NEW INEQUALITIES FOR SOME SPECIAL AND q-SPECIAL FUNCTIONS

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Abstract: In this paper, we give new inequalities involving some special (resp. q-special) functions, using their integral (resp. q-integral) representations and a technique developed by A. McD. Mercer in [11]. These inequalities generalize those given in [1], [2], [7] and [11].

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1. Introduction and Preliminaries

In [1], Alsina and M. S. Tomas studied a very interesting inequality involving the Gamma function and they proved the following double inequality

$$\frac{1}{n!} \leq \frac{\Gamma(1+x)^n}{\Gamma(1+nx)} \leq 1, \quad x \in [0,1],\ n \in \mathbb{N}, \tag{1.1}$$

by using geometric method.

In view of the interest in this type of inequalities, many authors extended this result to more general cases either for the classical Gamma function or the basic one, by using geometric or analytic approaches (see [2], [7], [12]).

In [11], A. McD. Mercer, developed a very interesting technique which was the source of some inequalities involving the Gamma, Beta and Zeta functions.

He considered a positive linear functional L defined on a subspace $C^*(I)$ of $C(I)$ (the space of continuous functions on I), where I is the interval $(0,a)$ with $a > 0$ or equal to $+\infty$, and he proved the following result:

Theorem 1.1. *For f, g in $C^*(I)$ such that $f(x) \to 0$, $g(x) \to 0$ as $x \to 0^+$ and $\frac{f}{g}$ is strictly increasing, put*

$$\phi = g\frac{L(f)}{L(g)}$$

and let F be defined on the ranges of f and g such that the compositions $F(f)$ and $F(g)$ each belong to $C^*(I)$.

a) If F is convex then

$$L\left[F(f)\right] \geq L\left[F(\phi)\right]. \tag{1.2}$$

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b) If F is concave then

$$L[F(f)] \leq L[F(\phi)]. \tag{1.3}$$

In this paper, using the previous theorem, we obtain some generalizations of inequalities involving some special and q-special functions.

Note that for $\alpha \in \mathbb{R}$, the function

$$F(t) = t^{\alpha}$$

is convex if $\alpha < 0$ or $\alpha > 1$ and concave if $0 < \alpha < 1$.

So, for f and g satisfying the conditions of the previous theorem, we have:

$$L(f^{\alpha}) > L(\phi^{\alpha}) \quad \text{if} \quad \alpha < 0 \quad \text{or} \quad \alpha > 1 \quad \text{and} \quad L(f^{\alpha}) < L(\phi^{\alpha}) \quad \text{if} \quad 0 < \alpha < 1.$$

Substituting for ϕ this reads:

$$\frac{[L(g)]^{\alpha}}{L(g^{\alpha})} > \quad (\text{resp.} \quad <) \frac{[L(f)]^{\alpha}}{L(f^{\alpha})},$$

if $\alpha < 0$ or $\alpha > 1$ (resp. $0 < \alpha < 1$). In particular, if we take $f(x) = x^{\beta}$ and $g(x) = x^{\delta}$ with $\beta > \delta > 0$, we obtain the following useful inequality:

$$\frac{[L(x^{\delta})]^{\alpha}}{L(x^{\alpha\delta})} \gtrless \frac{[L(x^{\beta})]^{\alpha}}{L(x^{\alpha\beta})}, \tag{1.4}$$

where, we follow the notations of [11], and $\gtrless$ correspond to the case ($\alpha < 0$ or $\alpha > 1$) and ($0 < \alpha < 1$) respectively.

Throughout this paper, we will fix $q \in]0, 1[$ and we will follow the terminology and notation of the book by G. Gasper and M. Rahman [4]. We denote, in particular,

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for $a \in \mathbb{C}$

$$[a]_q = \frac{1-q^a}{1-q}, \qquad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad n = 1, 2, \ldots, \infty.$$

The q-Jackson integrals from 0 to a and from 0 to ∞ are defined by (see [5])

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n, \tag{1.5}$$

$$\int_0^{\infty} f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n, \tag{1.6}$$

provided the sums converge absolutely.

The q-Jackson integral in a generic interval $[a,b]$ is given by (see [5])

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx. \tag{1.7}$$

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2. The Gamma Function

Theorem 2.1. *Let f be the function defined by*

$$f(x) = \frac{\left[\Gamma^{(2n)}(1+x)\right]^{\alpha}}{\Gamma^{(2n)}(1+\alpha x)} \tag{2.1}$$

then for all $0 < \alpha < 1$ (resp. $\alpha > 1$) f is increasing (resp. decreasing) on $(0, \infty)$.

