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SUFFICIENT CONDITIONS FOR STARLIKENESS AND CONVEXITY IN $|z| < \frac{1}{2}$

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ABSTRACT. For analytic functions $f(z)$ with $f(0) = f'(0) - 1 = 0$ in the open unit disc $\mathbb{E}$, T. H. MacGregor has considered some conditions for $f(z)$ to be starlike or convex. The object of the present paper is to discuss some interesting problems for $f(z)$ to be starlike or convex for $|z| < \frac{1}{2}$.

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1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disc $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$. A function $f \in \mathcal{A}$ is said to be starlike with respect to the origin in $\mathbb{E}$ if it satisfies

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (z \in \mathbb{E}).$$

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Also, a function $f \in \mathcal{A}$ is called as convex in $\mathbb{E}$ if it satisfies

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \qquad (z \in \mathbb{E}).$$

MacGregor [2] has shown the following.

Theorem A. *If $f \in \mathcal{A}$ satisfies*

$$\left|\frac{f(z)}{z} - 1\right| < 1 \qquad (z \in \mathbb{E}),$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 \qquad \left(|z| < \frac{1}{2}\right)$$

so that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad \left(|z| < \frac{1}{2}\right).$$

Therefore, $f(z)$ is univalent and starlike for $|z| < \frac{1}{2}$.

Also, MacGregor [3] had given the following results.

Theorem B. *If $f \in \mathcal{A}$ satisfies*

$$|f'(z) - 1| < 1 \qquad (z \in \mathbb{E}),$$

then

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \qquad \text{for } |z| < \frac{1}{2}.$$

Therefore, $f(z)$ is convex for $|z| < \frac{1}{2}$.

Theorem C. *If $f \in \mathcal{A}$ satisfies*

$$|f'(z) - 1| < 1 \qquad (z \in \mathbb{E}),$$

then $f(z)$ maps $|z| < \frac{2\sqrt{5}}{5} = 0.8944\ldots$ onto a domain which is starlike with respect to the origin,

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2} \qquad \text{for } |z| < \frac{2\sqrt{5}}{5}$$

or

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > 0 \qquad \text{for } |z| < \frac{2\sqrt{5}}{5}.$$

The condition domains of Theorem A, Theorem B and Theorem C are some circular domains whose center is the point $z = 1$.

It is the purpose of the present paper to obtain some sufficient conditions for starlikeness or convexity under the hypotheses whose condition domains are annular domains centered at the origin.

2. STARLIKENESS AND CONVEXITY

We start with the following result for starlikeness of functions $f(z)$.

Theorem 2.1. *Let $f \in \mathcal{A}$ and suppose that*

$$0.10583\cdots = \exp\left(-\frac{\pi^2}{4\log 3}\right) < \left|\frac{zf'(z)}{f(z)}\right| < \exp\left(\frac{\pi^2}{4\log 3}\right) = 9.44915\ldots \qquad (z \in \mathbb{E}). \tag{2.1}$$

Then $f(z)$ is starlike for $|z| < \frac{1}{2}$.

Proof. From the assumption (2.1), we get

$$f(z) \neq 0 \qquad (0 < |z| < 1).$$

From the harmonic function theory (cf. Duren [1]), we have

$$\begin{aligned} \log\left(\frac{zf'(z)}{f(z)}\right) &= \frac{1}{2\pi}\int_{|\zeta|=R}\left(\log\left|\frac{\zeta f'(\zeta)}{f(\zeta)}\right|\right)\frac{\zeta+z}{\zeta-z}d\varphi + i\arg\left(\frac{zf'(z)}{f(z)}\right)_{z=0} \\ &= \frac{1}{2\pi}\int_{|\zeta|=R}\left(\log\left|\frac{zf'(\zeta)}{f(\zeta)}\right|\right)\frac{\zeta+z}{\zeta-z}d\varphi \end{aligned}$$

where $|z| = r < |\zeta| = R < 1$, $z = re^{i\theta}$ and $\zeta = Re^{i\varphi}$.

It follows that

$$\begin{aligned} \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| &= \left|\frac{1}{2\pi}\int_{|\zeta|=R}\left(\log\left|\frac{\zeta f'(\zeta)}{f(\zeta)}\right|\right)\left(\operatorname{Im}\frac{\zeta+z}{\zeta-z}\right)d\varphi\right| \\ &\leq \frac{1}{2\pi}\int_0^{2\pi}\left|\log\left|\frac{\zeta f'(\zeta)}{f(\zeta)}\right|\right|\left|\frac{2Rr\sin(\varphi-\theta)}{R^2-2Rr\cos(\varphi-\theta)+r^2}\right|d\varphi \\ &< \frac{\pi^2}{4\log 3}\,\frac{1}{2\pi}\int_0^{2\pi}\frac{2Rr\,|\sin(\varphi-\theta)|}{R^2-2Rr\cos(\varphi-\theta)+r^2}d\varphi \\ &= \frac{\pi^2}{4\log 3}\,\frac{2}{\pi}\log\frac{R+r}{R-r}. \end{aligned}$$

