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ON HARDY-HILBERT'S INTEGRAL INEQUALITY

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ABSTRACT. In the present paper, by introducing some parameters, new forms of Hardy-Hilbert's inequalities are given.

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1. INTRODUCTION

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(t), g(t) \geq 0$, $0 < \int_0^\infty f^p(t)dt < \infty$, and $0 < \int_0^\infty g^q(t)dt < \infty$, then, the Hardy-Hilbert integral inequality is given by:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy \leq \frac{\pi}{\sin \frac{\pi}{p}} \left(\int_0^\infty f^p(t)dt \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(t)dt \right)^{\frac{1}{q}}, \tag{1.1}$$

where the constant $\frac{\pi}{\sin \frac{\pi}{p}}$ is the best possible (see [1]).

Yang [2] and [3] gave the following generalization of (1.1)

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y-2\alpha)^\lambda} dxdy \leq K_\lambda^{\frac{1}{p}}(p) K_\lambda^{\frac{1}{q}}(q) \left(\int_0^\infty (t-\alpha)^{1-\lambda} f^p(t)dt \right)^{\frac{1}{p}} \left(\int_0^\infty (t-\alpha)^{1-\lambda} g^q(t)dt \right)^{\frac{1}{q}}, \tag{1.2}$$

where

$$K_\lambda(r) = \int_0^\infty \frac{u^{\frac{1}{r}-1}}{(1+u)^\lambda} du = B\left(\frac{1}{r}, \lambda - \frac{1}{r} \right) \quad 0 < \lambda \leq 1,\ \lambda > \frac{1}{r} > 0,$$

B is the beta function defined by

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx, \quad p, q > 0$$

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and, for $0 < T < \infty$,

$$\int_0^T \int_0^T \frac{f(x)g(y)}{(x+y)^\lambda} dxdy \leq \beta\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left(\int_\alpha^T \left[1 - \frac{1}{2}\left(\frac{t}{T}\right)^{\frac{\lambda}{2}}\right] t^{1-\lambda} f^2(t)dt \right)^{\frac{1}{2}} \times \left(\int_\alpha^T \left[1 - \frac{1}{2}\left(\frac{t}{T}\right)^{\frac{\lambda}{2}}\right] t^{1-\lambda} g^2(t)dt \right)^{\frac{1}{2}}. \tag{1.3}$$

In the present paper, by introducing some parameters, new forms of Hardy-Hilbert's inequalities are given.

2. New Results

We state and prove the following:

Lemma 2.1. *Let* $\lambda > 0$, $p, q, r > 1$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, $f(t), g(t), h(t) \geq 0$, $\lambda_i(t) \geq 0$, $i = p, q, r$, *and assume that*

$$0 < \int_a^b \lambda_p^p(t) f^p(t)dt < \infty,$$
$$0 < \int_c^d \lambda_q^q(t) g^q(t)dt < \infty$$

and

$$0 < \int_c^d \lambda_r^r(t) h^r(t)dt < \infty.$$

Then the two following inequalities are equivalent

$$\int_a^b \int_c^d \int_e^k \frac{f(x)g(y)h(z)}{h^\lambda(x,y,z)} dxdydz \leq K \left(\int_a^b \lambda_p^p(t) f^p(t)dt\right)^{\frac{1}{p}} \left(\int_c^d \lambda_q^q(t) g^q(t)dt\right)^{\frac{1}{q}} \left(\int_e^k \lambda_r^r(t) h^r(t)dt\right)^{\frac{1}{r}}, \tag{2.1}$$

where $K = K(\lambda, p, q, r)$ *is a constant, and*

$$\int_a^b \lambda_p^{\frac{qr}{q+r}}(x) \left(\int_c^d \int_e^k \frac{g(y)h(z)}{h^\lambda(x,y,z)} dydz\right)^{\frac{qr}{q+r}} dx \leq K^{\frac{qr}{q+r}} \left(\int_c^d \lambda_q^q(t) g^q(t)dt\right)^{\frac{r}{q+r}} \left(\int_e^k \lambda_r^r(t) h^r(t)dt\right)^{\frac{q}{q+r}}. \tag{2.2}$$

