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GENERALIZED NEWTON–LIKE INEQUALITIES

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ABSTRACT. The notion of Newton–like inequalities is extended and an inductive approach is utilized to show that the generalized Newton–like inequalities hold on elementary symmetric functions with self–conjugate variables in the right half–plane.

Key words and phrases: Elementary symmetric functions, Newton's inequalities, Generalized Newton–like inequalities.

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1. INTRODUCTION

The k–th (normalized) elementary symmetric function with complex variables $x_1, x_2, \ldots, x_n \in \mathbb{C}$ is defined by

$$E_k(x_1, x_2, \ldots, x_n) = \frac{\sum_{1 \le j_1 < j_2 < \ldots < j_k \le n} x_{j_1} x_{j_2} \cdots x_{j_k}}{\binom{n}{k}},$$

where $k = 1, 2, \ldots, n$. By convention, $E_0(x_1, x_2, \ldots, x_n) = 1$. For the sake of brevity, we write such a function simply as E_k when there is no confusion over its variables.

It is well known that when $x_1, x_2, \ldots, x_n \in \mathbb{R}$, the sequence $\{E_k\}$ satisfies Newton's inequalities:

$$E_k^2 \ge E_{k-1}E_{k+1}, \quad 1 \le k \le n-1. \tag{1.1}$$

For background material regarding Newton's inequalities including some interesting historical notes, we refer the reader to [2, 6]. It should be pointed out, however, that a sequence with property (1.1) is also said to be log–concave or, more generally, Pólya frequency in literature [1, 8]. Furthermore, it is known that (1.1) holds if and only if

$$E_k E_l \ge E_{k-1} E_{l+1} \tag{1.2}$$

for all $k \le l$, provided that $E_k \ge 0$ for all k and that $\{E_k\}$ has no internal zeros, namely that for any $k < j < l$, $E_j \ne 0$ whenever $E_k, E_l \ne 0$.

For $\{E_k\}$ with variables $x_1, x_2, \ldots, x_n \in \mathbb{C}$, it is natural to require first that the non–real entries in $x_1, x_2, \ldots, x_n$ appear in conjugate pairs so as to guarantee that $\{E_k\} \subset \mathbb{R}$. A set

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of numbers that fulfills this requirement is said to be self-conjugate. In addition, we assume that for all j, $\mathrm{Re}\, x_j \geq 0$ unless stated otherwise. Consequently, $E_k \geq 0$ for all k. This latter requirement can be seen in Section 2 to arise naturally in a broader setting for the satisfaction of inequalities similar to Newton's.

When it comes to the question of whether Newton's inequalities continue to hold on $\{E_k\}$ with self–conjugate variables, the answer is, in general, negative. One observation is that if $x_j \neq 0$ for all j, then $\{E_k(x_1, x_2, \ldots, x_n)\}$ satisfies Newton's inequalities if and only if $\{E_k(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1})\}$ does [5]. In addition, it is shown in [3] that if $x_1, x_2, \ldots, x_n$ form the spectrum of an M– or inverse M–matrix, then $\{E_k\}$ satisfies Newton's inequalities. However, it is still an open question as to under what conditions Newton's inequalities carry over to the complex domain.

On the other hand, it is demonstrated in [4, 5] that when self-conjugate variables are allowed, $\{E_k\}$ satisfies the so-called Newton-like inequalities. Specifically, for $0 < \lambda \leq 1$, set

$$\Omega = \{z : |\arg z| \leq \cos^{-1}\sqrt{\lambda}\}, \tag{1.3}$$

and let $x_1, x_2, \ldots, x_n \in \Omega$ be self-conjugate, then according to [4],

$$E_k^2 \geq \lambda E_{k-1}E_{k+1} \tag{1.4}$$

for all k. We comment that (1.3) implies the dependence of λ on $x_1, x_2, \ldots, x_n$. Besides, it is illustrated in [5] that when $x_1, x_2, \ldots, x_n$ represent the spectrum of the Drazin inverse of a singular M–matrix, Newton-like inequalities hold in the form of (1.4) with $1/2 < \lambda \leq 1$ being independent of $x_1, x_2, \ldots, x_n$. It should be noted that Newton-like inequalities go back to Newton's when $\lambda = 1$.

