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ON MODULI OF EXPANSION OF THE DUALITY MAPPING OF SMOOTH BANACH SPACES

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ABSTRACT. Let X be a Banach space which is uniformly convex and uniformly smooth. We introduce the lower and upper moduli of expansion of the dual mapping J of the space X. Some estimation of certain well-known moduli (convexity, smoothness and flatness) and two new moduli introduced in [5] are described with this new moduli of expansion.

Key words and phrases: Uniformly convex (smooth) Banach space, Angle of modulus convexity (smoothness), Lower (upper) modulus of expansion.

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Let $(X, \|\cdot\|)$ be a real normed space, X^* its conjugate space, X^{**} the second conjugate of X and $S(X)$ the unit sphere in X $(S(X) = \{x \in X | \, \|x\| = 1\})$.

Moreover, we shall use the following definitions and notations.

The sign (S) denotes that X is smooth, (R) that X is reflexive, (US) that X is uniformly smooth, (SC) that X is strictly convex, and (UC) that X is uniformly convex.

The map $J : X \to 2^{X^*}$ is called the dual map if $J(0) = 0$ and for $x \in X$, $x \neq 0$,

$$J(x) = \{f \in X^* | f(x) = \|f\| \, \|x\|, \|f\| = \|x\|\}.$$

The dual map of X^* into $2^{X^{**}}$ we denote by J^*. The map τ is canonical linear isometry of X into X^{**}.

It is well known that functional

$$g(x, y) := \frac{\|x\|}{2} \left(\lim_{t \to -0} \frac{\|x + ty\| - \|x\|}{t} + \lim_{t \to +0} \frac{\|x + ty\| - \|x\|}{t} \right) \tag{1}$$

always exists on X^2. If X is (S), then (1) reduces to

$$g(x, y) = \|x\| \lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t};$$

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the functional g is linear in the second argument, $J(x)$ is a singleton and $g(x,\cdot) \in J(x)$. In this case we shall write $J(x) = Jx = f_x$. Then $[y,x] := g(x,y)$, defines a so called semi-inner product $[\cdot,\cdot]$ (s.i.p) on X^2 which generates the norm of X, $\left([x,x] = \|x\|^2\right)$, (see [1]). If X is an inner-product space (i.p. space) then $g(x,y)$ is the usual i.p. of the vector x and the vector y.

By the use of functional g we define the angle between vector x and vector y $(x \neq 0,\ y \neq 0)$ as

(2) $$\cos(x,y) := \frac{g(x,y) + g(y,x)}{2\|x\|\|y\|}$$

(see [3]). If $(X, (\cdot,\cdot))$ is an i.p. space, then (2) reduces to

$$\cos(x,y) = \frac{(x,y)}{\|x\|\|y\|}.$$

We say that X is a quasi-inner product space (q.i.p space) if the following equality holds

(3) $$\|x+y\|^4 - \|x-y\|^4 = 8\left[\|x\|^2 g(x,y) + \|y\|^2 g(y,x)\right], \quad (x,y \in X)^{1)}$$

The equality (3) holds in the space l^4, but does not hold in the space l^1. A q.i.p. space X is (SC) and (US) (see [6] and [4]).

Alongside the modulus of convexity of X, δ_X, and the modulus of smoothness of X, ρ_X, defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| \;\middle|\; x,y \in S(X);\ \|x-y\| \geq \varepsilon\right\};$$
$$\rho_X(\varepsilon) = \sup\left\{1 - \left\|\frac{x+y}{2}\right\| \;\middle|\; x,y \in S(X);\ \|x-y\| \leq \varepsilon\right\};$$

we have defined in [5] the angle modulus of convexity of X, δ'_X, and the angle modulus of smoothness of X, ρ'_X by:

$$\delta'_X(\varepsilon) = \inf\left\{\frac{1-\cos(x,y)}{2} \;\middle|\; x,y \in S(X);\ \|x-y\| \geq \varepsilon\right\};$$
$$\rho'_X(\varepsilon) = \sup\left\{\frac{1-\cos(x,y)}{2} \;\middle|\; x,y \in S(X);\ \|x-y\| \leq \varepsilon\right\}.$$

We also recall the known definition of modulus of flatness of X, η_X (Day's modulus):

$$\eta_X(\varepsilon) = \sup\left\{\frac{2-\|x+y\|}{\|x-y\|} \;\middle|\; x,y \in S(X);\ \|x-y\| \leq \varepsilon\right\}.$$

We now quote three known results.