Proof. Suppose (2.2) is satisfied, then

$$\begin{aligned}
&\int_a^b \int_c^d \int_e^k \frac{f(x)g(y)h(z)}{h^\lambda(x,y,z)} dxdydz \\
&= \int_a^b \lambda_p(x) f(x) \left(\lambda_p^{-1}(x) \int_c^d \int_e^k \frac{g(y)h(z)}{h^\lambda(x,y,z)} dydz \right) dx \\
&\le \left(\int_a^b \lambda_p^p(x) f^p(x) dx \right)^{\frac{1}{p}} \left(\int_a^b \lambda_p^{\frac{qr}{q+r}} \left[\int_c^d \int_e^k \frac{g(y)h(z)}{h^\lambda(x,y,z)} dydz \right]^{\frac{qr}{q+r}} dx \right)^{\frac{q+r}{qr}} \\
&\le K \left(\int_a^b \lambda_p^p(t) f^p(t) dt \right)^{\frac{1}{p}} \left(\int_c^d \lambda_q^q(t) g^q(t) dt \right)^{\frac{1}{q}} \left(\int_e^k \lambda_r^r(t) h^r(t) dt \right)^{\frac{1}{r}}.
\end{aligned}$$

Now, suppose that (2.1) is satisfied, then

$$\begin{aligned}
&\int_a^b \lambda_p^{\frac{qr}{q+r}} \left(\int_c^d \int_e^k \frac{g(y)h(z)}{h^\lambda(x,y,z)} dydz \right)^{\frac{qr}{q+r}} dx \\
&\quad = \int_a^b \int_c^d \int_e^k \frac{g(y)h(z)}{h^\lambda(x,y,z)} . \lambda_p^{\frac{qr}{q+r}} \left(\int_c^d \int_e^k \frac{g(y)h(z)}{h^\lambda(x,y,z)} dydz \right)^{\frac{qr}{q+r}-1} dxdydz \\
&\quad \le K \left(\int_c^d \lambda_q^q(y) g^q(y) dy \right)^{\frac{1}{q}} \left(\int_e^k \lambda_r^r(z) h^r(z) dz \right)^{\frac{1}{r}} \\
&\qquad \times \left(\int_a^b \lambda_p^p(x) \lambda_p^{-p\frac{qr}{q+r}}(x) \left(\int_c^d \int_e^k \frac{g(y)h(z)}{h^\lambda(x,y,z)} dydz \right)^{p(\frac{qr}{q+r}-1)} dx \right)^{\frac{1}{p}} \\
&\quad = K \left(\int_c^d \lambda_q^q(y) g^q(y) dy \right)^{\frac{1}{q}} \left(\int_e^k \lambda_r^r(z) h^r(z) dz \right)^{\frac{1}{r}} \\
&\qquad \times \left(\int_a^b \lambda_p^{\frac{qr}{q+r}}(x) \left(\int_c^d \int_e^k \frac{g(y)h(z)}{h^\lambda(x,y,z)} dydz \right)^{\frac{qr}{q+r}} dx \right)^{\frac{1}{p}},
\end{aligned}$$

therefore

$$\begin{aligned}
&\left(\int_a^b \lambda_p^{\frac{qr}{q+r}}(x) \left(\int_c^d \int_e^k \frac{g(y)h(z)}{h^\lambda(x,y,z)} dydz \right)^{\frac{qr}{q+r}} dx \right)^{\frac{1}{q}+\frac{1}{r}} \\
&\qquad\qquad \le k \left(\int_c^d \lambda_q^q(t) g^q(t) dt \right)^{\frac{1}{q}} \left(\int_e^k \lambda_r^r(t) h^r(t) dt \right)^{\frac{1}{r}},
\end{aligned}$$

and the desired equivalence is proved. □

Lemma 2.2. (a) *Let* $0 \le y \le 1$, $\alpha > 0$, $f \ge 0$. *If we define the function*

$$g(y) = y^{-\alpha} \int_0^y f(x) dx,$$

then $g(y) \ge g(1)$.

(b) *Let* $y \ge 1$, $\alpha > 0$, $f \ge 0$. *Defining the function,*

$$h(y) = y^{-\alpha} \int_0^y f(x) dx,$$

we have $h(y) \ge h(1)$.

Proof. (a) Let $x = \frac{1}{t}$, then

$$g(y) = y^{-\alpha} \int_{y^{-1}}^{\infty} \frac{f\left(\frac{1}{t}\right) dt}{t^2}.$$

We observe also that

$$g'(y) = y^{-\alpha}\left[-y^2 f(y)\right] + \left(\int_{y^{-1}}^{\infty} \frac{f\left(\frac{1}{t}\right)}{t^2} dt\right)(-\alpha)y^{-\alpha-1} \leq 0,$$

therefore g is non-increasing, which implies $g(y) \geq g(1)$.