In light of condition (1.2), we now extend the formulation of Newton-like inequalities. Suppose that $E_k \geq 0$ for all k. For the same $0 < \lambda \leq 1$ as in (1.4), we consider the following condition on $\{E_k\}$:

$$E_kE_l \geq \lambda E_{k-1}E_{l+1} \tag{1.5}$$

for all $k \leq l$. We observe that (1.5) leads to (1.4). Nevertheless, the converse is generally not true, thus the term generalized Newton-like inequalities for (1.5). In order to see that (1.5) is indeed a stronger condition than (1.4), we take the instance when $E_k > 0$.[1] From (1.4), it follows that

$$E_k^2E_{k+1} \geq \lambda E_{k-1}E_{k+1}^2 \geq \lambda^2 E_{k-1}E_kE_{k+2},$$

implying that

$$E_kE_{k+1} \geq \lambda^2 E_{k-1}E_{k+2}$$

instead of the tighter inequality $E_kE_{k+1} \geq \lambda E_{k-1}E_{k+2}$ from (1.5) on letting $l = k + 1$.

As another consequence of (1.5), it can be easily verified that for k being even, $E_k^{1/k} \geq \sqrt{\lambda}E_{k+2}^{1/(k+2)}$. This also turns out to be an improvement over the existing result in [4].

With the introduction of the generalized Newton-like inequalities in the form of (1.5), there is a quite intriguing question of whether they hold on $\{E_k\}$. Motivated by [2, 4, 6], we shall utilize an inductive argument to show that the answer is in fact affirmative for $\{E_k\}$ with self-conjugate variables in Ω. We mention that the proof of Newton's inequalities, see for example [2, 6, 7] for several variants, is essentially inductive, so is that of the Newton-like inequalities in [4]. The approach that we adopt in this work is mainly inspired by [2].

[1] This somehow amounts to the requirement of no internal zeros. However, it is clarified later that this requirement is actually met with self–conjugate variables in Ω.

2. PROOF OF GENERALIZED NEWTON–LIKE INEQUALITIES

Recall that $E_k = E_k(x_1, x_2, \ldots, x_n)$, where $x_1, x_2, \ldots, x_n \in \mathbb{C}$ are assumed to be self-conjugate. We begin with the following well-known observation.

Let $p(x) = \prod_{k=0}^{n}(x - x_k)$, the monic polynomial whose zeros are $x_1, x_2, \ldots, x_n$. Then, in terms of E_k, $p(x)$ can be expressed as

$$p(x) = \sum_{k=0}^{n}(-1)^k \binom{n}{k} E_k x^{n-k}. \tag{2.1}$$

The first few lemmas below validate the generalized Newton-like inequalities for the cases when $n = 2, 3$. Seeing the fact that Newton's inequalities are satisfied on $\{E_k\}$ with real variables, we only need to look at the cases in which one conjugate pair is present in the variables. In what follows, a, b, and c are all real numbers.

Lemma 2.1. *For $k = 0, 1, 2$, set $E_k = E_k(x_1, x_2)$, where $x_{1,2} = a \pm ib$ and $a^2 + b^2 > 0$. Then $E_1^2 \geq \lambda E_0 E_2$ for any $0 \leq \lambda \leq \frac{a^2}{a^2+b^2}$.*

Proof. Let $p(x) = (x - x_1)(x - x_2)$ be the monic polynomial with zeros x_1 and x_2. Clearly, $p(x) = x^2 - 2ax + a^2 + b^2$. Next, by comparing with (2.1), we obtain that $E_1 = a$ and $E_2 = a^2 + b^2$. Thus $E_1^2 - \lambda E_0 E_2 = a^2 - \lambda(a^2 + b^2) \geq 0$ for any $0 \leq \lambda \leq \frac{a^2}{a^2+b^2}$. □

The proof of Lemma 2.1 indicates that if $a^2+b^2 > 0$, then $a^2/(a^2+b^2)$ provides the best upper bound on λ in the generalized Newton-like inequalities for the case when $n = 2$. Alternatively, λ can be thought of as the best lower bound on $a^2/(a^2 + b^2)$ if λ is prescribed while a and b are allowed to vary. Besides, Lemma 2.1 indicates that the case of a purely imaginary conjugate pair should be excluded since they only lead to the trivial result.

Lemma 2.2. *Suppose that $b, c \geq 0$. For $0 \leq k \leq 3$, set $E_k = E_k(x_1, x_2, x_3)$, where $x_{1,2} = a \pm ib$ and $x_3 = c$. Then Newton's inequalities*

$$E_k^2 \geq E_{k-1}E_{k+1}, \quad k = 1, 2$$

hold if and only if either

$$\begin{cases} a - \sqrt{3}b \geq c, \\ \left(a - \frac{c}{2}\right)^2 + \left(b - \frac{\sqrt{3}}{2}c\right)^2 \geq c^2; \end{cases}$$

or

$$\begin{cases} a + \sqrt{3}b \leq c, \\ \left(a - \frac{c}{2}\right)^2 + \left(b + \frac{\sqrt{3}}{2}c\right)^2 \leq c^2. \end{cases}$$

