SHARP NORM INEQUALITY FOR BOUNDED SUBMARTINGALES AND STOCHASTIC INTEGRALS

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Abstract: Let $\alpha \in [0,1]$ be a fixed number and $f = (f_n)$ be a nonnegative submartingale bounded from above by 1. Assume $g = (g_n)$ is a process satisfying, with probability 1,

$$|dg_n| \leq |df_n|, \quad |\mathbb{E}(dg_{n+1}|\mathcal{F}_n)| \leq \alpha \mathbb{E}(df_{n+1}|\mathcal{F}_n), \qquad n = 0,\ 1,\ 2,\ \ldots.$$

We provide a sharp bound for the first moment of the process g. A related estimate for stochastic integrals is also established.

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1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(\mathcal{F}_n)_{n\geq 0}$ be a filtration, a nondecreasing sequence of sub-σ-algebras of $\mathcal{F}$. Throughout the paper, α is a fixed number belonging to the interval $[0, 1]$. Let $f = (f_n)_{n\geq 0}$, $g = (g_n)_{n\geq 0}$ denote adapted real-valued integrable processes, such that f is a submartingale and g is α-subordinate to f: for any $n = 0, 1, 2, \ldots$ we have, almost surely,

$$|dg_n| \leq |df_n| \tag{1.1}$$

and

$$|\mathbb{E}(dg_{n+1}|\mathcal{F}_n)| \leq \alpha\mathbb{E}(df_{n+1}|\mathcal{F}_n). \tag{1.2}$$

Here $df = (df_n)_{n\geq 0}$ and $dg = (dg_n)$ stand for the difference sequences of f and g, given by

$$df_0 = f_0, \quad df_n = f_n - f_{n-1}, \quad dg_0 = g_0, \quad dg_n = g_n - g_{n-1}, \qquad n = 1, 2, \ldots.$$

The main objective of this paper is to provide some bounds on the size of the process g under some additional assumptions on the boundedness of f. Let us provide some information about related estimates which have appeared in the literature. Let Φ be an increasing convex function on $[0, \infty)$ such that $\Phi(0) = 0$, the integral $\int_0^\infty \Phi(t)e^{-t}dt$ is finite and Φ is twice differentiable on $(0, \infty)$ with a strictly convex first derivative satisfying $\Phi'(0+) = 0$. For example, one can take $\Phi(t) = t^p$, $p > 2$, or $\Phi(t) = e^{at} - 1 - at$ for $a \in (0, 1)$.

In [2] Burkholder proved a sharp Φ-inequality

$$\sup_n \mathbb{E}\Phi(|g_n|) < \frac{1}{2}\int_0^\infty \Phi(t)e^{-t}dt$$

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under the assumption that f is a martingale (and so is g, by (1.2)), which is bounded in absolute value by 1. This inequality was later extended in [5] to the submartingale case: if f is a nonnegative submartingale bounded from above by 1 and g is 1-subordinate to f, then we have a sharp estimate

$$\sup_n \mathbb{E}\Phi\left(\frac{|g_n|}{2}\right) < \frac{2}{3}\int_0^\infty \Phi(t)e^{-t}dt.$$

Finally, Kim and Kim proved in [8], that if the 1-subordination is replaced by α-subordination, then we have

$$\sup_n \mathbb{E}\Phi\left(\frac{|g_n|}{1+\alpha}\right) < \frac{1+\alpha}{2+\alpha}\int_0^\infty \Phi(t)e^{-t}dt, \tag{1.3}$$

if f is a nonnegative submartingale bounded by 1.

There are other related results, concerning tail estimates of g. Let us state here Hammack's inequality, an estimate we will need later on. In [7] it is proved that if f is a submartingale bounded in absolute value by 1 and g is 1-subordinate to f, then, for $\lambda \geq 4$,

$$\mathbb{P}\left(\sup_n |g_n| \geq \lambda\right) \leq \frac{(8+\sqrt{2})e}{12}\exp(-\lambda/4). \tag{1.4}$$

For other similar results, see the papers by Burkholder [3] and Hammack [7].

A natural question arises: what can be said about the Φ-inequalities for other functions Φ? The purpose of this paper is to give the answer for $\Phi(t) = t$. The main result can be stated as follows.

