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ON CERTAIN CLASSES OF ANALYTIC FUNCTIONS

KHALIDA INAYAT NOOR

Mathematics Department
COMSATS Institute of Information Techonolgy
Islamabad, Pakistan

EMail: khalidanoor@hotmail.com

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Abstract

Let $\mathcal{A}$ be the class of functions $f : f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in the unit disk E. We introduce the class $B_k(\lambda, \alpha, \rho) \subset \mathcal{A}$ and study some of their interesting properties such as inclusion results and covering theorem. We also consider an integral operator for these classes.

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Contents

1. Introduction

Let $\mathcal{A}$ denote the class of functions

$$f : f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$ and let $S \subset \mathcal{A}$ be the class of functions univalent in E.

Let $P_k(\rho)$ be the class of functions $p(z)$ analytic in E satisfying the properties $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\operatorname{Re} p(z) - \rho}{1 - \rho} \right| d\theta \leq k\pi, \tag{1.1}$$

where $z = re^i\theta, \quad k \geq 2$ and $0 \leq \rho < 1$. This class has been introduced in [7]. We note that, for $\rho = 0$, we obtain the class P_k defined and studied in [8], and for $\rho = 0,\ k = 2$, we have the well known class P of functions with positive real part. The case $k = 2$ gives the class $P(\rho)$ of functions with positive real part greater than ρ.

From (1.1) we can easily deduce that $p \in P_k(\rho)$ if, and only if, there exist $p_1, p_2 \in P(\rho)$ such that, for E,

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z). \tag{1.2}$$

Let f and g be analytic in E with $f(z) = \sum_{m=0}^{\infty} a_m z^m$ and $g(z) = \sum_{m=0}^{\infty} b_m z^m$

in E. Then the convolution $\star$ (or Hadamard Product) of f and g is defined by

$$(f \star g)(z) = \sum_{m=0}^{\infty} a_m b_m z^m, \quad m \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}.$$

Definition 1.1. *Let $f \in \mathcal{A}$. Then $f \in B_k(\lambda, \alpha, \rho)$ if and only if*

$$\left[(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda \frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha}\right] \in P_k(\rho), \quad z \in E, \tag{1.3}$$

where $\alpha > 0, \lambda > 0, k \geq 2$ and $0 \leq \rho < 1$. The powers are understood as principal values.

For $k = 2$ and with different choices of λ, α and ρ, these classes have been studied in [2, 3, 4, 10]. In particular $B_2(1, \alpha, \rho)$ is the class of Bazilevic functions studied in [1].

We shall need the following results.

Lemma 1.1 ([9]). *If $p(z)$ is analytic in E with $p(0) = 1$ and if λ is a complex number satisfying $\operatorname{Re}\lambda \geq 0,\ (\lambda \neq 0)$, then*

$$\operatorname{Re}[p(z) + \lambda z p'(z)] > \beta \quad (0 \leq \beta < 1)$$

implies

$$\operatorname{Re} p(z) > \beta + (1-\beta)(2\gamma - 1),$$

where γ is given by

$$\gamma = \gamma_{\operatorname{Re}\lambda} = \int_0^1 (1 + t^{\operatorname{Re}\lambda})^{-1} dt.$$

Lemma 1.2 ([5]). *Let $c > 0, \lambda > 0, \rho < 1$ and $p(z) = 1 + b_1 z + b_2 z^2 + \cdots$ be analytic in E. Let $\operatorname{Re}[p(z) + c\lambda z p'(z)] > \rho$ in E, then*

$$\operatorname{Re}[p(z) + czp'(z)] \geq 2\rho - 1 + 2(1-\rho)\left(1 - \frac{1}{\lambda}\right)\frac{1}{c\lambda}\int_0^1 \frac{u^{\frac{1}{c\lambda}-1}}{1+u}du.$$

This result is sharp.

2. Main Results

Theorem 2.1. *Let* $\lambda, \alpha > 0,\ 0 \leq \rho < 1$ *and let* $f \in b_k(\lambda, \alpha, \rho)$. *Then* $\left(\frac{f(z)}{z}\right)^\alpha \in P_k(\rho_1)$, *where* ρ_1 *is given by*

$$\rho_1 = \rho + (1-\rho)(2\gamma - 1), \tag{2.1}$$

and

$$\gamma = \int_0^1 \left(1 + t^{\frac{\lambda}{\alpha}}\right)^{-1} dt.$$

Proof. Let

$$\left(\frac{f(z)}{z}\right)^\alpha = p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z).$$

Then $p(z) = 1 + \alpha a_2 z + \cdots$ is analytic in E, and

$$(f(z))^\alpha = z^\alpha p(z). \tag{2.2}$$

Differentiation of (2.2) and some computation give us

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^\alpha + \lambda \frac{zf'(z)}{f(z)} \left(\frac{f(z)}{z}\right)^\alpha = p(z) + \frac{\lambda}{\alpha} zp'(z).$$

Since $f \in B_k(\lambda, \alpha, \rho)$, so $\{p(z) + \frac{\lambda}{\alpha} zp'(z)\} \in P_k(\rho)$ for $z \in E$. This implies that

$$\operatorname{Re}\left[p_i(z) + \frac{\lambda}{\alpha} zp_i'(z)\right] > \rho, \quad i = 1, 2.$$

Using Lemma 1.1, we see that $\operatorname{Re}\{p_i(z)\} > \rho_1$, where ρ_1 is given by (2.1). Consequently $p \in P_k(\rho_1)$ for $z \in E$, and the proof is complete. □

Corollary 2.2. *Let $f = zF_1'$ and $f \in B_2(\lambda, 1, \rho)$. Then F_1 is univalent in E.*

Proceeding as in Theorem 2.1 and using Lemma 1.2, we have the following.