Lemma 1. *(Theorem 6 in* [7] *and Theorem 6 in* [1]*). Let X be a real normed space which is (S), (SC) and (R). Then for all $f \in X^*$ there exists a unique $x \in X$ such that*

$$f(y) = g(x,y), \quad (y \in X).$$

Lemma 2. *(Theorem 7 in* [1]*). Let X be a Banach space which is (US) and (UC) and let $[\cdot,\cdot]$ be an s.i.p. on X^2 which generates the norm on X (see* [1]*). Then the dual space X^* is (US) and (UC) and the functional*

$$\langle Jx, Jy\rangle := [y,x], \quad (x,y \in X),$$

is an s.i.p on $(X^*)^2$.

1) If $(\cdot,\cdot)$ is an i.p. on X^2 then $g(x,y) = (x,y)$ and the equality (3) is the parallelogram equality.

Lemma 3. *(Proposition 3 in* [2]*). Let X be a real normed space. Then for J, J^* and τ on their respective domains we have*

$$J^{-1} = \tau^{-1}J^* \text{ and } J = J^{*-1}\tau.$$

Remark 4. Under the hypothesis of Lemma 2, the mappings J, J^* and τ are bijective mappings. Then, by Lemma 3, Lemma 2 and Lemma 1, in this case, we have

$$\langle Jx, Jy\rangle = g(x,y) = g(f_y, f_x), \quad (x, y \in X).$$

Lemma 5. *Let X be a real normed space which is (S), (SC) and (R). Then for $x, y \in S(X)$ we have*

$$1 - \left\|\frac{x+y}{2}\right\| \le \frac{1-\cos(x,y)}{2} \le \frac{\|x-y\|\,\|f_x - f_y\|}{4}. \tag{4}$$

Proof. Under the hypothesis of Lemma 5, using Lemma 1, we have $f_x = g(x,\cdot)$ $(x \in X)$. Consequently,

$$\begin{aligned}\|f_x - f_y\| &= \sup\{|g(x,t) - g(y,t)| \mid t \in S(X)\}\\ &\ge g(x,t) - g(y,t) \quad (t \in S(X)).\end{aligned}$$

For $t = \frac{x-y}{\|x-y\|}$, $(x \ne y)$, we obtain

$$g\left(x, \frac{x-y}{\|x-y\|}\right) - g\left(y, \frac{x-y}{\|x-y\|}\right) \le \|f_x - f_y\|. \tag{5}$$

Since X is (S), the functional g is linear in the second argument. Hence, from (5) we get

$$1 - g(x,y) - g(y,x) + 1 \le \|x-y\|\,\|f_x - f_y\|. \tag{6}$$

Using the inequality

$$1 - \left\|\frac{x+y}{2}\right\| \le \frac{1-\cos(x,y)}{2} \le \frac{\|x-y\|}{2}$$

(see Lemma 1 in [5]) and the inequality (6) we obtain the inequality (4). □

Lemma 6. *Let X be a Banach space which is (US) and (UC). Let δ_{X^*} be the modulus of convexity of X^*. Then for each $\varepsilon > 0$ and for all $x, y \in S(X)$ the following implications hold*

$$\|x-y\| \le 2\delta_{X^*}(\varepsilon) \Longrightarrow \|f_x - f_y\| \le \varepsilon, \tag{7}$$

$$\|f_x - f_y\| \ge \varepsilon \Longrightarrow \|x-y\| \ge 2\delta_{X^*}(\varepsilon). \tag{8}$$

Proof. By Lemma 2, X^* is a Banach space which is (UC) and (US). Since X^* is (UC), for each $\varepsilon > 0$, we have $\delta_{X^*}(\varepsilon) > 0$ and, for all $x, y \in S(X)$,

$$\|f_x + f_y\| > 2 - 2\delta_{X^*}(\varepsilon) \Longrightarrow \|f_x - f_y\| < \varepsilon. \tag{9}$$

