Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 5, Issue 3, Article 64, 2004

INEQUALITIES OF BONFERRONI-GALAMBOS TYPE WITH APPLICATIONS TO THE TUTTE POLYNOMIAL AND THE CHROMATIC POLYNOMIAL

KLAUS DOHMEN AND PETER TITTMANN

DEPARTMENT OF MATHEMATICS
MITTWEIDA UNIVERSITY OF APPLIED SCIENCES
D-09648 MITTWEIDA, GERMANY
mathe@htwm.de
URL: http://www.mathe.htwm.de

Received 16 March, 2004; accepted 10 June, 2004
Communicated by S.S. Dragomir

ABSTRACT. In this paper, we generalize the classical Bonferroni inequalities and their improvements by Galambos to sums of type $\sum_{I\subseteq U}(-1)^{|I|}f(I)$ where U is a finite set and $f : 2^U \to \mathbb{R}$. The result is applied to the Tutte polynomial of a matroid and the chromatic polynomial of a graph.

Key words and phrases: Bonferroni inequalities, Inclusion-exclusion, Tutte polynomial, Chromatic polynomial, Graph, Matroid.

2000 *Mathematics Subject Classification.* Primary: 05A20, secondary: 05B35, 05C15, 60C05, 60E15.

1. INTRODUCTION

The classical inclusion-exclusion principle and its associated Bonferroni inequalities play an important role in combinatorial mathematics, probability theory, reliability theory, and statistics (see [4] for a detailed survey and [1] for some recent developments).

For any finite family of events $\{E_u\}_{u\in U}$ in some probability space $(\Omega, \mathcal{E}, P)$ the inclusion-exclusion principle (1.1) expresses the probability that none of the events E_u, $u \in U$, occurs as an alternating sum of $2^{|U|}$ terms each involving intersections of up to $|U|$ many events, while the classical Bonferroni inequalities (1.2) provide bounds on this sum for each choice of $r \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$:

$$P\left(\bigcap_{u\in U}\overline{E_u}\right) = \sum_{I\subseteq U}(-1)^{|I|}P\left(\bigcap_{i\in I}E_i\right), \tag{1.1}$$

$$(-1)^r P\left(\bigcap_{u\in U}\overline{E_u}\right) \leq (-1)^r \sum_{\substack{I\subseteq U\\|I|\leq r}}(-1)^{|I|}P\left(\bigcap_{i\in I}E_i\right). \tag{1.2}$$

ISSN (electronic): 1443-5756

058-04

The following bounds due to Galambos [3] improve the classical Bonferroni bounds by including additional terms based on the $(r+1)$-subsets of U:

$$(-1)^r P\left(\bigcap_{u\in U}\overline{E_u}\right) \leq (-1)^r \sum_{\substack{I\subseteq U\\|I|\leq r}}(-1)^{|I|}P\left(\bigcap_{i\in I}E_i\right) - \frac{r+1}{|U|}\sum_{\substack{I\subseteq U\\|I|=r+1}}P\left(\bigcap_{i\in I}E_i\right).$$

In view of (1.1), the preceding improvement over (1.2) can also be written as

$$(-1)^r \sum_{I\subseteq U}(-1)^{|I|}f(I) \leq (-1)^r \sum_{\substack{I\subseteq U\\|I|\leq r}}(-1)^{|I|}f(I) - \frac{r+1}{|U|}\sum_{\substack{I\subseteq U\\|I|=r+1}}f(I) \tag{1.3}$$

where $f(I) = P\left(\bigcap_{i\in I}E_i\right)$ for any $I\subseteq U$. This raises the question which other functions $f: 2^U\to\mathbb{R}^+$, where $\mathbb{R}^+ = \{x\in\mathbb{R}\,|\,x\geq 0\}$, satisfy the preceding inequality (1.3) for any possible choice of $r\in\mathbb{N}_0$, or its weaker form

$$(-1)^r \sum_{I\subseteq U}(-1)^{|I|}f(I) \leq (-1)^r \sum_{\substack{I\subseteq U\\|I|\leq r}}(-1)^{|I|}f(I). \tag{1.4}$$

Our main result provides a condition that ensures (1.3) (and thus (1.4)) to hold for any $r\in\mathbb{N}_0$, and which is easy to check. After establishing our main result in Section 2 and proving it in Section 3, we give another characterization of the class of relevant functions in Section 4. In Section 5 our main result is used to obtain bounds on the Tutte polynomial of a matroid which, as finally shown in Section 6, has applications to the chromatic polynomial of a graph.

