7, p. 469]) have proved that

$$C(n,\infty,x)=\frac{(x-a)^{n+1}+(b-x)^{n+1}}{n(n+1)!(b-a)},$$

while Fink [4] (see also [7, p. 473]) proved that the inequality (1.2) holds provided $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and $f^{(n)}\in L_p[a,b]$, with

$$C(n,p,x)=\frac{\left[(x-a)^{np'+1}+(b-x)^{np'+1}\right]^{\frac{1}{p'}}}{n!(b-a)}B\left((n-1)p'+1,p'+1\right)^{\frac{1}{p'}},$$

for $1<p\le\infty$, B is the beta function, and

$$C(n,1,x)=\frac{(n-1)^{n-1}}{n^n n!(b-a)}\max\left[(x-a)^n,(b-x)^n\right].$$

Recently, Pachpatte [10] and Dedić, Pečarić and Ujević [3] (see also [2]) have given some generalizations of Milovanić-Pečarić [6] and Fink [4] inequalities. Motivated by the results in [10] and [3], in this paper we establish new Ostrowski type inequalities involving the product of two functions. The analysis used in the proofs is based on the integral identity proved in [3] and our results provide new estimates on these types of inequalities.

2. Statement of Results

Let (P_n) be a harmonic sequence of polynomials, that is, $P_n'=P_{n-1}$, $n\ge 1$, $P_0=1$. Furthermore, let $I\subset\mathbb{R}$ be a segment and $h:I\to\mathbb{R}$ be such that $h^{(n-1)}$ is absolutely continuous for some $n\ge 1$. We use the notation

$$L[h(x)]=\frac{1}{n}\left[h(x)+\sum_{k=1}^{n-1}(-1)^k P_k(x)h^{(k)}(x)\right.$$
$$\left.+\sum_{k=1}^{n-1}\frac{(-1)^k(n-k)}{b-a}\left[P_k(a)h^{(k-1)}(a)-P_k(b)h^{(k-1)}(b)\right]\right],$$

to simplify the details of presentation. For $n=1$ the above sums are defined to be zero. In a recent paper [3], Dedić, Pečarić and Ujević proved the following identity (see also [2]):

$$L[h(x)]-\frac{1}{b-a}\int_a^b h(t)\,dt=\frac{(-1)^{n+1}}{n(b-a)}\int_a^b P_{n-1}(t)\,e(t,x)\,h^{(n)}(t)\,dt, \tag{2.1}$$

where

$$e(t,x)=\begin{cases} t-a & \text{if } t\in[a,x], \\ t-b & \text{if } t\in(x,b]. \end{cases} \tag{2.2}$$

For the harmonic sequence of polynomials $P_k(t)=\frac{(t-x)^k}{k!}$, $k\ge 0$ the identity (2.1) reduces to the main identity derived by Fink in [4] (see also [3, p. 177]).

Our main results are given in the following theorems.

Theorem 2.1. *Let (P_n) be a harmonic sequence of polynomials and $f, g : [a, b] \to \mathbb{R}$ be such that $f^{(n-1)}, g^{(n-1)}$ are absolutely continuous for some $n \geq 1$ and $f^{(n)}, g^{(n)} \in L_p[a, b]$, $1 \leq p \leq \infty$. Then the inequality*

$$\left| g(x) L[f(x)] + f(x) L[g(x)] - \frac{1}{b-a}\left[g(x) \int_a^b f(t)\,dt + f(x) \int_a^b g(t)\,dt \right] \right| \leq D(n,p,x)\left[|g(x)| \left\| f^{(n)} \right\|_p + |f(x)| \left\| g^{(n)} \right\|_p \right], \tag{2.3}$$

holds for all $x \in [a, b]$, where

$$D(n,p,x) = \frac{1}{n(b-a)} \left\| P_{n-1} e(\cdot, x) \right\|_{p'}, \tag{2.4}$$

$e(t, x)$ is given by (2.2) and p, p' are as explained in Section 1.

Theorem 2.2. *Let (P_n), f, g, $f^{(n)}, g^{(n)}$ and p be as in Theorem 2.1. Then the inequality*

$$\left| L[f(x)] L[g(x)] - \frac{1}{b-a}\left[L[g(x)] \int_a^b f(t)\,dt + L[f(x)] \int_a^b g(t)\,dt \right] + \left(\frac{1}{b-a} \int_a^b f(t)\,dt \right)\left(\frac{1}{b-a} \int_a^b g(t)\,dt \right) \right| \leq \{D(n,p,x)\}^2 \left\| f^{(n)} \right\|_p \left\| g^{(n)} \right\|_p, \tag{2.5}$$

holds for all $x \in [a, b]$, where $D(n, p, x)$ and p' are as in Theorem 2.1.

Remark 2.3. If we take $g(t) = 1$ and hence $g^{(n-1)}(t) = 0$ for $n \geq 2$ in Theorem 2.1, then we get a variant of the Ostrowski type inequality given by Dedić, Pečarić and Ujević in [3, p. 180]. We note that the inequality established in Theorem 2.2 is similar to the inequality given by Pachpatte in [9, Theorem 2].