(b) We obviously have

$$h'(y) = y^{\alpha} f(y) + \left(\int_0^y f(x)dx\right)\alpha y^{\alpha-1} \geq 0,$$

therefore h is non-decreasing, and hence $h(y) \geq h(1)$. □

The following result may be stated as well.

Theorem 2.3. *Let* $f(t),\ g(t),\ h(t) \geq 0,\ p,q,r > 1,\ \frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 1,\ 2 < \lambda < 3,\ \gamma > \mu \max\{p,q,r\}$, *and*

$$\max\left\{-\frac{1}{p}, -\frac{1}{q}, -\frac{1}{r}\right\} < \mu < \min\left\{\frac{\lambda-1}{p}, \frac{\lambda-1}{q}, \frac{\lambda-1}{r}\right\}.$$

$$0 < \int_{\alpha}^{T} (t-\alpha)^{2-\lambda} f^p(t)dt < \infty,$$

$$0 < \int_{\alpha}^{T} (t-\alpha)^{2-\lambda} g^q(t)dt < \infty$$

and

$$0 < \int_{\alpha}^{T} (t-\alpha)^{2-\lambda} h^r(t)dt < \infty,$$

then

$$\int_{\alpha}^{T}\int_{\alpha}^{T}\int_{\alpha}^{T} \frac{f(x)g(y)h(z)}{(x+y+z)^{\lambda}} dxdydz \leq \left(\int_{\alpha}^{T} \phi(t,\mu,\lambda,p)(t-\alpha)^{2-\lambda} f^p(t)dt\right)^{\frac{1}{p}} \times \left(\int_{\alpha}^{T} \phi(t,\mu,\lambda,q)(t-\alpha)^{2-\lambda} g^q(t)dt\right)^{\frac{1}{q}} \left(\int_{\alpha}^{T} \phi(t,\mu,\lambda,r)(t-\alpha)^{2-\lambda} h^r(t)dt\right)^{\frac{1}{r}}, \tag{2.3}$$

where

$$\begin{aligned}\phi(t,\mu,\lambda,j) &= B(\lambda-\mu j-1,\mu j+1)\left\{B(\lambda-2,1-\mu j) - \left(\frac{t-\alpha}{T-\alpha}\right)^{\gamma}\int_0^1 \frac{u^{\lambda-3}}{(1+u)^{\lambda-\mu j-1}}du\right\} - \left(\frac{t-\alpha}{T-\alpha}\right)^{2\gamma} \\ &\quad \times \int_0^1 \frac{u^{\lambda-\mu j-2}}{(1+u)^{\lambda}}du \int_0^1 \frac{u^{\gamma-\mu j-2}}{(1+u)^{\lambda-\mu j-1}}du, \quad j = p,q,r.\end{aligned}$$

Proof. The proof is as follows. We have

$$\begin{aligned}
&\int_\alpha^T\int_\alpha^T\int_\alpha^T \frac{f(x)g(x)h(z)}{(x+y+z)^\lambda}dxdydz\\
&=\int_\alpha^T\int_\alpha^T\int_\alpha^T \frac{f(x)\left(\frac{z-\alpha}{y-\alpha}\right)^\mu}{(x+y+z)^{\lambda/p}}\frac{g(y)\left(\frac{x-\alpha}{z-\alpha}\right)^\mu}{(x+y+z)^{\lambda/q}}\frac{h(z)\left(\frac{y-\alpha}{x-\alpha}\right)^\mu}{(x+y+z)^{\lambda/r}}dxdydz\\
&\le\left(\int_\alpha^T\int_\alpha^T\int_\alpha^T \frac{f^p(x)\left(\frac{z-\alpha}{y-\alpha}\right)^{\mu p}}{(x+y+z)^\lambda}dxdydz\right)^{\frac{1}{p}}\left(\int_\alpha^T\int_\alpha^T\int_\alpha^T \frac{g^q(y)\left(\frac{x-\alpha}{z-\alpha}\right)^{\mu q}}{(x+y+z)^\lambda}dxdydz\right)^{\frac{1}{q}}\\
&\times\left(\int_\alpha^T\int_\alpha^T\int_\alpha^T \frac{h^r(z)\left(\frac{y-\alpha}{x-\alpha}\right)^{\mu r}}{(x+y+z)^\lambda}dxdydz\right)^{\frac{1}{r}}\\
&=F^{\frac{1}{p}}G^{\frac{1}{q}}H^{\frac{1}{r}},\ \text{ say.}
\end{aligned}$$