Proof. First, we recall that the Gamma function is infinitely differentiable on $]0, +\infty[$ and we have

$$\forall x \in]0, +\infty[, \quad \forall n \in \mathbb{N}, \quad \Gamma^{(n)}(x) = \int_0^{\infty} t^{x-1}[\mathrm{Log}(t)]^n e^{-t} dt.$$

Now, we consider the subspace $C^*(I)$ obtained from $C(I)$ by requiring its members to satisfy:
(i) $w(x) = O(x^{\theta})$ (for any $\theta > -1$) as $x \to 0$,
(ii) $w(x) = O(x^{\varphi})$ (for any finite φ) as $x \to +\infty$.
For $w \in C^*(I)$, we define

$$L(w) = \int_0^{\infty} w(x)(\mathrm{Log}(x))^{2n} e^{-x} dx. \tag{2.2}$$

The linear functional L is well-defined on $C^*(I)$ and it is positive.

Then, by applying the inequality (1.4), we obtain for $\beta > \delta > 0$,

$$\frac{\left[\Gamma^{(2n)}(1+\delta)\right]^{\alpha}}{\Gamma^{(2n)}(1+\alpha\delta)} \gtrless \frac{\left[\Gamma^{(2n)}(1+\beta)\right]^{\alpha}}{\Gamma^{(2n)}(1+\alpha\beta)}. \tag{2.3}$$

This completes the proof. □

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In particular, we have the following result, which generalizes inequality (4.1) of [11].

Corollary 2.2. *For all $x \in [0,1]$ we have:*

$$\frac{\left[\Gamma^{(2n)}(2)\right]^{\alpha}}{\Gamma^{(2n)}(1+\alpha)} \leq \frac{\left[\Gamma^{(2n)}(1+x)\right]^{\alpha}}{\Gamma^{(2n)}(1+\alpha x)} \leq \left[\Gamma^{(2n)}(1)\right]^{\alpha-1} \quad \text{if} \quad \alpha \geq 1 \tag{2.4}$$

and

$$\left[\Gamma^{(2n)}(1)\right]^{\alpha-1} \leq \frac{\left[\Gamma^{(2n)}(1+x)\right]^{\alpha}}{\Gamma^{(2n)}(1+\alpha x)} \leq \frac{\left[\Gamma^{(2n)}(2)\right]^{\alpha}}{\Gamma^{(2n)}(1+\alpha)} \quad \text{if} \quad 0 < \alpha \leq 1. \tag{2.5}$$

Taking $n = 0$, one obtains:

Corollary 2.3. *For all $x \in [0,1]$,*

$$\frac{1}{\Gamma(1+\alpha)} \leq \frac{[\Gamma(1+x)]^{\alpha}}{\Gamma(1+\alpha x)} \leq 1, \quad \text{if} \quad \alpha \geq 1, \tag{2.6}$$

and

$$1 \leq \frac{[\Gamma(1+x)]^{\alpha}}{\Gamma(1+\alpha x)} \leq \frac{1}{\Gamma(1+\alpha)}, \quad \text{if} \quad 0 < \alpha \leq 1. \tag{2.7}$$

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3. The q-Gamma Function

Jackson [5] defined a q-analogue of the Gamma function by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \qquad x \neq 0, -1, -2, \dots. \tag{3.1}$$

It is well known that it satisfies

$$\Gamma_q(x+1) = [x]_q\Gamma_q(x), \quad \Gamma_q(1) = 1 \quad \text{and} \quad \lim_{q\to 1^-}\Gamma_q(x) = \Gamma(x), \ \Re(x) > 0. \tag{3.2}$$

It has the following q-integral representation (see [8])

$$\Gamma_q(s) = \int_0^{\frac{1}{1-q}} t^{s-1}E_q^{-qt}d_qt, \tag{3.3}$$

where

$$E_q^z =_0 \varphi_0(-;-;q,-(1-q)z) = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}}\frac{(1-q)^n}{(q;q)_n}z^n = (-(1-q)z;q)_\infty, \tag{3.4}$$

is a q-analogue of the exponential function (see [4] and [6]).

In [3], the authors proved that Γ_q is infinitely differentiable on $]0,+\infty[$ and we have

$$\forall x \in]0,+\infty[, \ \forall n \in \mathbb{N}, \ \Gamma_q^{(n)}(x) = \int_0^{\frac{1}{1-q}} t^{x-1}[\mathrm{Log}(t)]^n E_q^{-qt}dt. \tag{3.5}$$

Now, we are able to state a q-analogue of Theorem 2.1, and give generalizations of some inequalities studied in [7].