2. MAIN RESULT

Our main result, which is proved in Section 3, is as follows.

Theorem 2.1. *Let U be a finite non-empty set and $f: 2^U\to\mathbb{R}$ be a function such that for any disjoint subsets $J,K\subseteq U$,*

$$\sum_{I\subseteq K}(-1)^{|I|}f(I\cup J) \geq^* 0. \tag{2.1}$$

Then, for any $r\in\mathbb{N}_0$,

$$(-1)^r \sum_{I\subseteq U}(-1)^{|I|}f(I) \leq^* (-1)^r \sum_{\substack{I\subseteq U\\|I|\leq r}}(-1)^{|I|}f(I) - \frac{r+1}{|U|}\sum_{\substack{I\subseteq U\\|I|=r+1}}f(I). \tag{2.2}$$

Moreover, the theorem can be dualized by interchanging $\geq$ and $\leq$ at the starred $(^*)$ places.

Remark 2.2. It is easy to see that for non-disjoint subsets $J,K\subseteq U$ the left-hand side of (2.1) equals zero. Thus, the disjointness of J and K is not significant.

Remark 2.3. By putting $K=\emptyset$ we find that any function satisfying the requirements of Theorem 2.1 is non-negative. Similarly, any function satisfying the requirements of the dual version of Theorem 2.1 is non-positive. Thus, from (2.2) we may deduce the weaker inequality (1.4), respectively its dual.

Remark 2.4. By putting $K=\{u\}$ for some $u\in U$ we observe that any function satisfying the requirements of Theorem 2.1 is antitone. Likewise, any function satisfying the requirements of the dual version is monotone.

In verifying the requirements of Theorem 2.1 the following proposition is quite helpful. The example following the proposition demonstrates this.

Proposition 2.5. *Let U be a finite non-empty set, and let $f, g : 2^U \to \mathbb{R}^+$ be mappings such that for any subset $I \subseteq U$,*

$$f(I) = \sum_{J \supseteq I} g(J).$$

Then, f satisfies the requirements of Theorem 2.1.

Proof. For any disjoint sets $J, K \subseteq U$ we find that

$$\begin{aligned}\sum_{I \subseteq K} (-1)^{|I|} f(I \cup J) &= \sum_{I \subseteq K} (-1)^{|I|} \sum_{L \supseteq I \cup J} g(L) \\ &= \sum_{L \supseteq J} \sum_{I \subseteq K \cap L} (-1)^{|I|} g(L) \\ &= \sum_{L \supseteq J} g(L) \sum_{I \subseteq K \cap L} (-1)^{|I|} \\ &= \sum_{L \supseteq J} g(L) \delta(K \cap L, \emptyset) \geq 0,\end{aligned}$$

where $\delta(\cdot, \cdot)$ is the usual Kronecker delta. □

Example 2.1. For any non-empty and finite collection of events $\{E_u\}_{u \in U}$ in some probability space $(\Omega, \mathcal{E}, P)$, let $f, g : 2^U \to \mathbb{R}^+$ be defined by

$$f(I) := P\left(\bigcap_{i \in I} E_i\right), \quad g(I) := P\left(\bigcap_{i \notin I} \overline{E_i} \cap \bigcap_{i \in I} E_i\right).$$

Then, f and g satisfy the requirements of Proposition 2.5. For the present choice of f and g, the inequalities in (2.2) agree with those of Galambos.