Theorem 1.1. *Suppose f is a nonnegative submartingale such that $\sup_n f_n \leq 1$ almost surely and let g be α-subordinate to f. Then*

$$||g||_1 \leq \frac{(\alpha+1)(2\alpha^2+3\alpha+2)}{(2\alpha+1)(\alpha+2)}. \tag{1.5}$$

The constant on the right is the best possible.

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In the special case $\alpha = 1$, this leads to an interesting inequality for stochastic integrals. Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, filtered by a nondecreasing family $(\mathcal{F}_t)_{t\geq 0}$ of sub-σ-algebras of $\mathcal{F}$ and assume that $\mathcal{F}_0$ contains all the events A with $\mathbb{P}(A) = 0$. Let $X = (X_t)_{t\geq 0}$ be an adapted nonnegative right-continuous submartingale with left limits, satisfying $\mathbb{P}(X_t \leq 1) = 1$ for all t and let $H = (H_t)$ be a predictable process with values in $[-1, 1]$. Let $Y = (Y_t)$ be an Itô stochastic integral of H with respect to X, that is,

$$Y_t = H_0 X_0 + \int_{(0,t]} H_s dX_s.$$

Let $||Y||_1 = \sup_t ||Y_t||_1$.

Theorem 1.2. *For X, Y as above, we have*

$$||Y||_1 \leq \frac{14}{9} \tag{1.6}$$

and the constant is the best possible. It is already the best possible if H is assumed to take values in the set $\{-1, 1\}$.

The proofs are based on Burkholder's techniques which were developed in [2] and [3]. These enable us to reduce the proof of the submartingale inequality (1.5) to finding a special function, satisfying some convexity-type properties or, equivalently, to solving a certain boundary value problem.

The paper is organized as follows. In the next section we introduce the special function corresponding to the moment inequality and study its properties. Section 3 contains the proofs of inequalities (1.5) and (1.6). The sharpness of these estimates is postponed to the last section, Section 4.

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2. The Special Function

Let S denote the strip $[0,1]\times\mathbb{R}$. Consider the following subsets of S.

$$D_1 = \left\{(x,y)\in S : x\le \frac{\alpha}{2\alpha+1},\ x+|y|>\frac{\alpha}{2\alpha+1}\right\},$$
$$D_2 = \left\{(x,y)\in S : x\ge \frac{\alpha}{2\alpha+1},\ -x+|y|>-\frac{\alpha}{2\alpha+1}\right\},$$
$$D_3 = \left\{(x,y)\in S : x\ge \frac{\alpha}{2\alpha+1},\ -x+|y|\le-\frac{\alpha}{2\alpha+1}\right\},$$
$$D_4 = \left\{(x,y)\in S : x\le \frac{\alpha}{2\alpha+1},\ x+|y|\le\frac{\alpha}{2\alpha+1}\right\}.$$

Consider a function $H:\mathbb{R}^2\to\mathbb{R}$ defined by

$$H(x,y)=(|x|+|y|)^{1/(\alpha+1)}((\alpha+1)|x|-|y|).$$

Let $u:S\to\mathbb{R}$ be given by

$$u(x,y)=-\alpha x+|y|+\alpha+\exp\left[-\frac{2\alpha+1}{\alpha+1}\left(x+|y|-\frac{\alpha}{2\alpha+1}\right)\right]\left(x+\frac{1}{2\alpha+1}\right)$$

if $(x,y)\in D_1$,

$$u(x,y)=-\alpha x+|y|+\alpha+\exp\left[-\frac{2\alpha+1}{\alpha+1}\left(-x+|y|+\frac{\alpha}{2\alpha+1}\right)\right](1-x)$$

if $(x,y)\in D_2$,

$$u(x,y)=-(1-x)\log\left[\frac{2\alpha+1}{\alpha+1}(1-x+|y|)\right]+(\alpha+1)(1-x)+|y|$$

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if $(x,y) \in D_3$ and

$$u(x,y) = -\frac{\alpha^2}{(2\alpha+1)(\alpha+2)}\left[1+\left(\frac{2\alpha+1}{\alpha}\right)^{\frac{\alpha+2}{\alpha+1}} H(x,y)\right] + \frac{2\alpha^2}{2\alpha+1} + 1$$

if $(x,y) \in D_4$.

The key properties of the function u are described in the two lemmas below.