Proof. Similar to the proof of Lemma 2.1, we derive that $E_1 = \frac{2a+c}{3}$, $E_2 = \frac{a^2+b^2+2ac}{3}$, and $E_3 = c(a^2 + b^2)$. It is a matter of straightforward calculation to verify that $E_k^2 \geq E_{k-1}E_{k+1}$ for $k = 1, 2$ if and only if

$$\begin{cases} |a - c| \geq \sqrt{3}b, \\ |a^2 + b^2 - ac| \geq \sqrt{3}bc, \end{cases}$$

which leads to the conclusion. □

A similar conclusion can be reached for the case when $c \leq 0$. Note that b can always be assumed to be nonnegative. For any fixed $c > 0$, the region as characterized by the necessary and sufficient condition in Lemma 2.2 is illustrated in Figure 2.1.

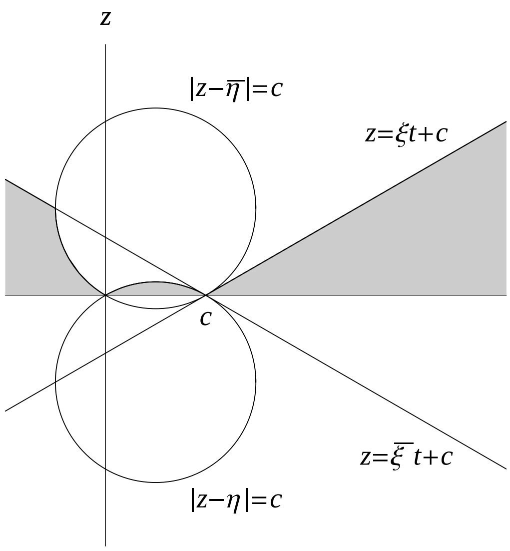

Figure 2.1: The shaded region, where t is a real parameter, $\xi = c + i\sqrt{3}c/3$, and $\eta = c/2 - i\sqrt{3}c/2$, represents the condition on a and b, with c being fixed, such that Newton's inequalities hold.

It can been seen from the formulas for E_k as given in the proof of Lemma 2.2 that $\{E_k\}$ has no internal zeros if we further assume that $x_1, x_2, x_3 \in \Omega$. In fact, such a property of $\{E_k\}$ can be readily verified to be true even when $x_1, x_2, x_3 \in \Omega$ are all real.

We also comment that according to [3], Newton's inequalities are upheld on $\{E_k\}$ with $x_1, x_2, \ldots, x_n$ being the spectrum of an M– or inverse M–matrix. Hence Lemma 2.2 also characterizes the region in which the eigenvalues of a 3×3 M– or inverse M–matrix are located. In Figure 2.1, this region is represented by the shaded part within the first quadrant.

The next lemma concerns the fulfillment of the generalized Newton–like inequalities when $n = 3$.

Lemma 2.3. *Suppose that $a, b, c \geq 0$ with $a^2 + b^2 > 0$. For $0 \leq k \leq 3$, set $E_k = E_k(x_1, x_2, x_3)$, where $x_{1,2} = a \pm ib$ and $x_3 = c$. Then for any λ such that $0 \leq \lambda \leq \frac{a^2}{a^2+b^2}$, $\{E_k\}$ satisfies the relationship that*

$$E_k E_l \geq \lambda E_{k-1} E_{l+1}$$

for all $k \leq l$.

Proof. It suffices to show the conclusion for the case $\lambda = \frac{a^2}{a^2+b^2}$. With the formulas for E_k, $k = 1, 2, 3$, as given in the proof of Lemma 2.2, we have that

$$E_1^2 - \frac{a^2}{a^2+b^2} E_0 E_2 = \frac{(a-c)^2}{9} + \frac{2ab^2c}{3(a^2+b^2)} \geq 0,$$

$$E_2^2 - \frac{a^2}{a^2+b^2} E_1 E_3 = \frac{1}{9}\left[a^2(a-c)^2 + 2a^2b^2 + 4ab^2c + b^4\right] \geq 0,$$

and

$$E_1 E_2 - \frac{a^2}{a^2+b^2} E_0 E_3 = \frac{1}{9}\left[2a(a-c)^2 + 2ab^2 + b^2c\right] \geq 0.$$

This completes the proof. □

Throughout the rest of this paper, we shall mainly focus on the scenario that $\operatorname{Re} x_j \geq 0$ for each j in addition to $x_1, x_2, \ldots, x_n$ being self-conjugate. Such a requirement plays a key role in the justification of Lemma 2.3. It also guarantees that $E_k(x_1, x_2, \ldots, x_n) \geq 0$. Recall that Ω is the region defined as in (1.3). We now fix $0 < \lambda \leq 1$ and assume that $x_1, x_2, \ldots, x_n \in \Omega$. Lemma 2.3 can then be rephrased as follows.