Under the hypothesis of Lemma 6, by Remark 4, we have $g(x,y) = g(f_y, f_x)$. Hence, by inequality

$$1 - \|x-y\| \le g(x,y) \le \|x+y\| - 1$$

(see Lemma 1 in [6]), we obtain

$$1 - \|x-y\| \le g(x,y) = g(f_y, f_x) \le \|f_x + f_y\| - 1, \tag{10}$$

so that we have

$$\|x-y\| + \|f_x + f_y\| \ge 2. \tag{11}$$

Now, let $x, y \in S(X)$ and $\|x-y\| < 2\delta_{X^*}(\varepsilon)$. Then, by (11) we obtain

$$\|f_x + f_y\| > 2 - 2\delta_{X^*}(\varepsilon).$$

Thus, by (9), we conclude that

$$\|x-y\| < 2\delta_{X^*}(\varepsilon) \Longrightarrow \|f_x - f_y\| < \varepsilon. \tag{12}$$

On the other hand if $\|x-y\| = 2\delta_{X^*}(\varepsilon)$ and $\|f_x - f_y\| > \varepsilon$, by (9), it follows

$$\|x-y\| + \|f_x + f_y\| \le 2.$$

So, by (11), we get

$$\|x-y\| + \|f_x + f_y\| = 2.$$

Hence, using (10), we conclude that $g(x,y) = 1 - \|x-y\|$, i.e., $g(x, x-y) = \|x\|\,\|x-y\|$. Thus, since X is (SC), using Lemma 5 in [1], we get $x = x - y$, which is impossible. So, the implication (7) is correct. The implication (8) follows from the implication (12). □

We now introduce a new definition.

According to the inequality (4), to make further progress in the estimates of the moduli $\delta_X, \delta'_X, \rho_X, \rho'_X$, it is convenient to introduce

Definition 1. Let X be (S) and $x, y \in S(X)$. The function $\underline{e_J}$: $[0,2] \to [0,2]$, defined by

$$\underline{e_J}(\varepsilon) := \inf\{\|f_x - f_y\| \mid \|x-y\| \ge \varepsilon\}$$

will be called the lower modulus of expansion of the dual mapping J.

The function $\overline{e_J} : [0,2] \to [0,2]$, defined as

$$\overline{e_J}(\varepsilon) := \sup\{\|f_x - f_y\| \mid \|x-y\| \le \varepsilon\}$$

is the upper modulus of expansion of the dual mapping J.

Now, we quote our new results. Firstly, we note some elementary properties of the moduli $\underline{e_J}$ and $\overline{e_J}$.

Theorem 7. *Let X be (S). Then the following assertions are valid.*

a) *The function $\underline{e_J}$ is nondecreasing on $[0,2]$.*
b) *The function $\overline{e_J}$ is nondecreasing on $[0,2]$.*
c) $\underline{e_J}(\varepsilon) \le \overline{e_J}(\varepsilon)$ $(\varepsilon \in [0,2])$.
d) *If X is a Hilbert space, then $\underline{e_J}(\varepsilon) = \overline{e_J}(\varepsilon)$.*

Proof. The assertions a) and b) follow from the implications

$$\varepsilon_1 < \varepsilon_2 \Longrightarrow \{(x,y) \mid \|x-y\| \ge \varepsilon_1\} \supset \{(x,y) \mid \|x-y\| \ge \varepsilon_2\} \quad (x,y \in S(X)),$$
$$\varepsilon_1 < \varepsilon_2 \Longrightarrow \{(x,y) \mid \|x-y\| \le \varepsilon_1\} \subset \{(x,y) \mid \|x-y\| \le \varepsilon_2\} \quad (x,y \in S(X)).$$

c) Assume, to the contrary, i.e., that there is an $\varepsilon \in [0,2]$ such that $\underline{e_J}(\varepsilon) > \overline{e_J}(\varepsilon)$. Then

$$\begin{aligned}\inf\{\|f_x - f_y\| \mid \|x-y\| = \varepsilon\} &\ge \inf\{\|f_x - f_y\| \mid \|x-y\| \ge \varepsilon\}\\ &> \sup\{\|f_x - f_y\| \mid \|x-y\| \le \varepsilon\}\\ &\ge \sup\{\|f_x - f_y\| \mid \|x-y\| = \varepsilon\},\end{aligned}$$

which is not possible.

d) In a Hilbert space, we have

$$\|f_x - f_y\| = \sup\{|(x,t) - (y,t)| \mid t \in S(X)\} \le \|x-y\|.$$

On the other hand, the functional $f_x - f_y$ attains its maximum in $t = \frac{x-y}{\|x-y\|} \in S(X)$.