Lemma 2.1. *The following statements hold true.*

(i) *The function u has continuous partial derivatives in the interior of S.*

(ii) *We have*

$$u_x \leq -\alpha|u_y|. \tag{2.1}$$

(iii) *For any real numbers x, h, y, k such that x, $x+h \in [0,1]$ and $|h| \geq |k|$ we have*

$$u(x+h, y+k) \leq u(x,y) + u_x(x,y)h + u_y(x,y)k. \tag{2.2}$$

Proof. Let us first compute the partial derivatives in the interiors D_i^o of the sets D_i, $i \in \{1, 2, 3, 4\}$. We have that $u_x(x,y)$ equals

$$\begin{cases} -\alpha + \exp\left[-\frac{2\alpha+1}{\alpha+1}\left(x+|y|-\frac{\alpha}{2\alpha+1}\right)\right]\left(-\frac{2\alpha+1}{\alpha+1}x+\frac{\alpha}{\alpha+1}\right), & (x,y) \in D_1^o, \\ -\alpha + \exp\left[-\frac{2\alpha+1}{\alpha+1}\left(-x+|y|+\frac{\alpha}{2\alpha+1}\right)\right]\left(-\frac{2\alpha+1}{\alpha+1}x+\frac{\alpha}{\alpha+1}\right), & (x,y) \in D_2^o, \\ \log\left[\frac{2\alpha+1}{\alpha+1}(1-x+|y|)\right] + \frac{1-x}{1-x+|y|} - (\alpha+1), & (x,y) \in D_3^o, \\ -\alpha\left(\frac{2\alpha+1}{\alpha}\right)^{\frac{1}{\alpha+1}}(x+|y|)^{-\frac{\alpha}{\alpha+1}}\left(x+\frac{\alpha}{\alpha+1}|y|\right), & (x,y) \in D_4^o, \end{cases}$$

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while $u_y(x,y)$ is given by

$$\begin{cases} y' - \frac{2\alpha+1}{\alpha+1}\exp\left[-\frac{2\alpha+1}{\alpha+1}\left(x+|y|-\frac{\alpha}{2\alpha+1}\right)\right]\left(x+\frac{1}{2\alpha+1}\right)y', & (x,y)\in D_1^o, \\ y' - \frac{2\alpha+1}{\alpha+1}\exp\left[-\frac{2\alpha+1}{\alpha+1}\left(-x+|y|+\frac{\alpha}{2\alpha+1}\right)\right](1-x)y', & (x,y)\in D_2^o, \\ \frac{y}{1-x+|y|}, & (x,y)\in D_3^o, \\ \left(\frac{2\alpha+1}{\alpha}\right)^{\frac{1}{\alpha+1}}(x+|y|)^{-\frac{\alpha}{\alpha+1}}\frac{\alpha}{\alpha+1}y, & (x,y)\in D_4^o. \end{cases}$$

Here $y'=y/|y|$ is the sign of y. Now we turn to the properties (i) - (iii).

(i) This follows immediately by the formulas for u_x, u_y above.

(ii) We have that $u_x(x,y)+\alpha|u_y(x,y)|$ equals

$$\begin{cases} -\exp\left[-\frac{2\alpha+1}{\alpha+1}\left(x+|y|-\frac{\alpha}{2\alpha+1}\right)\right](2\alpha+1)x, & (x,y)\in D_1, \\ -\exp\left[-\frac{2\alpha+1}{\alpha+1}\left(-x+|y|+\frac{\alpha}{2\alpha+1}\right)\right]\left(\frac{2\alpha+1}{\alpha+1}x(1-\alpha)+\frac{2\alpha^2}{\alpha+1}\right), & (x,y)\in D_2, \\ -\alpha+\log\left[\frac{2\alpha+1}{\alpha+1}(1-x+|y|)\right]-\frac{|y|(1-\alpha)}{1-x+|y|}, & (x,y)\in D_3, \\ -\alpha\left(\frac{2\alpha+1}{\alpha}\right)^{1/(\alpha+1)}(x+|y|)^{-\alpha/(\alpha+1)}x, & (x,y)\in D_4 \end{cases}$$

and all the expressions are clearly nonpositive.