Lemma 2.4. *For self-conjugate $x_1, x_2, x_3 \in \Omega$, denote $E_k = E_k(x_1, x_2, x_3)$, where $0 \leq k \leq 3$. Then*

$$E_k E_l \geq \lambda E_{k-1} E_{l+1}$$

for all $k \leq l$.

Following an inductive approach, the main question now is whether the generalized Newton-like inequalities continue to hold as the number of variables n increases, with the assumption that such inequalities hold on $\{E_k\}$ with variables $x_1, x_2, \ldots, x_n$.

The lemma below updates the elementary symmetric functions when a nonnegative variable c is added to the existing variables $\{x_1, x_2, \ldots, x_n\}$.

Lemma 2.5. *Suppose that $x_1, x_2, \ldots, x_n \in \mathbb{C}$ are self-conjugate such that $E_k = E_k(x_1, x_2, \ldots, x_n) \geq 0$ for all k. Let $\widetilde{E}_k = E_k(x_1, x_2, \ldots, x_n, c)$, where $c \geq 0$. Then $\widetilde{E}_k \geq 0$ for all k. Moreover,*

$$\widetilde{E}_k = \frac{(n+1-k)E_k + ckE_{k-1}}{n+1}, \quad 0 \leq k \leq n+1. \tag{2.2}$$

In particular, $\widetilde{E}_0 = E_0$, and $\widetilde{E}_{n+1} = cE_n$.[1]

Proof. Similar to the proof of Lemma 2.1, we set $p(x) = \prod_{j=1}^{n}(x - x_j)$. Denote by $\widetilde{p}(x)$ the monic polynomial whose zeros are $x_1, x_2, \ldots, x_n$, and c. Note that according to (2.1), $p(x)$ and $\widetilde{p}(x)$ can be expressed in terms of E_k and $\widetilde{E}_k$, respectively. The conclusion follows by comparing the coefficients on both sides of the identity $\widetilde{p}(x) = (x - c)p(x)$. □

Note that formula (2.2) also shows that $\{\widetilde{E}_k\}$ has no internal zeros if the same is true for $\{E_k\}$. Moreover, it can be seen from (2.2) that the number of internal zeros, if present, tends to diminish while passing from $\{E_k\}$ to $\{\widetilde{E}_k\}$.

Continuing with E_k and $\widetilde{E}_k$ as considered in Lemma 2.5, we demonstrate next that the generalized Newton-like inequalities carry over from $\{E_k\}$ to $\{\widetilde{E}_k\}$ whenever $c \geq 0$. For the sake of simplicity, we define that

$$D_{k,l} = E_k E_l - \lambda E_{k-1} E_{l+1}. \tag{2.3}$$

By the inductive assumption, $D_{k,l} \geq 0$ for all $k \leq l$.

Theorem 2.6. *Let $x_1, x_2, \ldots, x_n \in \mathbb{C}$ be self-conjugate. Suppose that for all k, $E_k = E_k(x_1, x_2, \ldots, x_n) \geq 0$. Set $\widetilde{E}_k = E_k(x_1, x_2, \ldots, x_{n+1})$, where $x_{n+1} = c \geq 0$. If there exists some $0 \leq \lambda \leq 1$ such that $E_k E_l \geq \lambda E_{k-1} E_{l-1}$ for all $1 \leq k \leq l \leq n-1$, then*

$$\widetilde{E}_k \widetilde{E}_l \geq \lambda \widetilde{E}_{k-1} \widetilde{E}_{l+1} \tag{2.4}$$

for all $1 \leq k \leq l \leq n$.

[1] We follow the convention that $E_k = 0$ if $k < 0$ or $k > n$. This kind of interpretation is adopted throughout whenever a subscript goes beyond its range.