Hence $\|x-y\| = \|f_x - f_y\|$. Because of that, we have $\underline{e_J}(\varepsilon) = \overline{e_J}(\varepsilon) = \varepsilon$. □

In the next theorems some relation between moduli $\delta'_X, \rho'_X, \underline{e_J}, \overline{e_J}$ are given.

Theorem 8. *Let X be (S), (SC) and (R). Then, for $\varepsilon \in (0,2]$ we have*

a) $\delta'_X(\varepsilon) \le \frac{1}{2}\underline{e_J}(\varepsilon)$

b) $\rho'_X(\varepsilon) \le \frac{\varepsilon}{4}\overline{e_J}(\varepsilon)$,

c) $\frac{2}{\varepsilon}\rho_X(\varepsilon) \le \eta_X(\varepsilon) \le \frac{1}{2}\overline{e_J}(\varepsilon)$.

Proof. The proof of the assertions a) and b) follows immediately using the definitions of the functions δ'_X and ρ'_X and the inequality (4).

c) Let $x, y \in S(X)$, $x \neq y$. By Lemma 5, we have

$$\begin{aligned}\frac{2-\|x+y\|}{\|x-y\|} &= \frac{2}{\|x-y\|}\left(1-\frac{\|x+y\|}{2}\right)\\ &\le \frac{1-\cos(x,y)}{\|x-y\|}\\ &\le \frac{\|x-y\|\,\|f_x-f_y\|}{2\,\|x-y\|}\\ &= \frac{\|f_x-f_y\|}{2}.\end{aligned}$$

So

$$\frac{2-\|x+y\|}{\|x-y\|} \le \frac{\|f_x-f_y\|}{2}.$$

Using the definition of η_X and $\overline{e_J}$, we obtain

$$\eta_X(\varepsilon) \le \frac{1}{2}\overline{e_J}(\varepsilon).$$

On the other hand

$$(0 < \|x-y\| \le \varepsilon) \Longrightarrow \left(\frac{1}{\|x-y\|} \ge \frac{1}{\varepsilon}\right) \Longrightarrow \frac{2-\|x+y\|}{\|x-y\|} \ge \frac{2}{\varepsilon}\left(1-\frac{\|x+y\|}{2}\right).$$

Because of that we have

$$\eta_X(\varepsilon) \ge \frac{2}{\varepsilon}\rho_X(\varepsilon).$$

□

Remark 9. The last inequality is true for an arbitrary space X.

Corollary 10. *For a q.i.p. space, it holds that*

$$\underline{e_J}(\varepsilon) \ge \left(\frac{\varepsilon}{2}\right)^4 \quad (\varepsilon \in [0,2]). \tag{13}$$

Proof. By a) of Theorem 8 and the inequality $\frac{\varepsilon^4}{32} \le \delta'_X(\varepsilon)$ (see Corollary 2 in [5]), we get (13). □

Corollary 11. *If X is (S), (SC) and (R) then*

a) $\delta'_{X^*}(\varepsilon) \le \frac{1}{2}\underline{e_J}(\varepsilon)$,

b) $\rho'_{X^*} \le \frac{1}{2}\overline{e_{J^*}}(\varepsilon)$,

c) $\frac{2}{3}\rho_{X^*}(\varepsilon) \le \eta_{X^*}(\varepsilon) \le \frac{1}{2}\overline{e_{J^*}}(\varepsilon)$.

Proof. It is well-known that if X is (S), (SC) and (R) then X^* is (S), (SC) and (R). Hence Theorem 8 is valid for X^*. □

Theorem 12. *Let X be a Banach space which is (UC) and (US). Then, for all $\varepsilon > 0$, we have the following estimations:*

a) $\rho'_X(2\delta_{X^*}(\varepsilon)) \leq \dfrac{\varepsilon\delta_{X^*}(\varepsilon)}{2}$,

b) $\rho'_{X^*}(2\delta_X(\varepsilon)) \leq \dfrac{\varepsilon\delta_X(\varepsilon)}{2}$,

c) $\underline{e_{J^*}}(\varepsilon) \geq 2\delta_{X^*}(\varepsilon)$,

d) $\overline{e_J}(2\delta_{X^*}(\varepsilon)) \leq \varepsilon, \quad (\overline{e_{J^*}}(2\delta_X(\varepsilon)) \leq \varepsilon)$.