(iii) There is a well-known procedure to establish (2.2). Fix x, y, h and k satisfying the conditions of (iii) and consider a function $G=G_{x,y,h,k}:t\mapsto u(x+th,y+tk)$, defined on $\{t:0\le x+th\le 1\}$. The inequality (2.2) reads $G(1)\le G(0)+G'(0)$, so in order to prove it, it suffices to show that G is concave. Since u is of class C^1, it is enough to check $G''(t)\le 0$ for those t, for which $(x+th,y+tk)$ belongs to the interior of D_1, D_2, D_3 or D_4. Furthermore, by translation argument (we have $G''_{x,y,h,k}(t)=G''_{x+th,y+tk,h,k}(0)$), we may assume $t=0$.

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If $(x,y)\in D_1^o$, we have

$$G''(0)=\frac{2\alpha+1}{\alpha+1}\exp\left[-\frac{2\alpha+1}{\alpha+1}\left(x+|y|-\frac{\alpha}{2\alpha+1}\right)\right]$$
$$\times(h+k)\left\{\left[\frac{2\alpha+1}{\alpha+1}\left(x+\frac{1}{2\alpha+1}\right)-2\right]h+\frac{2\alpha+1}{\alpha+1}\left(x+\frac{1}{2\alpha+1}\right)k\right\},$$

which is nonpositive; this is due to

$$|h|\geq|k|,\quad \frac{2\alpha+1}{\alpha+1}\left(x+\frac{1}{2\alpha+1}\right)-2\leq-1\ \text{ and }\ \frac{2\alpha+1}{\alpha+1}\left(x+\frac{1}{2\alpha+1}\right)\leq 1.$$

If $(x,y)\in D_2^o$, then

$$G''(0)=\frac{2\alpha+1}{\alpha+1}\exp\left[-\frac{2\alpha+1}{\alpha+1}\left(-x+|y|+\frac{\alpha}{2\alpha+1}\right)\right]$$
$$\times(h-k)\left\{\left[\frac{2\alpha+1}{\alpha+1}(1-x)-2\right]h-\frac{2\alpha+1}{\alpha+1}(1-x)k\right\}\leq 0,$$

since

$$|h|\geq|k|,\quad \frac{2\alpha+1}{\alpha+1}(1-x)-2\leq-1\quad\text{and}\quad\frac{2\alpha+1}{\alpha+1}(1-x)\leq 1.$$

For $(x,y)\in D_3^o$ we have

$$G''(0)=\frac{-h+k}{1-x+|y|}\left[\left(2-\frac{1-x}{1-x+|y|}\right)h+\frac{1-x}{1-x+|y|}k\right]\leq 0,$$

because

$$|h|\geq|k|,\quad 2-\frac{1-x}{1-x+|y|}\geq 1\quad\text{and}\quad\frac{1-x}{1-x+|y|}\leq 1.$$

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Finally, for $(x,y) \in D_4^o$, this follows by the result of Burkholder: the function $t \mapsto -H(x+th, y+tk)$ is concave, see page 17 of [3]. $\square$

Lemma 2.2. *Let* $(x,y) \in S$.

(i) *We have*

$$u(x,y) \geq |y|. \tag{2.3}$$

(ii) *If* $|y| \leq x$, *then*

$$u(x,y) \leq u(0,0) = \frac{(\alpha+1)(2\alpha^2+3\alpha+2)}{(2\alpha+1)(\alpha+2)}. \tag{2.4}$$

Proof. (i) Since for any $(x,y) \in S$ the function $G(t) = u(x+t, y+t)$ defined on $\{t : x+t \in [0,1]\}$ is concave, it suffices to prove (2.3) on the boundary of the strip S. Furthermore, by symmetry, we may restrict ourselves to $(x,y) \in \partial S$ satisfying $y \geq 0$. We have, for $y \in [0, \alpha/(2\alpha+1)]$,

$$u(0,y) \geq -\frac{\alpha^2}{(2\alpha+1)(\alpha+2)} + \frac{2\alpha^2}{2\alpha+1} + 1 \geq 1 \geq y,$$

while for $y > \alpha/(2\alpha+1)$, the inequality $u(0,y) \geq y$ is trivial. Finally, note that we have $u(1,y) = y$ for $y \geq 0$. Thus (2.3) follows.