Letting $R \to 1$, we have

$$\begin{aligned} \left|\arg\frac{zf'(z)}{f(z)}\right| &< \frac{\pi}{2\log 3}\log\frac{1+r}{1-r} \\ &< \frac{\pi}{2\log 3}\log 3 \\ &= \frac{\pi}{2} \qquad \left(|z| = r < \frac{1}{2}\right). \end{aligned}$$

This completes the proof of the theorem. □

Next we derive the following

Theorem 2.2. *Let $f \in \mathcal{A}$ and suppose that*

$$\begin{aligned} 0.472367\ldots = \exp\left(-\frac{3}{4}\right) &< \left|\frac{f(z)}{z}\right| \\ &< \exp\left(\frac{3}{4}\right) = 2.177\ldots \qquad (z \in \mathbb{E}). \end{aligned} \tag{2.2}$$

Then we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 \qquad \left(|z| < \frac{1}{2}\right),$$

or $f(z)$ is starlike for $|z| < \frac{1}{2}$.

Proof. From the assumption (2.2), we have

$$f(z) \neq 0 \qquad (0 < |z| < 1).$$

Applying the harmonic function theory (cf. Duren [1]), we have

$$\log\left(\frac{f(z)}{z}\right) = \frac{1}{2\pi}\int_{|\zeta|=R}\left(\log\left|\frac{f(\zeta)}{\zeta}\right|\right)\frac{\zeta+z}{\zeta-z}d\varphi,$$

where $|z| = r < |\zeta| = R < 1$, $z = re^{i\theta}$ and $\zeta = Re^{i\varphi}$.
Then, it follows that

$$\frac{zf'(z)}{f(z)} - 1 = \frac{1}{2\pi}\int_{|\zeta|=R}\left(\log\left|\frac{f(\zeta)}{\zeta}\right|\right)\frac{2\zeta z}{(\zeta-z)^2}d\varphi.$$

This gives us

$$\begin{aligned} \left|\frac{zf'(z)}{f(z)} - 1\right| &\leq \frac{1}{2\pi}\int_{|\zeta|=R}\left|\log\left|\frac{f(\zeta)}{\zeta}\right|\right|\frac{2Rr}{R^2 - 2Rr\cos(\varphi-\theta) + r^2}d\varphi \\ &< \frac{3}{4}\,\frac{1}{2\pi}\int_{|\zeta|=R}\frac{2Rr}{R^2 - 2Rr\cos(\varphi-\theta) + r^2}d\varphi \\ &= \frac{3}{4}\,\frac{2Rr}{R^2 - r^2}. \end{aligned}$$

Making $R \to 1$, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{3}{4}\,\frac{2r}{1-r^2} < 1 \qquad \left(|z| = r < \frac{1}{2}\right),$$

which completes the proof of the theorem. □

For convexity of functions $f(z)$, we show the following corollary without the proof.

Corollary 2.3. *Let $f \in \mathcal{A}$ and suppose that*

$$0.472367\cdots = \exp\left(-\frac{3}{4}\right) < |f'(z)| < \exp\left(\frac{3}{4}\right) = 2.117\ldots \quad (z \in \mathbb{E}). \tag{2.3}$$

Then $f(z)$ is convex for $|z| < \frac{1}{2}$.

Next our result for the convexity of functions $f(z)$ is contained in

Theorem 2.4. *Let $f \in \mathcal{A}$ and suppose that*

$$0.778801\cdots = \exp\left(-\frac{1}{4}\right) < \left|\frac{zf'(z)}{f(z)}\right| < \exp\left(\frac{1}{4}\right) = 1.28403\ldots \quad (z \in \mathbb{E}). \tag{2.4}$$

Then $f(z)$ is convex for $|z| < \frac{1}{2}$.

Proof. From the condition (2.4) of the theorem, we have

$$\frac{zf'(z)}{f(z)} \neq 0 \qquad \text{in } \mathbb{E}.$$

Then, it follows that

$$\log\frac{zf'(z)}{f(z)} = \frac{1}{2\pi}\int_{|\zeta|=R}\left(\log\frac{\zeta f'(\zeta)}{f(\zeta)}\right)\frac{\zeta+z}{\zeta-z}d\varphi, \tag{2.5}$$

where $|z| = r < |\zeta| = R < 1$, $z = re^{i\theta}$ and $\zeta = Re^{i\varphi}$.
Differentiating (2.5) and multiplying by z, we obtain that

$$1+\frac{zf''(z)}{f'(z)} = \frac{zf'(z)}{f(z)} + \frac{1}{2\pi}\int_{|\zeta|=R}\left(\log\left|\frac{\zeta f'(\zeta)}{f(\zeta)}\right|\right)\frac{2\zeta z}{(\zeta-z)^2}d\varphi.$$