Proof. By Lemma 2.5, we have that

$$\begin{aligned}
&(n+1)^2(\widetilde{E}_k\widetilde{E}_l - \lambda \widetilde{E}_{k-1}\widetilde{E}_{l+1})\\
&= \big[(n+1-k)E_k + ckE_{k-1}\big]\big[(n+1-l)E_l + clE_{l-1}\big]\\
&\qquad - \lambda\big[(n+2-k)E_{k-1} + c(k-1)E_{k-2}\big]\big[(n-l)E_{l+1} + c(l+1)E_l\big]\\
&= (n+2-k)(n-l)D_{k,l} + c^2(k-1)(l+1)D_{k-1,l-1} + c(n+1-k)lD_{k,l-1}\\
&\qquad + c(k-1)(n-l)D_{k-1,l} + (1+l-k)(E_kE_l + c^2E_{k-1}E_{l-1})\\
&\qquad - \lambda c(n+2+l-k)E_{k-1}E_l + c(n-l+k)E_{k-1}E_l.
\end{aligned}$$

Note that $D_{k,l-1} \geq 0$ even when $l = k$. It remains to show that the sum of the last three terms above is nonnegative, which can be done by observing that

$$\begin{aligned}
(1+l-k)(E_kE_l + c^2E_{k-1}E_{l-1}) &\geq 2c(1+l-k)\sqrt{E_kE_lE_{k-1}E_{l-1}}\\
&\geq 2c(1+l-k)\lambda E_{k-1}E_l
\end{aligned}$$

and, consequently, that the sum of those last three terms is bounded below by $c(1-\lambda)(n-l+k)E_{k-1}E_l \geq 0$. $\square$

It should be mentioned that the proof of Theorem 2.6 is basically in the same fashion as that of Theorem 51 in [2]. Our result here, however, is more general in that it involves the generalized Newton-like inequalities on $\{E_k\}$ with $x_1, x_2, \dots, x_n \in \mathbb{C}$.

Next we proceed to the case when a conjugate complex pair $x_{n+1,n+2} = a \pm ib$, where $a \geq 0$, is added to the existing variables $\{x_1, x_2, \dots, x_n\}$. In a way similar to Lemma 2.5, the result below provides a connection between $E_k = E_k(x_1, x_2, \dots, x_n)$ and $\widetilde{E}_k = E_k(x_1, x_2, \dots, x_{n+2})$. It also indicates that $\{\widetilde{E}_k\}$ is free of internal zeros if $\{E_k\}$ is, assuming that $x_{n+1,n+2} \in \Omega$.

Lemma 2.7. *Suppose that $x_1, x_2, \dots, x_n \in \mathbb{C}$ are self–conjugate such that $E_k = E_k(x_1, x_2, \dots, x_n) \geq 0$ for all k. Let $x_{n+1,n+2} = a \pm ib$ be a conjugate pair such that $a \geq 0$. Denote $\widetilde{E}_k = E_k(x_1, x_2, \dots, x_{n+2})$. Then $\widetilde{E}_k \geq 0$ for all k. Moreover, for $0 \leq k \leq n+2$,*

$$\widetilde{E}_k = \frac{(n+1-k)(n+2-k)E_k + 2a(n+2-k)kE_{k-1} + (a^2+b^2)k(k-1)E_{k-2}}{(n+1)(n+2)}. \tag{2.5}$$

In particular, $\widetilde{E}_0 = E_0$, $\widetilde{E}_1 = \frac{nE_1 + 2aE_0}{n+2}$, $\widetilde{E}_{n+1} = \frac{2aE_n + n(a^2+b^2)E_{n-1}}{n+2}$, and $\widetilde{E}_{n+2} = (a^2+b^2)E_n$.

Proof. The proof of this conclusion is similar to that of Lemma 2.5. Denote by $p(x)$ the monic polynomial with zeros at $x_1, x_2, \dots, x_n$. Set $\widetilde{p}(x) = (x - x_{n+1})(x - x_{n+2})p(x)$, which reduces to $\widetilde{p}(x) = (x^2 - 2ax + a^2 + b^2)p(x)$. A comparison of the coefficients, in terms of E_k and $\widetilde{E}_k$ in accordance with (2.1), on both sides of this latter identity yields (2.5). $\square$

If $a > 0$, then on letting

$$F_k = \frac{(n+1-k)E_k + akE_{k-1}}{n+1} \tag{2.6}$$

and

$$G_k = \frac{(n+1-k)E_k + \frac{a^2+b^2}{a}kE_{k-1}}{n+1}, \tag{2.7}$$

we can rewrite (2.5) as

$$\widetilde{E}_k = \frac{(n+2-k)F_k + akG_{k-1}}{n+2}. \tag{2.8}$$

It is obvious that Theorem 2.6 applies to both F_k and G_k. Moreover, there is the following connection between F_k and G_k.

Lemma 2.8. *Assuming that $a > 0$, F_k and G_k as defined in (2.6) and (2.7), respectively, satisfy*

$$F_k \leq G_k \leq \frac{a^2+b^2}{a^2} F_k \tag{2.9}$$

for all k.