3. PROOF OF THE MAIN RESULT

For the proof of Theorem 2.1 some preliminary notations and results are needed. For any function $f : 2^U \to \mathbb{R}$ and any $u \in U$ define

$$\begin{aligned} f_u &: 2^{U \setminus \{u\}} \to \mathbb{R}, \quad f_u(I) := f(I), \\ f^u &: 2^{U \setminus \{u\}} \to \mathbb{R}, \quad f^u(I) := f(I \cup \{u\}). \end{aligned}$$

Lemma 3.1. *Let U be a finite set and $f : 2^U \to \mathbb{R}$ be a function. Then, for any $u \in U$ and any $J, K \subseteq U \setminus \{u\}$,*

$$\sum_{I \subseteq K} (-1)^{|I|} (f_u - f^u)(I \cup J) = \sum_{I \subseteq K \cup \{u\}} (-1)^{|I|} f(I \cup J). \tag{3.1}$$

Proof. Evidently, the left-hand side of (3.1) is equal to

$$\begin{aligned}&\sum_{I \subseteq K} (-1)^{|I|} f(I \cup J) - \sum_{I \subseteq K} (-1)^{|I|} f(I \cup J \cup \{u\}) \\ &\qquad = \sum_{\substack{I \subseteq K \cup \{u\} \\ u \notin I}} (-1)^{|I|} f(I \cup J) + \sum_{\substack{I \subseteq K \cup \{u\} \\ u \in I}} (-1)^{|I|} f(I \cup J)\end{aligned}$$

which immediately gives the right hand side of (3.1). □

Lemma 3.2. *If $f : 2^U \to \mathbb{R}$ is a function satisfying (2.1) for any disjoint $J, K \subseteq U$, then the same applies to f_u, f^u, and $f_u - f^u$ for any $u \in U$.*

Proof. For f_u and f^u the statement is immediately clear, while for $f_u - f^u$ it is implied by Lemma 3.1. □

Although (1.4) is an immediate consequence of Theorem 2.1, our forthcoming proof of Theorem 2.1 requires (1.4) to be shown first.

Lemma 3.3. *Under the requirements of Theorem 2.1, (1.4) holds for any* $r \in \mathbb{N}_0$.

Proof. The proof is by induction on $|U|$. Evidently, the statement holds if $|U| = 1$. In what follows, we may assume that $|U| > 1$ and that the statement holds for all proper non-empty subsets of U. Let $u \in U$ be chosen arbitrarily. By applying Lemma 3.1 with $K = U \setminus \{u\}$ and $J = \emptyset$ we obtain

$$(-1)^r \sum_{I \subseteq U} (-1)^{|I|} f(I) = (-1)^r \sum_{I \subseteq U \setminus \{u\}} (-1)^{|I|} f_u(I) + (-1)^{r-1} \sum_{I \subseteq U \setminus \{u\}} (-1)^{|I|} f^u(I).$$

By Lemma 3.2 both f_u and f^u satisfy the requirements of Theorem 2.1. Thus, by the induction hypothesis, these two functions both satisfy (1.4) and hence,

$$\begin{aligned}(-1)^r \sum_{I \subseteq U \setminus \{u\}} (-1)^{|I|} f_u(I) &\leq (-1)^r \sum_{\substack{I \subseteq U \setminus \{u\} \\ |I| \leq r}} (-1)^{|I|} f_u(I), \\ (-1)^{r-1} \sum_{I \subseteq U \setminus \{u\}} (-1)^{|I|} f^u(I) &\leq (-1)^{r-1} \sum_{\substack{I \subseteq U \setminus \{u\} \\ |I| \leq r-1}} (-1)^{|I|} f^u(I),\end{aligned}$$

where, of course, the conclusion for f^u requires that $r \geq 1$. However, due to requirement (2.1) (with f^u in place of f, $K = U \setminus \{u\}$, $J = \emptyset$) the preceding inequality for f^u also holds for $r = 0$, and so for all $r \in \mathbb{N}_0$ we find that

$$\begin{aligned}(-1)^r \sum_{I \subseteq U} (-1)^{|I|} f(I) &\leq (-1)^r \sum_{\substack{I \subseteq U \setminus \{u\} \\ |I| \leq r}} (-1)^{|I|} f_u(I) + (-1)^{r-1} \sum_{\substack{I \subseteq U \setminus \{u\} \\ |I| \leq r-1}} (-1)^{|I|} f^u(I) \\ &= (-1)^r \sum_{\substack{I \subseteq U,\, u \notin I \\ |I| \leq r}} (-1)^{|I|} f(I) + (-1)^r \sum_{\substack{I \subseteq U,\, u \in I \\ |I| \leq r}} (-1)^{|I|} f(I),\end{aligned}$$

which immediately gives the right-hand side of (1.4). □

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. Let $u \in U$ be chosen uniformly at random. By applying Lemma 3.1 with $K = U \setminus \{u\}$ and $J = \emptyset$ we obtain

$$(-1)^r \sum_{I \subseteq U} (-1)^{|I|} f(I) = (-1)^r \sum_{I \subseteq U \setminus \{u\}} (-1)^{|I|} \left(f_u - f^u\right)(I). \tag{3.2}$$