(ii) As one easily checks, we have $u_y(x,y) \geq 0$ for $y \geq 0$ and hence, by symmetry, it suffices to prove (2.4) for $x = y$. The function $G(t) = u(t,t)$, $t \in [0,1]$, is concave and satisfies $G'(0+) = 0$. Thus $G \leq G(0)$ and the proof is complete. $\square$

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3. Proofs of the Inequalities (1.5) and (1.6)

Proof of inequality (1.5). Let f, g be as in the statement and fix a nonnegative integer n. Furthermore, fix $\beta \in (0,1)$ and set $f' = \beta f$, $g' = \beta g$. Clearly, g' is α-subordinate to f', so the inequality (2.2) implies that, with probability 1,

$$u(f'_{n+1}, g'_{n+1}) \leq u(f'_n, g'_n) + u_x(f'_n, g'_n) df'_{n+1} + u_y(f'_n, g'_n) dg'_{n+1}. \tag{3.1}$$

Both sides are integrable: indeed, since f is bounded by 1, so is f'; furthermore, we have $\mathbb{P}(|df_k| \leq 1) = 1$ and hence $\mathbb{P}(|dg_k| \leq 1) = 1$ by (1.1). This gives $|g'_n| = \beta|g_n| \leq \beta n$ almost surely and now it suffices to note that u is locally bounded on $[0, \beta] \times \mathbb{R}$ and the partial derivatives u_x, u_y are bounded on this set.

Therefore, taking the conditional expectation of (3.1) with respect to $\mathcal{F}_n$ yields

$$\begin{aligned}
&\mathbb{E}(u(f'_{n+1}, g'_{n+1})|\mathcal{F}_n) \\
&\leq u(f'_n, g'_n) + u_x(f'_n, g'_n)\mathbb{E}(df'_{n+1}|\mathcal{F}_n) + u_y(f'_n, g'_n)\mathbb{E}(dg'_{n+1}|\mathcal{F}_n) \\
&\leq u(f'_n, g'_n) + u_x(f'_n, g'_n)\mathbb{E}(df'_{n+1}|\mathcal{F}_n) + |u_y(f'_n, g'_n)| \cdot |\mathbb{E}(dg'_{n+1}|\mathcal{F}_n)|.
\end{aligned}$$

By α-subordination, this can be further bounded from above by

$$u(f'_n, g'_n) + (u_x(f'_n, g'_n) + \alpha|u_y(f'_n, g'_n)|)\mathbb{E}(df'_{n+1}|\mathcal{F}_n) \leq u(f'_n, g'_n),$$

the latter inequality being a consequence of (2.1). Thus, taking the expectation, we obtain

$$\mathbb{E}u(f'_{n+1}, g'_{n+1}) \leq \mathbb{E}u(f'_n, g'_n). \tag{3.2}$$

Combining this with (2.3), we get

$$\mathbb{E}|g'_n| \leq \mathbb{E}u(f'_n, g'_n) \leq \mathbb{E}u(f'_0, g'_0).$$

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But $|g_0'| \leq f_0'$ by (1.1); hence (2.4) implies

$$\beta \mathbb{E}|g_n| = \mathbb{E}|g_n'| \leq \frac{(\alpha+1)(2\alpha^2+3\alpha+2)}{(2\alpha+1)(\alpha+2)}.$$

Since n and $\beta \in (0,1)$ were arbitrary, the proof is complete. $\square$

Proof of the inequality (1.6). This follows by an approximation argument. See Section 16 of [2], where it is shown how similar inequalities for stochastic integrals are implied by their discrete-time analogues combined with the result of Bichteler [1]. $\square$

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4. Sharpness

We start with the inequality (1.5). For $\alpha = 0$ simply take constant processes $f = g = (1, 1, 1, \ldots)$ and note that both sides are equal in (1.5). Suppose then, that α is a positive number. We will construct an appropriate example; this will be done in a few steps. Denote $\gamma = \alpha/(2\alpha + 1)$ and fix $\varepsilon > 0$.