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Theorem 3.1. *Let f be the function defined by*

$$f(x) = \frac{\left[\Gamma_q^{(2n)}(1+x)\right]^{\alpha}}{\Gamma_q^{(2n)}(1+\alpha x)} \tag{3.6}$$

then for all $0 < \alpha < 1$ (resp. $\alpha > 1$) f is increasing (resp. decreasing) on $(0, \infty)$.

Proof. We consider $I = \left(0, \frac{1}{1-q}\right)$ and the subspace $C^*(I)$ obtained from $C(I)$ by requiring its members to satisfy:
(i) $w(x) = O(x^\theta)$ (for any $\theta > -1$) as $x \to 0$,
(ii) $w(x) = O(1)$ as $x \to \frac{1}{1-q}$.
For $w \in C^*(I)$, we define

$$L(w) = \int_0^{\frac{1}{1-q}} w(x)(\operatorname{Log}(x))^{2n} E_q^{-qx} d_q x. \tag{3.7}$$

L is well-defined on $C^*(I)$ and it is a positive linear functional on $C^*(I)$.
From the inequality (1.4) and the relation (3.5), we obtain for $\beta > \delta > 0$

$$\frac{\left[\Gamma_q^{(2n)}(1+\delta)\right]^{\alpha}}{\Gamma_q^{(2n)}(1+\alpha\delta)} \gtrless \frac{\left[\Gamma_q^{(2n)}(1+\beta)\right]^{\alpha}}{\Gamma_q^{(2n)}(1+\alpha\beta)}, \tag{3.8}$$

which achieves the proof. □

In particular, we have the following result.

Corollary 3.2. *For all $x \in [0, 1]$ we have*

$$\frac{\left[\Gamma_q^{(2n)}(2)\right]^{\alpha}}{\Gamma_q^{(2n)}(1+\alpha)} \leq \frac{\left[\Gamma_q^{(2n)}(1+x)\right]^{\alpha}}{\Gamma_q^{(2n)}(1+\alpha x)} \leq \left[\Gamma_q^{(2n)}(1)\right]^{\alpha-1}, \quad \textit{if} \quad \alpha \geq 1 \tag{3.9}$$

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and

$$\left[\Gamma_q^{(2n)}(1)\right]^{\alpha-1} \le \frac{\left[\Gamma_q^{(2n)}(1+x)\right]^{\alpha}}{\Gamma_q^{(2n)}(1+\alpha x)} \le \frac{\left[\Gamma_q^{(2n)}(2)\right]^{\alpha}}{\Gamma_q^{(2n)}(1+\alpha)}, \quad \textit{if} \quad 0<\alpha\le 1. \tag{3.10}$$

Corollary 3.3. *For all $x \in [0,1]$,*

$$\frac{1}{\Gamma_q(1+\alpha)} \le \frac{[\Gamma_q(1+x)]^{\alpha}}{\Gamma_q(1+\alpha x)} \le 1, \quad \textit{if} \quad \alpha \ge 1, \tag{3.11}$$

and

$$1 \le \frac{[\Gamma_q(1+x)]^{\alpha}}{\Gamma_q(1+\alpha x)} \le \frac{1}{\Gamma_q(1+\alpha)}, \quad \textit{if} \quad 0<\alpha\le 1. \tag{3.12}$$

Proof. By taking $n=0$ in Corollary 3.2 we obtain the inequalities (3.11) and (3.12). □

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4. The q-Beta function

The q-Beta function is defined by (see [4], [8])

$$B_q(t,s)=\int_0^1 x^{t-1}\frac{(xq;q)_\infty}{(xq^s;q)_\infty}d_qx, \qquad \Re(s)>0, \Re(t)>0 \tag{4.1}$$

and we have

$$B_q(t,s)=\frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t+s)}. \tag{4.2}$$

Since B_q is a q-analogue of the classical Beta function, we can see the following results as generalizations of those given in [11].

Theorem 4.1. *For $s>0$, let f be the function defined by*

$$f(x)=\frac{[B_q(1+x,s)]^\alpha}{B_q(1+\alpha x,s)}. \tag{4.3}$$

If $0<\alpha<1$, f is increasing on $[0,+\infty[$.
If $\alpha>1$ f is decreasing on $[0,+\infty[$.