In view of Theorem 2.1, $f(z)$ is starlike for $|z| < \frac{1}{2}$ and therefore, we have

$$\operatorname{Re}\frac{zf'(z)}{f(z)} \geq \frac{1-r}{1+r} \qquad \left(|z| = r < \frac{1}{2}\right).$$

Then, we have

$$\begin{aligned}1+\operatorname{Re}\frac{zf''(z)}{f'(z)} &= \operatorname{Re}\frac{zf'(z)}{f(z)} + \frac{1}{2\pi}\int_{|\zeta|=R}\left(\log\left|\frac{\zeta f'(\zeta)}{f(\zeta)}\right|\right)\left(\operatorname{Re}\frac{2\zeta z}{(\zeta-z)^2}\right)d\varphi \\ &> \frac{1-r}{1+r} - \frac{1}{2\pi}\int_{|\zeta|=R}\frac{1}{4}\,\frac{2Rr}{|\zeta-z|^2}d\varphi \\ &= \frac{1-r}{1+r} - \frac{1}{4}\,\frac{2Rr}{R^2-r^2}.\end{aligned}$$

Letting $R \to 1$, we see that

$$\begin{aligned}1+\operatorname{Re}\frac{zf''(z)}{f'(z)} &> \frac{1-r}{1+r} - \frac{1}{4}\,\frac{2r}{1-r^2} \\ &= \frac{1}{3} - \frac{1}{4}\cdot\frac{4}{3} \\ &= 0 \qquad \left(|z| = r < \frac{1}{2}\right),\end{aligned}$$

which completes the proof of our theorem. □

Finally, we prove

Theorem 2.5. *Let $f \in \mathcal{A}$ and suppose that*

$$\begin{aligned}0.10583\ldots = \exp&\left(-\frac{\pi^2}{4\log 3}\right) \\ &< \left|\frac{zf'(z)}{f(z)}\right| < \exp\left(\frac{\pi^2}{4\log 3}\right) = 9.44915\ldots \quad (z \in \mathbb{E}).\end{aligned}$$

Then $f(z)$ is convex in $|z| < r_0$ where r_0 is the root of the equation

$$(4\log 3)r^2 - 2(4\log 3 + \pi^2)r + 4\log 3 = 0,$$

$$r_0 = \frac{\pi^2 - 4\log 3 - \pi\sqrt{\pi^2 + 8\log 3}}{4\log 3} = 0.15787\dots.$$

Proof. Applying the same method as the proof of Theorem 2.5, we have

$$\begin{aligned} 1 + \operatorname{Re}\frac{zf''(z)}{f'(z)} &= \operatorname{Re}\frac{zf'(z)}{f(z)} + \frac{1}{2\pi}\int_{|\zeta|=R}\left(\log\left|\frac{\zeta f'(\zeta)}{f(\zeta)}\right|\right)\left(\operatorname{Re}\frac{2\zeta z}{(\zeta - z)^2}\right)d\varphi \\ &> \frac{1-r}{1+r} - \frac{\pi^2}{4\log 3}\,\frac{2Rr}{R^2 - r^2} \end{aligned}$$

where $|z| = r < |\zeta| = R < 1$, $z = re^{i\theta}$ and $\zeta = Re^{i\varphi}$.
Putting $R \to 1$, we have

$$\begin{aligned} 1 + \operatorname{Re}\frac{zf''(z)}{f'(z)} &> \frac{1-r}{1+r} - \frac{\pi^2}{4\log 3}\,\frac{2r}{1-r^2} \\ &= \frac{1}{(1-r^2)4\log 3}\left\{(4\log 3)r^2 - 2(4\log 3 + \pi^2)r + 4\log 3\right\} \\ &> 0 \qquad (|z| < r_0). \end{aligned}$$

□

Remark 1. The condition in Theorem A by MacGregor [2] implies that

$$0 < \operatorname{Re}\left(\frac{f(z)}{z}\right) < 2 \qquad (z \in \mathbb{E}).$$

However, the condition in Theorem 2.2 implies that

$$-2.117\cdots < \operatorname{Re}\left(\frac{f(z)}{z}\right) < 2.117\dots \qquad (z \in \mathbb{E}).$$

Furthermore, the condition in Theorem B by MacGregor [3] implies that

$$0 < \operatorname{Re} f'(z) < 2 \qquad (z \in \mathbb{E}).$$

However, the condition in Corollary 2.3 implies that

$$-2.117\cdots < \operatorname{Re} f'(z) < 2.117\dots \qquad (z \in \mathbb{E}).$$

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