Step 1. Using the ideas of Choi [6] (which go back to Burkholder's examples from [4]), one can show that there exists a pair (F, G) of processes starting from $(0, 0)$ such that F is a nonnegative submartingale, G is α-subordinate to F and, for some N, (F_{3N}, G_{3N}), takes values in the set $\{(\gamma, 0), (0, \pm\gamma)\}$ with

$$\left|\mathbb{P}((F_{3N}, G_{3N}) = (\gamma, 0)) - \frac{1}{\alpha + 2}\right| \leq \varepsilon, \quad \left|\mathbb{P}((F_{3N}, G_{3N}) = (0, \gamma)) - \frac{\alpha + 1}{2(\alpha + 2)}\right| \leq \varepsilon$$

and $\mathbb{P}((F_{3N}, G_{3N}) = (0, \gamma)) = \mathbb{P}((F_{3N}, G_{3N}) = (0, -\gamma))$. Furthermore, if $\alpha = 1$, then G can be taken to be a ± 1 transform of F, that is, $dF_n = \pm dG_n$ for any nonnegative integer n.

Step 2. Consider the following two-dimensional Markov process (f, g), with a certain initial distribution concentrated on the set $\{(\gamma, 0), (0, \gamma), (0, -\gamma)\}$. To describe the transity function, let M be a (large) nonnegative integer and $\delta \in (0, \gamma/3)$; both numbers will be specified later. Assume for $k = 0, 1, 2, \ldots, M - 1$ and any $\hat{\varepsilon} \in \{-1, 1\}$ that the conditions below are satisfied.

- The state $(0, \hat{\varepsilon}(\gamma + k(\alpha + 1)\delta))$ leads to $(\delta, \hat{\varepsilon}(\gamma + k(\alpha + 1)\delta + \alpha\delta))$ with probability 1.
- The state $(\delta, \hat{\varepsilon}(\gamma + k(\alpha + 1)\delta + \alpha\delta))$ leads to $(0, \hat{\varepsilon}(\gamma + (k + 1)(\alpha + 1)\delta))$ with probability $1 - \delta/\gamma$ and to $(\gamma, \hat{\varepsilon}(k + 1)(\alpha + 1)\delta)$ with probability δ/γ.

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- The state $(\gamma, \hat{\varepsilon}(k+1)(\alpha+1)\delta)$ leads to $(1, \hat{\varepsilon}((k+1)(\alpha+1)\delta + 1 - \gamma))$ with probability
$$\frac{(\alpha+1)\delta}{2 - 2\gamma + (\alpha+1)\delta}$$
and to $(\gamma - (\alpha+1)\delta/2, \hat{\varepsilon}(k+1/2)(\alpha+1)\delta)$ with probability
$$1 - \frac{(\alpha+1)\delta}{2 - 2\gamma + (\alpha+1)\delta}.$$
- The state $(\gamma - (\alpha+1)\delta/2, \hat{\varepsilon}(k+1/2)(\alpha+1)\delta)$ leads to $(0, \hat{\varepsilon}(\gamma + k(\alpha+1)\delta))$ with probability $(\alpha+1)\delta/(2\gamma)$ and to $(\gamma, \hat{\varepsilon}k(\alpha+1)\delta)$ with probability $1-(\alpha+1)\delta/(2\gamma)$.
- The state $(\gamma, 0)$ leads to $(1, 1-\gamma)$ with probability γ and to $(0, -\gamma)$ with probability $1-\gamma$.
- The state $(0, \hat{\varepsilon}(\gamma + M(\alpha+1)\delta))$ is absorbing.
- The states lying on the line $x = 1$ are absorbing.

It is easy to check that f is a nonnegative submartingale bounded by 1 and g satisfies
$$|dg_n| \leq |df_n| \;\text{ and }\; |\mathbb{E}(dg_n|\mathcal{F}_{n-1})| \leq \alpha\mathbb{E}(df_n|\mathcal{F}_{n-1}), \;\; n = 1,\ 2,\ \ldots$$
almost surely. Furthermore, if $\alpha = 1$, then g is a ± 1 transform of f: $df_n = \pm dg_n$ for $n \geq 1$ (note that this fails for $n = 0$).

Step 3. Let $(\mathcal{G}_n)$ be the natural filtration generated by the process (f, g) and set $K = \gamma + M(1+\alpha)\delta$. Introduce the stopping time $\tau = \inf\{k : f_k = 1 \;\text{ or }\; g_k = \pm K\}$. The purpose of this step is to establish a bound for the first moment of τ.