Then we have

$$F=\int_\alpha^T(x-\alpha)^{2-\lambda}f^p(x)dx\int_\alpha^T\frac{\left(\frac{y-\alpha}{x-\alpha}\right)^{-\mu p}}{(1+\frac{y-\alpha}{x-\alpha})^{\lambda-\mu p-1}}\frac{dy}{x-\alpha}\int_\alpha^T\frac{\left(\frac{z-\alpha}{x+y-2\alpha}\right)^{-\mu p}}{(1+\frac{z-\alpha}{x+y-2\alpha})^{\lambda}}\frac{dz}{x+y-2\alpha}.$$

Now by Lemma 2.2, we can state that

$$\begin{aligned}
&\int_0^T\frac{\left(\frac{z-\alpha}{x+y-2\alpha}\right)^{\mu p}}{(1+\frac{z-\alpha}{x+y-2\alpha})^{\lambda}}\frac{dz}{x+y-2\alpha}\\
&\qquad=\int_0^{\frac{T-\alpha}{x+y-2\alpha}}\frac{u^{\mu p}}{(1+u)^\lambda}du\\
&\qquad\le\int_0^{\frac{T-\alpha}{y-\alpha}}\frac{u^{\mu p}}{(1+u)^\lambda}du\\
&\qquad=\int_{\frac{y-\alpha}{T-\alpha}}^\infty\frac{u^{\lambda-\mu p-2}}{(1+u)^\lambda}du\\
&\qquad=\left(\int_0^\infty-\left(\frac{y-\alpha}{T-\alpha}\right)^\gamma\left(\frac{y-\alpha}{T-\alpha}\right)^{-\gamma}\int_0^{\frac{y-\alpha}{T-\alpha}}\right)\frac{u^{\lambda-\mu p-2}}{(1+u)^\lambda}du\\
&\qquad\le\left[B(\lambda-\mu p-1,\mu p+1)-\left(\frac{y-\alpha}{T-\alpha}\right)^\gamma\int_0^1\frac{u^{\lambda-\mu p-2}}{(1+u)^\lambda}du\right].
\end{aligned}$$

Therefore

$$\begin{aligned}
F&\le\int_\alpha^T(x-\alpha)^{2-\lambda}f^p(x)dx\int_\alpha^T\frac{\left(\frac{y-\alpha}{x-\alpha}\right)^{-\mu p}}{(1+\frac{y-\alpha}{x-\alpha})^{\lambda-\mu p-1}}\frac{dy}{x-\alpha}\\
&\qquad\times\left[B(\lambda-\mu p-1,\mu p+1)-\left(\frac{y-\alpha}{T-\alpha}\right)^\gamma\int_0^1\frac{u^{\lambda-\mu p-2}}{(1+u)^\lambda}du\right]\\
&=\int_\alpha^T(x-\alpha)^{2-\lambda}f^p(x)dx\int_0^{\frac{T-\alpha}{x-\alpha}}\frac{u^{-\mu p}}{(1+u)^{\lambda-\mu p-1}}du
\end{aligned}$$

$$\times\left[B(\lambda-\mu p-1,\mu p+1)-\left(\frac{x-\alpha}{T-\alpha}\right)^{\gamma}u^{\gamma}\int_0^1\frac{u^{\lambda-\mu p-2}}{(1+u)^{\lambda}}du\right]$$
$$=\int_\alpha^T(x-\alpha)^{2-\lambda}f^p(x)dx\times\left[B(\lambda-\mu p-1,\mu p+1)-\int_0^{\frac{T-\alpha}{x-\alpha}}\frac{u^{-\mu p}}{(1+u)^{\lambda-\mu p-1}}\right.$$
$$\left.-\left(\frac{x-\alpha}{T-\alpha}\right)^{\gamma}\int_0^1\frac{u^{\lambda-\mu p-2}}{(1+u)^{\lambda}}du\int_0^{\frac{T-\alpha}{x-\alpha}}\frac{u^{\gamma-\mu p}}{(1+u)^{\lambda-\mu p-1}}du\right]$$
$$=\int_\alpha^T(x-\alpha)^{2-\lambda}f^p(x)dx\times\left[B(\lambda-\mu p-1,\mu p+1)-\int_{\frac{x-\alpha}{T-\alpha}}^{\infty}\frac{u^{\lambda-3}}{(1+u)^{\lambda-\mu p-1}}du\right.$$
$$\left.-\left(\frac{x-\alpha}{T-\alpha}\right)^{\gamma}\int_0^1\frac{u^{\lambda-\mu p-2}}{(1+u)^{\lambda}}du\int_0^{\frac{T-\alpha}{x-\alpha}}\frac{u^{\gamma-\mu p}}{(1+u)^{\lambda-\mu p-1}}du\right]$$