By Lemma 3.2 $f_u - f^u$ satisfies the requirements of Theorem 2.1. Hence, we may apply Lemma 3.3 to $f_u - f^u$, which gives

$$(-1)^r \sum_{I \subseteq U \setminus \{u\}} (-1)^{|I|} \left(f_u - f^u\right)(I) \leq (-1)^r \sum_{\substack{I \subseteq U \setminus \{u\} \\ |I| \leq r}} (-1)^{|I|} \left(f_u - f^u\right)(I). \tag{3.3}$$

By combining (3.2) and (3.3) we obtain

$$\begin{aligned}(-1)^r \sum_{I\subseteq U}(-1)^{|I|}f(I) &\le (-1)^r \sum_{\substack{I\subseteq U\setminus\{u\}\\|I|\le r}} (-1)^{|I|}\left(f_u - f^u\right)(I)\\ &= (-1)^r \sum_{\substack{I\subseteq U,\, u\notin I\\|I|\le r}} (-1)^{|I|}f(I) + (-1)^r \sum_{\substack{I\subseteq U, u\in I\\|I|\le r+1}} (-1)^{|I|}f(I)\\ &= (-1)^r \sum_{\substack{I\subseteq U\\|I|\le r}}(-1)^{|I|}f(I) - \sum_{\substack{I\subseteq U, u\in I\\|I|=r+1}} f(I)\\ &= (-1)^r \sum_{\substack{I\subseteq U\\|I|\le r}}(-1)^{|I|}f(I) - \sum_{\substack{I\subseteq U\\|I|=r+1}} f(I)1_I(u),\end{aligned}$$

where 1_I denotes the indicator function of I. We thus have

$$(-1)^r \sum_{I\subseteq U}(-1)^{|I|}f(I) \le (-1)^r \sum_{\substack{I\subseteq U\\|I|\le r}}(-1)^{|I|}f(I) - \sum_{\substack{I\subseteq U\\|I|=r+1}} f(I)\, 1_I\,. \tag{3.4}$$

Now, (2.2) follows by taking the expectation on both sides of (3.4). The dual version of the theorem is finally obtained by moving from f to $-f$. □

4. CHARACTERIZATION

The following theorem characterizes the class of functions satisfying the requirements of Theorem 2.1.

Theorem 4.1. *The class of functions satisfying the requirements of Theorem 2.1 is the smallest class of functions $\mathcal{F}$ such that*

(1) *all functions $f : 2^U \to \mathbb{R}^+$ where $|U| = 1$ belong to $\mathcal{F}$,*
(2) *if $f^u \in \mathcal{F}$ and $f_u - f^u \in \mathcal{F}$ for some function $f : 2^U \to \mathbb{R}^+$, where U is finite and non-empty, and $u \in U$, then $f \in \mathcal{F}$.*

Proof. Let $\mathcal{D}$ be the class of functions satisfying the requirements of Theorem 2.1. Then, $\mathcal{D}$ contains all functions $f : 2^U \to \mathbb{R}^+$ where $|U| = 1$ and as shown subsequently, it contains all functions $f : 2^U \to \mathbb{R}^+$ where U is finite and non-empty and both f^u and $f_u - f^u$ are in $\mathcal{D}$ for some $u \in U$. Let f be such a function. Since $\mathcal{D}$ is closed under taking sums of functions on the same domain, $f_u = f^u + (f_u - f^u) \in \mathcal{D}$. Now, in order to show that $f \in \mathcal{D}$, we show that (2.1) holds for all disjoint $J, K \subseteq U$. We consider three cases:
Case 1. If $u \notin K$ and $u \notin J$, then $J, K \subseteq U \setminus \{u\}$ and hence, since $f_u \in \mathcal{D}$,

$$\sum_{I\subseteq K}(-1)^{|I|}f(I\cup J) = \sum_{I\subseteq K}(-1)^{|I|}f_u(I\cup J) \ge 0.$$