Theorem 2.3. *Let $\alpha > 0,\ \lambda > 0,\ 0 \le \rho < 1$ and let $f \in B_k(\lambda, \alpha, \rho)$. Then $\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} \in P_k(\rho_2)$, where*

$$\rho_2 = 2\rho - 1 + \frac{1-\rho}{\lambda} + 2(1-\rho)\left(1 - \frac{1}{\lambda}\right)\frac{\alpha}{\lambda}\int_0^1 \frac{u^{\frac{\alpha}{\lambda}-1}}{1+u}du.$$

This result is sharp.

For $k = 2$, we note that f is univalent, see [1].

Theorem 2.4. *Let, for $\alpha > 0, \lambda > 0,\ 0 \le \rho < 1,\ f \in B_k(\lambda, \alpha, \rho)$ and define $I(f) : \mathcal{A} \longrightarrow \mathcal{A}$ as*

$$I(f) = F(z) = \left[\frac{1}{\lambda} z^{\alpha - \frac{1}{\lambda}} \int_0^z t^{\frac{1}{\lambda}-1-\alpha} (f(z))^{\alpha}\, dt\right]^{\frac{1}{\alpha}}, \quad z \in E. \tag{2.3}$$

Then $F \in B_k(\alpha\lambda, \alpha, \rho_1)$ for $z \in E$, where ρ_1 is given by (2.1).

Proof. Differentiating (2.3), we have

$$(1-\alpha\lambda)\left(\frac{F(z)}{z}\right)^{\alpha} + \alpha\lambda\frac{zF'(z)}{F(z)}\left(\frac{F(z)}{z}\right)^{\alpha} = \left(\frac{f(z)}{z}\right)^{\alpha}.$$

Now, using Theorem 2.1, we obtain the required result. □

Theorem 2.5. *Let*

$$f : f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in B_k(\lambda, \alpha, \rho).$$

Then

$$|a_n| \leq \frac{k(1-\rho)}{\lambda + \alpha}.$$

The function $f_{\lambda,\alpha,\rho}(z)$ *defined as*

$$\left(\frac{f_{\lambda,\alpha,\rho}(z)}{z}\right)^{\alpha} = \frac{\alpha}{\lambda}\int_0^1 \left[\left(\frac{k}{4}+\frac{1}{2}\right) u^{\frac{\alpha}{\lambda}-1}\frac{1+(1-2\rho)uz}{1-uz} - \left(\frac{k}{4}-\frac{1}{2}\right) u^{\frac{\alpha}{\lambda}-1}\frac{1-(1-2\rho)uz}{1+uz}\right] du$$

shows that this inequality is sharp.

Proof. Since $f \in B_k(\lambda, \alpha, \rho)$, so

$$(1-\lambda)\left(1+\sum_{n=2}^{\infty} a_n z^{n-1}\right)^{\alpha} + \lambda\left(1+\sum_{n=2}^{\infty} n a_n z^{n-1}\right)\left(1+\sum_{n=2}^{\infty} a_n z^{n-1}\right)^{\alpha}$$
$$= H(z) = \left(1+\sum_{n=1}^{\infty} c_n z^n\right) \in P_k(\rho).$$

It is known that $|c_n| \leq k(1-\rho)$ for all n and using this inequality, we prove the required result. $\square$

Different choices of k, λ, α and ρ yield several known results.

Theorem 2.6 (Covering Theorem). *Let $\lambda > 0$ and $0 < \rho < 1$. Let $f = zF_1' \in B_2(\lambda, 1, \rho)$. If D is the boundary of the image of E under F_1, then every point of D has a distance of at least $\frac{\lambda+1}{(3+2\lambda-\rho)}$ from the origin.*

Proof. Let $F_1(z) \neq w_0,\ w_0 \neq 0$. Then $f_1(z) = \frac{w_0 F_1(z)}{w_0 + F_1(z)}$ is univalent in E since F_1 is univalent. Let

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad F_1(z) = z + \sum_{n=2}^{\infty} b_n z^n.$$

Then $a_2 = 2b_2$. Also

$$f_1(z) = z + \left(b_2 + \frac{1}{w_0}\right) z^2 + \cdots,$$

and so $|b_2 + \frac{1}{w_0}| \leq 2$. Since, by Theorem 2.5, $|b_2| \leq \frac{1-\rho}{1+\lambda}$, we obtain $|w_0| \geq \frac{\lambda+1}{3+2\lambda-\rho}$. $\square$

Theorem 2.7. *For each $\alpha > 0$, $B_k(\lambda_1, \alpha, \rho) \subset B_k(\lambda_2, \alpha, \rho)$ for $0 \leq \lambda_2 < \lambda_1$.*

Proof. For $\lambda_2 = 0$, the proof is immediate. Let $\lambda_2 > 0$ and let $f \in B_k(\lambda_1, \alpha, \rho)$. Then there exist two functions $h_1, h_2 \in P_k(\rho)$ such that, from Definition 1.1 and Theorem 2.1,

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda_1 \frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} = h_1(z),$$

and

$$\left(\frac{f(z)}{z}\right)^{\alpha} = h_2(z).$$

Hence

$$(1-\lambda_2)\left(\frac{f(z)}{z}\right)^{\alpha}+\lambda_2\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} = \frac{\lambda_2}{\lambda_1}h_1(z)+\left(1-\frac{\lambda_2}{\lambda_1}\right)h_2(z). \tag{2.4}$$

Since the class $P_k(\rho)$ is a convex set, see [6], it follows that the right hand side of (2.4) belongs to $P_k(\rho)$ and this proves the result. □

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