$$=\int_\alpha^T(x-\alpha)^{2-\lambda}f^p(x)dx$$
$$\times\left[B(\lambda-\mu p-1,\mu p+1)-\left(\int_0^{\infty}-\int_0^{\frac{x-\alpha}{T-\alpha}}\right)\frac{u^{\lambda-3}}{(1+u)^{\lambda-\mu p-1}}du\right.$$
$$\left.-\left(\frac{x-\alpha}{T-\alpha}\right)^{\gamma}\int_0^1\frac{u^{\lambda-\mu p-2}}{(1+u)^{\lambda}}du\int_0^{\frac{T-\alpha}{x-\alpha}}\frac{u^{\gamma-\mu p}}{(1+u)^{\lambda-\mu p-1}}du\right]$$
$$=\int_\alpha^T(x-\alpha)^{2-\lambda}f^p(x)dx$$
$$\times\left[B(\lambda-\mu p-1,\mu p+1)\left\{B(\lambda-2,1-\mu p)-\left(\frac{x-\alpha}{T-\alpha}\right)^{\gamma}\left(\frac{x-\alpha}{T-\alpha}\right)^{-\gamma}\right.\right.$$
$$\left.\times\int_0^{\frac{x-\alpha}{T-\alpha}}\frac{u^{\lambda-3}}{(1+u)^{\lambda-\mu p-1}}du\right\}-\left(\frac{x-\alpha}{T-\alpha}\right)^{2\gamma}\int_0^1\frac{u^{\lambda-\mu p-2}}{(1+u)^{\lambda}}du$$
$$\left.\times\left(\frac{T-\alpha}{x-\alpha}\right)^{\gamma}\int_0^{\frac{T-\alpha}{x-\alpha}}\frac{u^{\gamma-\mu p}}{(1+u)^{\lambda-\mu p-1}}\right]$$
$$\leq\int_\alpha^T(x-\alpha)^{2-\lambda}f^p(x)dx\times\left[B(\lambda-\mu p-1,\mu p+1)\left\{B(\lambda-2,1-\mu p)-\left(\frac{x-\alpha}{T-\alpha}\right)^{\gamma}\right.\right.$$
$$\left.\left.\times\int_0^1\frac{u^{\lambda-3}}{(1+u)^{\lambda-\mu p-1}}du\right\}-\left(\frac{x-\alpha}{T-\alpha}\right)^{2\gamma}\int_0^1\frac{u^{\lambda-\mu p-2}}{(1+u)^{\lambda}}du\int_0^1\frac{u^{\gamma-\mu p}}{(1+u)^{\lambda-\mu p-1}}\right]$$
$$=\int_\alpha^T\phi(x,\lambda,\mu,p)(x-\alpha)^{2-\lambda}f^p(x)dx,$$

where

$$\phi(x,\lambda,\mu,p)$$
$$=\left[B(\lambda-\mu p-1,\mu p+1)\left\{B(\lambda-2,1-\mu p)-\left(\frac{x-\alpha}{T-\alpha}\right)^{\gamma}\int_0^1\frac{u^{\lambda-3}}{(1+u)^{\lambda-\mu p-1}}du\right\}\right.$$
$$\left.-\left(\frac{x-\alpha}{T-\alpha}\right)^{2\gamma}\int_0^1\frac{u^{\lambda-\mu p-2}}{(1+u)^{\lambda}}du\int_0^1\frac{u^{\gamma-\mu p}}{(1+u)^{\lambda-\mu p-1}}du\right].$$