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Let n be a nonnegative integer and set $\kappa = 4^{-3\delta M/(2\gamma)}$. We will prove that

$$\mathbb{P}(\tau \leq n + 2M + 1|\mathcal{G}_n) \geq \kappa\gamma. \tag{4.1}$$

We will need the following estimate

$$\left(1 - \frac{3\delta}{2\gamma}\right)^M \geq \kappa, \tag{4.2}$$

which immediately follows from the facts that the function $h : (0, 1/2] \to \mathbb{R}_+$ given by $h(x) = (1-x)^{1/x}$ is decreasing and $\delta < \gamma/3$.

Let $A \neq \emptyset$ be an atom of $\mathcal{G}_n$. We will consider three cases.

1°. If we have $f_n = 0$ or $f_n = \delta$ on A, consider the event

$$A' = A \cap \{|g_{n+k+1}| \geq |g_{n+k}|,\ \ k = 0,\ 1,\ \ldots, 2M-1\}.$$

Clearly, in view of the transity function described above, we have $A' \subseteq \{|g_{n+2M}| = K\} \subseteq \{\tau \leq n + 2M\}$ and

$$\begin{aligned}\mathbb{P}\left(\tau \leq n + 2M + 1|\mathcal{G}_n\right) &\geq \mathbb{P}(\tau \leq n + 2M|\mathcal{G}_n)\\ &\geq \frac{\mathbb{P}(A')}{\mathbb{P}(A)} \geq (1 - \delta/\gamma)^M > \kappa > \kappa\gamma \quad \text{on } A,\end{aligned}$$

in view of (4.2).

2°. If we have $f_n = \gamma$ or $f_n = \gamma - (\alpha+1)\delta/2$ on A, consider the event

$$A' = A \cap \{|g_{n+k+1}| < |g_{n+k}| \quad \text{or} \quad (f_{n+k+1}, g_{n+k+1}) = (1, 1-\gamma), \quad k = 0,\ 1,\ \ldots\}.$$

In other words, A' contains those paths of $(f_{n+k}, g_{n+k})_{k\geq 0}$, for which $|g|$ decreases to 0 and then, in the next step, (f, g) moves to $(1, 1-\gamma)$. It follows from the definition

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of the transity function, that, on A, it is impossible for $|g|$ to be decreasing $2M+1$ times in a row; that is to say, we have $f_{n+2M+1}=1$ on A' and hence

$$\begin{aligned}\mathbb{P}(\tau \le n+2M+1|\mathcal{G}_n) &\ge \frac{\mathbb{P}(A')}{\mathbb{P}(A)}\\ &\ge \left[\left(1-\frac{(\alpha+1)\delta}{2\gamma}\right)\left(1-\frac{(\alpha+1)\delta}{2-2\gamma+(\alpha+1)\delta}\right)\right]^M\gamma\\ &=\left(1-\frac{(2\alpha+1)\delta}{(2+(2\alpha+1)\delta)\gamma}\right)^M\gamma \ge \left(1-\frac{3\delta}{2\gamma}\right)^M\gamma\ge\kappa\gamma,\end{aligned}$$

by (4.2).

3°. Finally, if $f_n=1$ on A, we have

$$\mathbb{P}(\tau \le n+2M+1|\mathcal{G}_n)=1\ge\kappa\gamma.$$

Therefore the inequality (4.1) is established. It implies that

$$\mathbb{P}(\tau > n+2M+1)\le(1-\kappa\gamma)\mathbb{P}(\tau>n),$$

which leads to

$$\mathbb{E}\tau \le \frac{2M+1}{\kappa\gamma}<\frac{2K}{\kappa\gamma\delta}=\frac{2K}{\gamma\delta}\cdot 4^{3(K-\gamma)/2\gamma(1+\alpha)}. \tag{4.3}$$

This implies that $\tau<\infty$ with probability 1 and the pointwise limits f_∞, g_∞ exist almost surely.

Step 4. Let us establish an exponential bound for $\mathbb{P}(f_\infty=0)$. We have $\{f_\infty=0\}\subseteq\{g_\infty\ge K\}$ and g is clearly 1-subordinate to f (as it is α-subordinate to f). Therefore, we may use Hammack's result (1.4): we have

$$\mathbb{P}(f_\infty=0)\le\frac{(8+\sqrt{2})e}{12}\exp(-K/4), \tag{4.4}$$

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provided $K \geq 4$.

Step 5. Consider a process $(u(f_n, g_n))_n$ and observe the following.