Proof. a) Using, in succession, the definition of the function ρ'_X, the inequality (4) in Lemma 2 and the implication (7), we obtain:

$$\begin{aligned}\rho'_X(2\delta_{X^*}(\varepsilon)) &= \sup\left\{\frac{1-\cos(x,y)}{2}\,\middle|\; \|x-y\| \leq 2\delta_{X^*}(\varepsilon)\right\}\\ &\leq \frac{1}{4}\sup\{\|x-y\|\,\|f_x-f_y\| \mid \|x-y\| \leq 2\delta_{X^*}(\varepsilon)\}\\ &\leq \frac{1}{4}2\varepsilon\delta_{X^*}(\varepsilon)\\ &= \frac{\varepsilon\delta_{X^*}(\varepsilon)}{2}.\end{aligned}$$

b) If, in a), we set X^* instead of X (X^{**} instead of X^*), we get

$$\rho'_{X^*}(2\delta_{X^{**}}(\varepsilon)) \leq \frac{\varepsilon\delta_{X^{**}}(\varepsilon)}{2}. \tag{14}$$

Let $F, G \in S(X^{**})$. Under the hypothesis of Theorem 12, we have

$$\begin{aligned}\delta_{X^{**}}(\varepsilon) &= \inf\left\{1-\frac{\|F+G\|}{2}\,\middle|\; \|F-G\| \geq \varepsilon\right\}\\ &= \inf\left\{1-\frac{\|\tau x+\tau y\|}{2}\,\middle|\; \|\tau x-\tau y\| \geq \varepsilon\right\}\\ &= \inf\left\{1-\frac{\|\tau(x+y)\|}{2}\,\middle|\; \|\tau(x-y)\| \geq \varepsilon\right\}\\ &= \inf\left\{1-\frac{\|x+y\|}{2}\,\middle|\; \|x-y\| \geq \varepsilon\right\}\\ &= \delta_X(\varepsilon).\end{aligned}$$

Consequently the inequality (14) is equivalent to the inequality b).

c) Using, in succession, the definition of $\underline{e_J}$, Lemma 3, and the implication (8), we get

$$\begin{aligned}\underline{e_{J^*}}(\varepsilon) &= \inf\{\|J^*f_x - J^*f_y\| \mid \|f_x-f_y\| \geq \varepsilon\}\\ &= \inf\{\|\tau x-\tau y\| \mid \|f_x-f_y\| \geq \varepsilon\}\\ &\geq 2\delta_{X^*}(\varepsilon).\end{aligned}$$

d) Using the definition of $\overline{e_J}$ and the implication (7), we get

$$\overline{e_J}(2\delta_{X^*}(\varepsilon)) = \sup\{\|f_x-f_y\| \mid \|x-y\| \leq 2\delta_{X^*}(\varepsilon)\} \leq \varepsilon.$$

Replacing, here, X^* with X^{**} and J with J^*, we get the second inequality.

□

Since in a Banach space X we have

$$\delta_X(\varepsilon) \leq 1 - \sqrt{1 - \frac{\varepsilon^2}{4}} \text{ and } \delta_X(\varepsilon) \leq \delta'_X(\varepsilon)$$

(see Theorem 1 in [5]), using b) and a) of Theorem 12, we obtain

Corollary 13. *Under the hypothesis of Theorem 12, we have*

a) $\dfrac{2}{\varepsilon}\rho'_{X^*}(2\delta_X(\varepsilon)) \leq \delta_X(\varepsilon) \leq \dfrac{2}{\varepsilon}\delta'_X(\varepsilon)$,

b) $\rho'_X(2\delta_{X^*}(\varepsilon)) \leq \dfrac{\varepsilon}{2}\left(1 - \sqrt{1 - \dfrac{\varepsilon^2}{4}}\right)$.

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