Similarly

$$G = \int_{\alpha}^{T} \phi(y, \lambda, \mu, q)(y - \alpha)^{2-\lambda} g^q(y) dy,$$

and

$$H = \int_{\alpha}^{T} \phi(z, \lambda, \mu, r)(z - \alpha)^{2-\lambda} h^r(z) dz.$$

This completes the proof. □

Corollary 2.4. *Let* $f(t), g(t), h(z) \geq 0$*,* $p, q, r > 1$*,* $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$*,* $2 < \lambda < 3$*, and*

$$\max\left\{-\frac{1}{p}, -\frac{1}{q}, -\frac{1}{r}\right\} < \mu < \min\left\{\frac{\lambda - 1}{p}, \frac{\lambda - 1}{q}, \frac{\lambda - 1}{r}\right\}.$$

If

$$0 < \int_{\alpha}^{\infty} (t - \alpha)^{2-\lambda} f^p(t) dt < \infty,$$
$$0 < \int_{\alpha}^{\infty} (t - \alpha)^{2-\lambda} g^q(t) dt < \infty$$

and

$$0 < \int_{\alpha}^{\infty} (t - \alpha)^{2-\lambda} h^r(t) dt < \infty,$$

then we have the inequality

$$\begin{aligned} (2.4) \quad & \int_{\alpha}^{\infty}\int_{\alpha}^{\infty}\int_{\alpha}^{\infty} \frac{f(x)g(y)h(z)}{(x+y+z)^{\lambda}} dx dy dz \\ & \leq K \left(\int_{\alpha}^{\infty} (t - \alpha)^{2-\lambda} f^p(t) dt\right)^{\frac{1}{p}} \left(\int_{\alpha}^{\infty} (t - \alpha)^{2-\lambda} g^q(t) dt\right)^{\frac{1}{q}} \\ & \qquad \times \left(\int_{\alpha}^{\infty} (t - \alpha)^{2-\lambda} h^r(t) dt\right)^{\frac{1}{r}}, \end{aligned}$$

where

$$K = \prod_{j=p,q,r} B^{1/j}(\lambda - \mu j - 1, \mu j + 1) B^{1/j}(\lambda - 2, 1 - \mu j)$$

and

$$\begin{aligned} (2.5) \quad & \int_{\alpha}^{\infty} (x - \alpha)^{\frac{qr(\lambda-2)}{p(q+r)}} \left(\int_{\alpha}^{\infty}\int_{\alpha}^{\infty} \frac{g(y)h(z)}{(x+y+z)^{\lambda}} dx dy dz\right)^{\frac{qr}{q+r}} dx \\ & \leq K^{\frac{qr}{q+r}} \left(\int_{\alpha}^{\infty} (t - \alpha)^{2-\lambda} g^q(t) dt\right)^{\frac{r}{q+r}} \left(\int_{\alpha}^{\infty} (t - \alpha)^{2-\lambda} h^r(t) dt\right)^{\frac{q}{q+r}}. \end{aligned}$$

The inequalities (2.4)and (2.5) are equivalent.

Proof. Follows from Theorem 2.3 and Lemma 2.1, on choosing $\gamma = 1$, $T = \infty$, and $\lambda_j(t) = (t - \alpha)^{\frac{2-\lambda}{j}}$. We omit the details. □

Corollary 2.5. *Let* $f(t), g(t), h(t) \geq 0$, $2 < \lambda < 3$, $\gamma > 3\mu$, *and* $-\frac{1}{3} < \mu < \frac{\lambda-1}{3}$. *If*

$$0 < \int_{\alpha}^{T} (t-\alpha)^{2-\lambda} f^3(t)dt < \infty,$$
$$0 < \int_{\alpha}^{T} (t-\alpha)^{2-\lambda} g^3(t)dt < \infty$$

and

$$0 < \int_{\alpha}^{T} (t-\alpha)^{2-\lambda} h^3(t)dt < \infty,$$

then

$$\int_{\alpha}^{T}\int_{\alpha}^{T}\int_{\alpha}^{T} \frac{f(x)g(x)h(z)}{(x+y+z)^{\lambda}}dxdydz \leq \left(\int_{\alpha}^{T} \phi(t,\lambda,\mu,3)(t-\alpha)^{2-\lambda} f^3(t)dt\right)^{\frac{1}{3}} \left(\int_{\alpha}^{T} \phi(t,\lambda,\mu,3)(t-\alpha)^{2-\lambda} g^3(t)dt\right)^{\frac{1}{3}} \times \left(\int_{\alpha}^{T} \phi(t,\lambda,\mu,3)(t-\alpha)^{2-\lambda} h^3(t)dt\right)^{\frac{1}{3}}, \tag{2.6}$$