Case 2. If $u \notin K$ and $u \in J$, then $K \subseteq U \setminus \{u\}$ and $J \setminus \{u\} \subseteq U \setminus \{u\}$ and hence, since $f^u \in \mathcal{D}$, we find that

$$\sum_{I\subseteq K}(-1)^{|I|}f(I\cup J) = \sum_{I\subseteq K}(-1)^{|I|}f^u\left(I\cup (J\setminus\{u\})\right) \ge 0.$$

Case 3. If $u \in K$ and $u \notin J$, then $J \subseteq U \setminus \{u\}$ and $K \setminus \{u\} \subseteq U \setminus \{u\}$. Hence, by Lemma 3.1 and the assumption that $f_u - f^u \in \mathcal{D}$, we have

$$\sum_{I\subseteq K}(-1)^{|I|}f(I\cup J) = \sum_{I\subseteq K\setminus\{u\}}(-1)^{|I|}(f_u - f^u)(I\cup J) \ge 0.$$

In all three cases it turns out that $f \in \mathcal{D}$. To establish the minimality of $\mathcal{D}$, we show that $\mathcal{D} \subseteq \mathcal{F}$ for any class $\mathcal{F}$ satisfying conditions 1 and 2 above. Let $\mathcal{F}$ be such a class. By induction on $|U|$ we show that any $f : 2^U \to \mathbb{R}^+$ which is in $\mathcal{D}$ must be in $\mathcal{F}$. If $|U| = 1$, then $f \in \mathcal{F}$ by condition 1. Let $|U| > 1$, and $u \in U$. By Lemma 3.2, $f^u, f_u - f^u \in \mathcal{D}$. By the induction hypothesis, $f^u, f_u - f^u \in \mathcal{F}$ and hence, by condition 2, $f \in \mathcal{F}$. Hence, $\mathcal{D} \subseteq \mathcal{F}$. $\square$

5. THE TUTTE POLYNOMIAL

In this section, our main result in Section 2 is applied to the Tutte polynomial of a general matroid. In the following, we briefly review the necessary concepts. For a detailed exposition, the reader is referred to Welsh [5].

Definition 5.1. A *matroid* is a pair $M = (U, \varrho)$ consisting of a finite set U and a function $\varrho : 2^U \to \mathbb{N}_0$ (*rank function*) such that for any $X, Y \subseteq U$,

$$\begin{aligned} &\text{(i)} \quad \varrho(X) \leq |X|, \\ &\text{(ii)} \quad X \subseteq Y \Rightarrow \varrho(X) \leq \varrho(Y), \\ &\text{(iii)} \quad \varrho(X \cup Y) + \varrho(X \cap Y) \leq \varrho(X) + \varrho(Y). \end{aligned}$$

The *Tutte polynomial* $T(M; x, y)$ of matroid $M = (U, \varrho)$ is defined by

$$T(M; x, y) := \sum_{I \subseteq U} (x-1)^{\varrho(U)-\varrho(I)} (y-1)^{|I|-\varrho(I)},$$

where x and y are independent variables, and the *rank polynomial* by

$$R(M; x, y) := T(M; x+1, y+1).$$

Example 5.1. Let $G = (V, U)$ be a finite undirected graph. For any subset I of the edge-set U of G let $G[I]$ denote the edge-subgraph induced by I, and let $n(G[I])$ and $c(G[I])$ denote its number of vertices and connected components, respectively. Let $\varrho : 2^U \to \mathbb{N}_0$ be defined by

$$\varrho(I) := n(G[I]) - c(G[I]). \tag{5.1}$$

Then, $M(G) := (U, \varrho)$ is a matroid, which is called the *cycle matroid* of G. Specializations of the Tutte or rank polynomial associated with $M(G)$ count various objects associated with G, e.g., subgraphs, spanning trees, acyclic orientations and proper λ-colorings (see Section 6). It is also related to network reliability. For details and further applications, see Welsh [5].