In the following several technical lemmas we suppose that there exists some $0 < \lambda \leq 1$ such that $\{E_k\}$ satisfies the generalized Newton-like inequalities (1.5). Furthermore, as motivated by [4] as well as by the discussion in Lemmas 2.1 and 2.3, we assume that the following additional condition holds on a and b:

$$\frac{a}{\sqrt{a^2+b^2}} \geq \sqrt{\lambda}, \tag{2.10}$$

which implies that $x_{n+1,n+2} \in \Omega$, where Ω is defined as in (1.3). Note that such a condition also implies that $a > 0$.

Lemma 2.9. *For all $k \leq l$,*

$$F_k G_{l-1} \geq \lambda F_{k-1} G_l, \tag{2.11}$$

provided that condition (2.10) holds on a and b.

Proof. We first verify the case when $k = l$, namely $F_k G_{k-1} \geq \lambda F_{k-1} G_k$. By Lemma 2.8 and condition (2.10), it follows that

$$F_k G_{k-1} \geq \frac{a^2}{a^2+b^2} F_{k-1} G_k \geq \lambda F_{k-1} G_k.$$

For the case when $k < l$, using (2.6) and (2.7), we obtain that

$$\begin{aligned}
&(n+1)^2 (F_k G_{l-1} - \lambda F_{k-1} G_l) \\
&= \big[(n+1-k)E_k + akE_{k-1}\big] \left[(n+2-l)E_{l-1} + \frac{a^2+b^2}{a}(l-1)E_{l-2}\right] \\
&\qquad - \lambda\big[(n+2-k)E_{k-1} + a(k-1)E_{k-2}\big] \left[(n+1-l)E_l + \frac{a^2+b^2}{a} l E_{l-1}\right] \\
&= (n+2-k)(n+1-l)D_{k,l-1} + (a^2+b^2)(k-1)lD_{k-1,l-2} \\
&\qquad + \frac{a^2+b^2}{a}(n+1-k)(l-1)D_{k,l-2} + a(k-1)(n+1-l)D_{k-1,l-1} \\
&\qquad + (l-k)\big[E_k E_{l-1} + (a^2+b^2)E_{k-1}E_{l-2}\big] - \lambda \frac{a^2+b^2}{a}(n+1+l-k)E_{k-1}E_{l-1} \\
&\qquad + a(n+1-l+k)E_{k-1}E_{l-1},
\end{aligned}$$

where $D_{k,l}$ is defined as in (2.3). Note that $D_{k,l-2} \geq 0$ even when $l = k+1$. It therefore suffices to show that the sum of the last three terms above, denoted by S, is nonnegative. Clearly,

$$\begin{aligned}
S \geq 2(l-k)\sqrt{\lambda(a^2+b^2)}E_{k-1}E_{l-1} - \lambda \frac{a^2+b^2}{a}(n+1+l-k)E_{k-1}E_{l-1} \\
+ a(n+1-l+k)E_{k-1}E_{l-1}.
\end{aligned}$$

Set $t = \frac{\sqrt{\lambda(a^2+b^2)}}{a}$. Thus

$$\begin{aligned} S &\geq a\big[2(l-k)t - (n+1+l-k)t^2 + n + 1 - (l-k)\big]E_{k-1}E_{l-1} \\ &= a(1-t)\big[(n+1)(1+t) - (l-k)(1-t)\big]E_{k-1}E_{l-1} \geq 0 \end{aligned}$$

since $0 < t \leq 1$. □

Lemma 2.10. *For all* $k \leq l$*,*

$$G_{k-1}F_l \geq \lambda G_{k-2}F_{l+1}. \tag{2.12}$$

Proof. With (2.6) and (2.7) we compute as follows.

$$\begin{aligned} &(n+1)^2(G_{k-1}F_l - \lambda G_{k-2}F_{l+1}) \\ &= \left[(n+2-k)E_{k-1} + \frac{a^2+b^2}{a}(k-1)E_{k-2}\right]\big[(n+1-l)E_l + alE_{l-1}\big] \\ &\qquad - \lambda\left[(n+3-k)E_{k-2} + \frac{a^2+b^2}{a}(k-2)E_{k-3}\right]\big[(n-l)E_{l+1} + a(l+1)E_l\big] \\ &= (n+3-k)(n-l)D_{k-1,l} + (a^2+b^2)(k-2)(l+1)D_{k-2,l-1} \\ &\qquad + a(n+2-k)lD_{k-1,l-1} + \frac{a^2+b^2}{a}(k-2)(n-l)D_{k-2,l} \\ &\qquad + (2+l-k)\big[E_{k-1}E_l + (a^2+b^2)E_{k-2}E_{l-1}\big] - \lambda a(n+3+l-k)E_{k-2}E_l \\ &\qquad + \frac{a^2+b^2}{a}(n-1-l+k)E_{k-2}E_l. \end{aligned}$$