and

$$\int_{\alpha}^{T} \phi^{-\frac{1}{2}}(t,\lambda,\mu,3)(x-\alpha)^{\frac{\lambda}{2}-1}\left(\int_{\alpha}^{T}\int_{\alpha}^{T} \frac{g(y)h(z)}{(x+y+z)^{\lambda}}dydz\right)^{\frac{1}{2}}dx \leq \left(\int_{\alpha}^{T} \phi(t,\lambda,\mu,3)(t-\alpha)^{2-\lambda} g^3(t)dt\right)^{\frac{1}{3}} \left(\int_{\alpha}^{T} \phi(t,\lambda,\mu,3)(t-\alpha)^{2-\lambda} h^3(t)dt\right)^{\frac{1}{3}}. \tag{2.7}$$

The inequalities (2.6) and (2.7) are equivalent.

Proof. Follows from Theorem 2.3 and Lemma 2.1, by putting $p = q = r = 3$, and $\lambda_j(t) = (t-\alpha)^{\frac{2-\lambda}{3}}\phi^{\frac{1}{3}}(t,\lambda,\mu,3)$, $j = p, q, r$. ☐

Note. In Corollary 2.5, we may take as a special case $\mu = 1 - \frac{\lambda}{3}$ to obtain

$$\phi(t,\lambda,1-\lambda/3,3) = B(2\lambda-4, 4-\lambda)B(\lambda-2,\lambda-2)\left\{1-\frac{1}{2}\left(\frac{t-\alpha}{T-\alpha}\right)^{\gamma}\right\} - \left(\frac{t-\alpha}{T-\alpha}\right)^{2\gamma}\int_0^1 \frac{u^{2\lambda-5}}{(1+u)^{\lambda}}du\int_0^1 \frac{u^{\gamma+\lambda-3}}{(1+u)^{2\lambda-4}}du.$$

Corollary 2.6. *Let* $f(t), g(t), h(t) \geq 0$, $2 < \lambda < 3$, *and* $-\frac{1}{3} < \mu < \frac{\lambda-1}{3}$. *If*

$$0 < \int_{\alpha}^{\infty} (t-\alpha)^{2-\lambda} f^3(t)dt < \infty,$$
$$0 < \int_{\alpha}^{\infty} (t-\alpha)^{2-\lambda} g^3(t)dt < \infty$$

and

$$0 < \int_{\alpha}^{\infty} (t-\alpha)^{2-\lambda} h^3(t)dt < \infty,$$

then

$$\int_\alpha^\infty \int_\alpha^\infty \int_\alpha^\infty \frac{f(x)g(y)h(z)}{(x+y+z)^\lambda} dxdydz \le K \left(\int_\alpha^\infty (t-\alpha)^{2-\lambda} f^3(t)dt \right)^{\frac{1}{3}} \left(\int_\alpha^\infty (t-\alpha)^{2-\lambda} g^3(t)dt \right)^{\frac{1}{3}} \times \left(\int_\alpha^\infty (t-\alpha)^{2-\lambda} h^3(t)dt \right)^{\frac{1}{3}}, \tag{2.8}$$

where

$$K = (B(\lambda - 3\mu - 1, 3\mu + 1)B(\lambda - 2, 1 - 3\mu)$$

and

$$\int_\alpha^\infty (x-\alpha)^{-\frac{1}{2}} \left(\int_\alpha^\infty \int_\alpha^\infty \frac{g(y)h(z)}{(x+y+z)^\lambda} dydz \right)^{\frac{1}{2}} dx \le K^{3/2} \left(\int_\alpha^\infty (t-\alpha)^{2-\lambda} g^3(t)dt \right)^{\frac{1}{2}} \times \left(\int_\alpha^\infty (t-\alpha)^{2-\lambda} h^3(t)dt \right)^{\frac{1}{2}}. \tag{2.9}$$

The inequalities (2.8) and (2.9) are equivalent.

Proof. Follows from Corollary 2.4 and Lemma 2.1, by putting $p = q = r = 3$, $T = \infty$. □

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