Our main result in this section is simplified by the following definition.

Definition 5.2. For any matroid $M = (U, \varrho)$ and any $X \subseteq U$ the *deletion of X from M* is defined by $M \setminus X := (U \setminus X, \varrho \,|\, 2^{U \setminus X})$. The *contraction of X from M* is defined by $M/X := (U \setminus X, \varrho_X)$ where the function $\varrho_X : 2^{U \setminus X} \to \mathbb{N}_0$ is defined by $\varrho_X(I) := \varrho(X \cup I) - \varrho(X)$ for any $I \subseteq U \setminus X$. Finally, the *restriction of M to X* is defined by $M|X := M \setminus (U \setminus X)$. (Note that $M \setminus X$, M/X and $M|X$ are again matroids.)

As the rank polynomial gives rise to shorter expressions than the Tutte polynomial, the results below are stated in terms of the rank polynomial.

Theorem 5.1. *Let $M = (U, \varrho)$ be a matroid on some finite non-empty set U, and let $x, y \in \mathbb{R}$ such that for any disjoint subsets $J, K \subseteq U$,*

$$(-1)^{|J|} x^{\varrho(U)-\varrho(J \cup K)} y^{|J|-\varrho(J)} R\left((M/J)|K; x, y\right) \geq^* 0. \tag{5.2}$$

Then, for any $r \in \mathbb{N}_0$,

$$(5.3) \qquad (-1)^r\, R(M;x,y) \leq^* (-1)^r \sum_{\substack{I\subseteq U\\ |I|\leq r}} x^{\varrho(U)-\varrho(I)} y^{|I|-\varrho(I)} + (-1)^r \frac{r+1}{|U|} \sum_{\substack{I\subseteq U\\ |I|=r+1}} x^{\varrho(U)-\varrho(I)} y^{|I|-\varrho(I)}.$$

Moreover, the theorem can be dualized by interchanging $\geq$ *and* $\leq$ *in the starred (***) places.*

Proof. In order to apply Theorem 2.1 we write

$$R(M;x,y) = \sum_{I\subseteq U} (-1)^{|I|} f(I),$$

where $f : 2^U \to \mathbb{R}$ is defined by

$$(5.4) \qquad f(I) := (-1)^{|I|} x^{\varrho(U)-\varrho(I)} y^{|I|-\varrho(I)} \quad (I \subseteq U)\,.$$

For any disjoint subsets $J, K \subseteq U$ we find that

$$\begin{aligned}
\sum_{I\subseteq K} (-1)^{|I|} f(I\cup J) &= \sum_{I\subseteq K} (-1)^{|J|} x^{\varrho(U)-\varrho(I\cup J)} y^{|I|+|J|-\varrho(I\cup J)} \\
&= (-1)^{|J|} x^{\varrho(U)-\varrho(J)-\varrho_J(K)} y^{|J|-\varrho(J)} \sum_{I\subseteq K} x^{\varrho_J(K)-\varrho_J(I)} y^{|I|-\varrho_J(I)} \\
&= (-1)^{|J|} x^{\varrho(U)-\varrho(J\cup K)} y^{|J|-\varrho(J)} R\left((M/J)|K;x,y\right) \geq 0,
\end{aligned}$$

where the last inequality comes from condition (5.2) above. Hence, f satisfies the requirements of Theorem 2.1, and thus (5.3) follows from (2.2). Similarly, the dual version follows by applying the dual version of Theorem 2.1. □

Remark 5.2. By using (1.4) instead of (2.2) in the proof of Theorem 5.1 one would, under the requirements of Theorem 5.1, obtain the weaker inequality

$$(5.5) \qquad (-1)^r R(M;x,y) \leq (-1)^r \sum_{\substack{I\subseteq U\\ |I|\leq r}} x^{\varrho(U)-\varrho(I)} y^{|I|-\varrho(I)}.$$

This weaker inequality is also a direct consequence of Theorem 5.1: Since $f(I)$, as defined in (5.4), satisfies the requirements of Theorem 2.1, it must be non-negative due to the second remark following Theorem 2.1 and hence,

$$(5.6) \qquad (-1)^r \frac{r+1}{|U|} \sum_{\substack{I\subseteq U\\ |I|=r+1}} x^{\varrho(U)-\varrho(I)} y^{|I|-\varrho(I)} \leq 0.$$

Now, from (5.3) and (5.6) the weaker inequality (5.5) follows. Under the dual requirements simply replace $\leq$ by $\geq$ in (5.5) and (5.6). The latter inequality (5.6) and its dual will be used in deriving the subsequent corollary.