We again set S to be the sum of the last three terms in the above expression.

$$\begin{aligned} S &\geq 2(2+l-k)\sqrt{\lambda(a^2+b^2)}E_{k-2}E_l - \lambda a(n+3+l-k)E_{k-2}E_l \\ &\qquad + \frac{a^2+b^2}{a}(n-1-l+k)E_{k-2}E_l \\ &= \lambda a\left[2(2+l-k)t - (n+3+l-k) + (n-1-l+k)t^2\right]E_{k-2}E_l, \end{aligned}$$

where $t = \frac{1}{a}\sqrt{\frac{a^2+b^2}{\lambda}} \geq 1$. Hence,

$$S \geq \lambda a(t-1)\big[(n-1)(t+1) - (l-k)(t-1) + 4\big]E_{k-2}E_l \geq 0,$$

which concludes the proof. □

We comment that, unlike Lemma 2.9, Lemma 2.10 does not require condition (2.10) to hold on a and b.

Lemma 2.11. *For all* $k \leq l$*,*

$$F_lG_{k-1} \geq \lambda F_{k-1}G_l, \tag{2.13}$$

provided that a *and* b *satisfy condition (2.10).*

Proof. By (2.6) and (2.7), it is clear that

$$\begin{aligned}
&(n+1)^2(F_lG_{k-1}-\lambda F_{k-1}G_l)\\
&=\big[(n+1-l)E_l+alE_{l-1}\big]\left[(n+2-k)E_{k-1}+\frac{a^2+b^2}{a}(k-1)E_{k-2}\right]\\
&\qquad -\lambda\big[(n+2-k)E_{k-1}+a(k-1)E_{k-2}\big]\left[(n+1-l)E_l+\frac{a^2+b^2}{a}lE_{l-1}\right]\\
&=(1-\lambda)(n+2-k)(n+1-l)E_{k-1}E_l+(1-\lambda)(a^2+b^2)(k-1)lE_{k-2}E_{l-1}\\
&\qquad +a(k-1)(n+1-l)\left(\frac{a^2+b^2}{a^2}-\lambda\right)E_{k-2}E_l\\
&\qquad +a(n+2-k)l\left(1-\lambda\frac{a^2+b^2}{a^2}\right)E_{k-1}E_{l-1}\\
&\ge 0,
\end{aligned}$$

thus verifying the claim. □

For E_k and $\widetilde{E}_k$ as defined in Lemma 2.7, the next conclusion shows that the generalized Newton-like inequalities still carry over from $\{E_k\}$ to $\{\widetilde{E}_k\}$ as long as a and b satisfy condition (2.10).

Theorem 2.12. *Let $x_1, x_2, \dots, x_n \in \mathbb{C}$ be self–conjugate such that $E_k = E_k(x_1, x_2, \dots, x_n) \ge 0$ for all k and that for some $0 \le \lambda \le 1$, $E_kE_l \ge \lambda E_{k-1}E_{l+1}$ for all $k \le l$. Suppose that a and b satisfy $b \ge 0$, $a^2+b^2 > 0$, and condition (2.10), i.e. $\frac{a}{\sqrt{a^2+b^2}} \ge \sqrt{\lambda}$. Set $\widetilde{E}_k = E_k(x_1, x_2, \dots, x_{n+2})$, where $x_{n+1,n+2} = a \pm ib$. Then*

$$\widetilde{E}_k\widetilde{E}_l \ge \lambda\widetilde{E}_{k-1}\widetilde{E}_{l+1} \tag{2.14}$$

for all $1 \le k \le l \le n+1$.

Proof. The above conclusion holds trivially if $a = 0$.

Suppose next that $a > 0$. Using (2.8), we see that

$$\begin{aligned}
&(n+2)^2(\widetilde{E}_k\widetilde{E}_l-\lambda\widetilde{E}_{k-1}\widetilde{E}_{l+1})\\
&=\big[(n+2-k)F_k+akG_{k-1}\big]\big[(n+2-l)F_l+alG_{l-1}\big]\\
&\qquad -\lambda\big[(n+3-k)F_{k-1}+a(k-1)G_{k-2}\big]\big[(n+1-l)F_{l+1}+a(l+1)G_l\big]\\
&=(n+3-k)(n+1-l)(F_kF_l-\lambda F_{k-1}F_{l+1})\\
&\qquad +a^2(k-1)(l+1)(G_{k-1}G_{l-1}-\lambda G_{k-2}G_l)+a(n+2-k)l(F_kG_{l-1}-\lambda F_{k-1}G_l)\\
&\qquad +a(k-1)(n+1-l)(G_{k-1}F_l-\lambda G_{k-2}F_{l+1})+(1+l-k)(F_kF_l+a^2G_{k-1}G_{l-1})\\
&\qquad -a\lambda(n+3+l-k)F_{k-1}G_l+a(n+1-l+k)G_{k-1}F_l.
\end{aligned}$$