3. PROOFS OF THEOREMS 2.1 AND 2.2

Proof of Theorem 2.1. From the hypotheses we have the following identities (see [3, p. 176]):

$$L[f(x)] - \frac{1}{b-a} \int_a^b f(t)\,dt = \frac{(-1)^{n-1}}{n(b-a)} \int_a^b P_{n-1}(t)\, e(t,x)\, f^{(n)}(t)\,dt \tag{3.1}$$

and

$$L[g(x)] - \frac{1}{b-a} \int_a^b g(t)\,dt = \frac{(-1)^{n-1}}{n(b-a)} \int_a^b P_{n-1}(t)\, e(t,x)\, g^{(n)}(t)\,dt. \tag{3.2}$$

Multiplying (3.1) and (3.2) by $g(x)$ and $f(x)$ respectively and adding the resulting identities we have

$$g(x) L[f(x)] + f(x) L[g(x)] - \frac{1}{b-a}\left[g(x) \int_a^b f(t)\,dt + f(x) \int_a^b g(t)\,dt \right] = \frac{(-1)^{n-1}}{n(b-a)} \left[g(x) \int_a^b P_{n-1}(t)\, e(t,x)\, f^{(n)}(t)\,dt + f(x) \int_a^b P_{n-1}(t)\, e(t,x)\, g^{(n)}(t)\,dt \right]. \tag{3.3}$$

From (3.3) and using the properties of modulus and Hölder's integral inequality we have

$$\begin{aligned}
&\left| g(x) L[f(x)] + f(x) L[g(x)] - \frac{1}{b-a}\left[g(x) \int_a^b f(t)\,dt + f(x)\int_a^b g(t)\,dt \right] \right| \\
&\le \frac{1}{n(b-a)}\left[|g(x)| \int_a^b \left| P_{n-1}(t)\, e(t,x)\, f^{(n)}(t) \right| dt + |f(x)| \int_a^b \left| P_{n-1}(t)\, e(t,x)\, g^{(n)}(t)\right| dt \right] \\
&\le \frac{1}{n(b-a)}\left[|g(x)| \left\{ \int_a^b |P_{n-1}(t)\, e(t,x)|^{p'} dt \right\}^{\frac{1}{p'}} \left\{ \int_a^b \left| f^{(n)}(t) \right|^p dt \right\}^{\frac{1}{p}} \right. \\
&\qquad \left. + |f(x)| \left\{ \int_a^b |P_{n-1}(t)\, e(t,x)|^{p'} dt \right\}^{\frac{1}{p'}} \left\{ \int_a^b \left| g^{(n)}(t) \right|^p dt \right\}^{\frac{1}{p}} \right] \\
&= D(n,p,x)\left[|g(x)| \left\| f^{(n)} \right\|_p + |f(x)| \left\| g^{(n)} \right\|_p \right].
\end{aligned}$$

The proof of Theorem 2.1 is complete. □

Proof of Theorem 2.2. Multiplying the left sides and the right sides of (3.1) and (3.2) we get

$$\begin{aligned}
(3.4)\quad & L[f(x)]\, L[g(x)] - \frac{1}{b-a}\left[L[g(x)] \int_a^b f(t)\,dt + L[f(x)] \int_a^b g(t)\,dt \right] \\
&\quad + \left(\frac{1}{b-a}\int_a^b f(t)\,dt \right)\left(\frac{1}{b-a}\int_a^b g(t)\,dt \right) \\
&\qquad = \frac{(-1)^{2n-2}}{n^2 (b-a)^2} \left\{ \int_a^b P_{n-1}(t)\, e(t,x)\, f^{(n)}(t)\,dt \right\} \\
&\qquad\qquad \times \left\{ \int_a^b P_{n-1}(t)\, e(t,x)\, g^{(n)}(t)\,dt \right\}.
\end{aligned}$$

From (3.4) and following the proof of Theorem 2.1 given above with suitable modifications, we get the required inequality in (2.4). The proof of Theorem 2.2 is complete. □

Remark 3.1. Dividing both sides of (3.3) and (3.4) by $(b-a)$ and integrating the resulting identities with respect to x over $[a,b]$, then using the properties of modulus and Hölder's integral inequality, we get the following inequalities

$$\begin{aligned}
(3.5)\quad & \left| \frac{1}{b-a}\int_a^b [g(x) L[f(x)] + f(x) L[g(x)]]dx \right. \\
&\qquad \left. -2\left(\frac{1}{b-a}\int_a^b f(t)\,dt \right)\left(\frac{1}{b-a}\int_a^b g(t)\,dt \right) \right| \\
&\qquad\qquad \le \frac{1}{b-a}\int_a^b D(n,p,x)\left[|g(x)| \left\| f^{(n)} \right\|_p + |f(x)| \left\| g^{(n)} \right\|_p \right] dx,
\end{aligned}$$

and

$$\begin{aligned}(3.6)\quad &\left|\frac{1}{b-a}\int_a^b L\left[f\left(x\right)\right]L\left[g\left(x\right)\right]dx\right.\\ &-\left[\left(\frac{1}{b-a}\int_a^b L\left[f\left(x\right)\right]dx\right)\left(\frac{1}{b-a}\int_a^b g\left(x\right)dx\right)\right.\\ &\left.+\left(\frac{1}{b-a}\int_a^b L\left[g\left(x\right)\right]dx\right)\left(\frac{1}{b-a}\int_a^b f\left(x\right)dx\right)\right]\\ &\left.+\left(\frac{1}{b-a}\int_a^b f\left(x\right)dx\right)\left(\frac{1}{b-a}\int_a^b g\left(x\right)dx\right)\right|\\ &\leq \frac{1}{b-a}\left\|f^{(n)}\right\|_p\left\|g^{(n)}\right\|_p\int_a^b\left\{D\left(n,p,x\right)\right\}^2dx.\end{aligned}$$

We note that the inequalities obtained in (3.5) and (3.6) are respectively similar to the well known Grüss [5] and Čebyšev [1] inequalities (see also [8]) and we believe that these inequalities are new to the literature.

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