- For $y \geq \gamma$, the function $G(t) = u(t, y - t)$, $t \in [0, 1]$, is continuously differentiable and linear on $[0, \gamma]$.
- For $y \geq -\gamma$, the function $G(t) = u(t, y + t)$, $t \in [0, 1]$, is continuously differentiable and linear on $[\gamma, 1]$.
- For $y \geq \gamma$, the function $G(t) = u(t, y + \alpha t)$, $t \in [0, 1]$, satisfies $G'(0+) = 0$.
- The function u is locally bounded on $\overline{D_1 \cup D_2}$ and its partial derivatives are bounded on this set.

These four facts, together with the symmetry of u, imply that there exists a constant $\eta(\delta, K)$ such that $\eta(\delta, K)/\delta \to 0$ as $\delta \to 0$ and, for any n,

$$u(f_{n+1}, g_{n+1}) \geq u(f_n, g_n) + u_x(f_n, g_n)df_{n+1} + u_y(f_n, g_n)dg_{n+1} - \eta(\delta, K)\chi_{\{\tau > n\}}.$$

Both sides of this inequality are integrable: indeed, it suffices to use the fourth property above and the fact that (f_n, g_n) is bounded and belongs to $\overline{D_1 \cup D_2}$. Therefore, we may take the expectation to obtain

$$\mathbb{E}u(f_{n+1}, g_{n+1}) \geq \mathbb{E}u(f_n, g_n) - \eta(\delta, K)\mathbb{P}(\tau > n).$$

This implies

$$\mathbb{E}u(f_\infty, g_\infty) \geq \mathbb{E}u(f_0, g_0) - \eta(\delta, K)\mathbb{E}\tau,$$

or

$$\begin{aligned}\mathbb{E}|g_\infty| + \left\{\alpha + \exp\left[-\frac{2\alpha+1}{\alpha+1}\left(K - \frac{\alpha}{2\alpha+1}\right)\right] \cdot \frac{1}{2\alpha+1}\right\}\mathbb{P}(f_\infty = 0) \\ \geq \mathbb{E}u(f_0, g_0) - \eta(\delta, K)\mathbb{E}\tau.\end{aligned}$$

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By (4.4), we may fix $K \geq 4$ such that

$$\left\{\alpha + \exp\left[-\frac{2\alpha+1}{\alpha+1}\left(K + \frac{\alpha}{2\alpha+1}\right)\right] \cdot \frac{1}{2\alpha+1}\right\} \mathbb{P}(f_\infty = 0) \leq \varepsilon.$$

Now we specify the numbers δ and M, as promised at the beginning of Step 2. By (4.3), we may choose $\delta > 0$ such that $\eta(\delta, K)\mathbb{E}\tau \leq \varepsilon$ and, clearly, we may also ensure that $M = (K-\gamma)/(1+\alpha)\delta$ is an integer. Thus we obtain

$$\mathbb{E}|g_\infty| \geq \mathbb{E}u(f_0, g_0) - 2\varepsilon. \tag{4.5}$$

Step 6. Now we put all the things together. Let $(f, g) = ((f_n, g_n))_{n\geq 0}$ be a process which coincides with (F, G) from Step 1 for $n \leq 3N$ and which, for $n > 3N$, conditionally on $\mathcal{F}_{3N}$, moves according to the transitics described in Step 2. We have, by (4.5),

$$\mathbb{E}|g_\infty| \geq \mathbb{E}u(F_{3N}, G_{3N}) - 2\varepsilon.$$

However, since u is nonnegative (due to (2.3)),

$$\begin{aligned}\mathbb{E}u(F_{3N}, G_{3N}) &\geq u(\gamma, 0)\left(\frac{1}{\alpha+2} - \varepsilon\right) + u(0, \gamma)\left(\frac{\alpha+1}{\alpha+2} - \varepsilon\right) \\ &= \frac{(\alpha+1)(2\alpha^2+3\alpha+2)}{(2\alpha+1)(\alpha+2)} - (u(\gamma, 0) + u(0, \gamma))\varepsilon.\end{aligned}$$

Since ε was arbitrary, this implies that the constant in (1.5) is the best possible. This also establishes the sharpness of the estimate (1.6), even in the special case $H \in \{-1, 1\}$: if $\alpha = 1$, then the processes f, g constructed above satisfy $|df_k| = |dg_k|$ for all k. The proofs of Theorems 1.1 and 1.2 are complete.

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