By Theorem 2.6 and Lemmas 2.9 and 2.10, the terms in the last expression are all nonnegative except possibly the sum of the last three. For convenience, we designate this sum by S again. Note that

$$\begin{aligned}
S &\ge 2a(1+l-k)\sqrt{F_kF_lG_{k-1}G_{l-1}}-a\lambda(n+3+l-k)F_{k-1}F_l\\
&\qquad +a(n+1-l+k)G_{k-1}F_l\\
&\ge 2a(1+l-k)\sqrt{\lambda F_{k-1}G_lF_lG_{k-1}}-2a\lambda(1+l-k)F_{k-1}G_l
\end{aligned}$$

by Theorem 2.6, Lemma 2.8, and condition (2.10). Continuing with S, we note further that

$$S = 2a\sqrt{\lambda F_{k-1}G_l}(\sqrt{F_l G_{k-1}} - \sqrt{\lambda F_{k-1}G_l}) \geq 0$$

by Lemma 2.11, which consequently yields (2.14). □

Combining Lemma 2.4 with Theorems 2.6 and 2.12, together with Newton's inequalities for the case of real variables, we are now in a position to state the following main result, thus concluding the inductive proof of the generalized Newton–like inequalities:

Theorem 2.13. *Let Ω be the region in the complex plane as defined in (1.3). For any self-conjugate $x_1, x_2, \ldots, x_n \in \Omega$, set $E_k = E_k(x_1, x_2, \ldots, x_n)$, where $k = 0, 1, \ldots, n$. Then*

$$E_k E_l \geq \lambda E_{k-1} E_{l+1}$$

for all $k \leq l$. In particular, $E_k^2 \geq \lambda E_{k-1} E_{k+1}$ for $1 \leq k \leq n-1$.

3. CONCLUDING REMARKS

In this paper we introduce the notion of generalized Newton-like inequalities on elementary symmetric functions with self-conjugate variables $x_1, x_2, \ldots, x_n$ and show that such inequalities are satisfied as $x_1, x_2, \ldots, x_n$ range, essentially, in the right half-plane. The main conclusion of this work also includes as its special cases Newton-like inequalities [4, 5] as well as the celebrated Newton's inequalities on elementary symmetric functions with nonnegative variables.

The methodology of this paper is an inductive argument. It is motivated largely by the proof in [2] of Newton's inequalities as well as several recent results on Newton's and Newton-like inequalities [4, 6, 7]. It, however, differs from previous works mostly in that no argument involving mean value theorems, either Rolle's or Gauss-Lucas', is required. It therefore serves as an alternative which may turn out to be useful for the further investigation of some related problems, particularly problems regarding higher order Newton's inequalities, Newton's and Newton-like inequalities on elementary symmetric functions with respect to eigenvalues of matrices, and such inequalities over the complex domain.

REFERENCES

[1] F. BRENTI, Log-concave and unimodal sequences in algebra, combinatorics, and geometry: an update, *Contemp. Math.*, **178** (1994), 71–89.

[2] G.H. HARDY, J.E. LITTLEWOOD, AND G. PÓLYA, *Inequalities*, 2nd ed., Cambridge Mathematical Library, 1952.

[3] O. HOLTZ, M–matrices satisfy Newton's inequalities, *Proc. Amer. Math. Soc.*, **133**(3) (2005), 711–716.

[4] V. MONOV, Newton's inequalities for families of complex numbers, *J. Inequal. Pure and Appl. Math.*, **6**(3) (2005), Art. 78.

[5] M. NEUMANN AND J. XU, A note on Newton and Newton–like inequalities for M–matrices and for Drazin inverses of M–matrices, *Electron. J. Lin. Alg.*, **15** (2006), 314–328.

[6] C.P. NICULESCU, A new look at Newton's inequalities, *J. Inequal. Pure and Appl. Math.*, **1**(2) (2000), Art. 17. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=111`].

[7] S. ROSSET, Normalized symmetric functions, Newton's inequalities, and a new set of stronger inequalities, *Amer. Math. Month.*, **96** (1989), 815–820.

[8] R.P. STANLEY, Log–concave and unimodal sequences in algebra, combinatorics, and geometry, *Ann. New York Acad. Sci.*, **576** (1989), 500–534.