Proof. We consider the interval $I=(0,1)$ and the subspace $C^*(I)$ obtained from $C(I)$ by requiring its members to satisfy:
(i) $w(x)=O(x^\theta)$ (for any $\theta>-1$) as $x\to 0$,
(ii) $w(x)=O(1)$ as $x\to 1$.
For $s>0$, we put for $w\in C^*(I)$,

$$L(w)=\int_0^1 w(x)\frac{(xq;q)_\infty}{(xq^s;q)_\infty}d_qx. \tag{4.4}$$

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It is easy to see that L is well-defined on $C^*(I)$ and it is a positive linear functional on $C^*(I)$.

Then, from the inequality (1.4), we obtain for $\beta > \delta > 0$

$$\frac{[B_q(1+\delta,s)]^\alpha}{B_q(1+\alpha\delta,s)} \gtrless \frac{[B_q(1+\beta,s)]^\alpha}{B_q(1+\alpha\beta,s)}. \tag{4.5}$$

This achieves the proof. □

Corollary 4.2. *For all $x \in [0,1]$, $s > 0$*

$$\frac{[\alpha+s]_q}{[\alpha]_q[s]_q^\alpha[s+1]_q^\alpha B_q(\alpha,s)} \le \frac{[B_q(1+x,s)]^\alpha}{B_q(1+\alpha x,s)} \le \frac{1}{[s]_q^{\alpha-1}}, \quad \textit{if} \quad \alpha \ge 1. \tag{4.6}$$

Proof. It is a consequence of the previous theorem and the relations:

$$B_q(1,s) = \frac{1}{[s]_q}, \quad B_q(2,s) = \frac{1}{[s]_q[s+1]_q}$$

and

$$B_q(1+\alpha,s) = \frac{[\alpha]_q}{[\alpha+s]_q} B_q(\alpha,s).$$

□

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5. The q- Zeta Function

For $x > 0$, we put

$$\alpha(x) = \frac{\text{Log}(x)}{\text{Log}(q)} - E\left(\frac{\text{Log}(x)}{\text{Log}(q)}\right)$$

and

$$\{x\}_q = \frac{[x]_q}{q^{x+\alpha([x]_q)}},$$

where $E\left(\frac{\text{Log}(x)}{\text{Log}(q)}\right)$ is the integer part of $\frac{\text{Log}(x)}{\text{Log}(q)}$.

In [3], the authors defined the q-Zeta function as follows

$$\zeta_q(s) = \sum_{n=1}^{\infty} \frac{1}{\{n\}_q^s} = \sum_{n=1}^{\infty} \frac{q^{(n+\alpha([n]_q))s}}{[n]_q^s}. \tag{5.1}$$

They proved that it is a q-analogue of the classical Riemann Zeta function and in the additional assumption $\frac{\text{Log}(1-q)}{\text{Log}(q)} \in \mathbb{Z}$, we have for all $s \in \mathbb{C}$ such that $\Re(s) > 1$,

$$\zeta_q(s) = \frac{1}{\widetilde{\Gamma}_q(s)} \int_0^{\infty} t^{s-1} Z_q(t) d_q t,$$

where for all $t > 0$,

$$Z_q(t) = \sum_{n=1}^{\infty} e_q^{-\{n\}_q t} \quad \text{and} \quad \widetilde{\Gamma}_q(t) = \frac{\Gamma_q(t)(-q^t, -q^{1-t}; q)_\infty}{(-q, -1; q)_\infty}.$$

Now, we consider the subspace $C^*(I)$ obtained from $C(I)$ by requiring its members to satisfy:

$(i)\ \ w(x) = O(x^\theta)\quad$ (for any $\theta > -1$) as $x \to 0$,

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(ii) $w(x) = O(x^\varphi)$ (for any finite φ) as $x \to +\infty$.

For $w \in C^*(I)$, we define

$$L(w) = \int_0^\infty w(x) Z_q(x) d_q x. \tag{5.2}$$

L is a positive linear functional on $C^*(I)$. So, by application of the inequality (1.4), we obtain for all $\beta > \delta > 0$,

$$\frac{\left[\widetilde{\Gamma}_q(1+\delta)\zeta_q(1+\delta)\right]^\alpha}{\widetilde{\Gamma}_q(1+\alpha\delta)\zeta_q(1+\alpha\delta)} \gtrless \frac{\left[\widetilde{\Gamma}_q(1+\beta)\zeta_q(1+\beta)\right]^\alpha}{\widetilde{\Gamma}_q(1+\alpha\beta)\zeta_q(1+\alpha\beta)}.$$

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