Definition 5.3. Let $M = (U, \varrho)$ be a matroid. A subset $I \subseteq U$ is *dependent in* M if $\varrho(I) < |I|$. The *girth of* M, $g(M)$ for short, is the smallest size of a dependent set in M if such a set exists; otherwise $g(M) := +\infty$.

Corollary 5.3. *Under the requirements of Theorem 5.1 for* $0 \leq r < g(M)$,

$$(5.7) \qquad (-1)^r R(M;x,y) \leq (-1)^r \sum_{k=0}^{r} \binom{|U|}{k} x^{\varrho(U)-k} + (-1)^r \binom{|U|-1}{r} x^{\varrho(U)-r-1}.$$

The corollary can be dualized by interchanging $\geq$ and $\leq$ in (5.2) and (5.7).

Proof. For any $I \subseteq U$, $\varrho(I) = |I|$ if $|I| < g(M)$, and $\varrho(I) \leq |I|$ if $|I| \geq g(M)$. Thus, the inequality follows from (5.3) and (5.6), respectively their dual. □

Remark 5.4. Using (5.5) instead of (5.3) in the proof of the preceding corollary would, under the requirements of Theorem 5.1, give the weaker inequality

$$(-1)^r R(M;x,y) \leq (-1)^r \sum_{k=0}^{r} \binom{|U|}{k} x^{\varrho(U)-k} \quad (0 \leq r < g(M)),$$

respectively its dual (under the dual requirements).

6. The Chromatic Polynomial

Let $G = (V, U)$ be a finite undirected graph, and let $M(G)$ denote its cycle matroid (see Example 5.1). It is well-known (cf. [5]) that for any $\lambda \in \mathbb{N}$,

$$P_G(\lambda) := (-1)^{\varrho(U)} \lambda^{c(G)} T(M(G); 1-\lambda, 0) = (-1)^{\varrho(U)} \lambda^{c(G)} R(M(G); -\lambda, -1)$$

counts the number of proper λ-colorings of G, that is, the number of mappings $f : V \to \{1, \dots, \lambda\}$ such that $f(v) \neq f(w)$ if $v \neq w$ and v and w are adjacent in G. The polynomial $P_G(\lambda)$ is called the *chromatic polynomial* of G.

Theorem 6.1. *Let $G = (V, U)$ be a finite undirected graph having at least one edge (that is, $U \neq \emptyset$). Then, for any $\lambda \in \mathbb{N}$ and any $r \in \mathbb{N}_0$ we have*

$$(-1)^r P_G(\lambda) \leq (-1)^r \sum_{\substack{I \subseteq U \\ |I| \leq r}} (-1)^{|I|} \lambda^{c(V,I)} - \frac{r+1}{|U|} \sum_{\substack{I \subseteq U \\ |I| = r+1}} \lambda^{c(V,I)}, \tag{6.1}$$

where $c(V, I)$ denotes the number of connected components of the graph (V, I) having vertex-set V and edge-set I.

Proof. Theorem 6.1 is deduced from Theorem 5.1 and its dual. For any disjoint subsets $J, K \subseteq U$ the left-hand side of (5.2) is equal to

$$\begin{aligned}(-1)^{|J|}(-\lambda)^{\varrho(U)-\varrho(J\cup K)}(-1)^{|J|-\varrho(J)} R\left((M/J)|K; -\lambda, -1\right) \\ = (-1)^{\varrho(U)} \lambda^{\varrho(U)-\varrho(J\cup K)} \lambda^{c((G/J)[K])} P_{(G/J)[K]}(\lambda),\end{aligned} \tag{6.2}$$

where ϱ is the rank function of the cycle matroid as defined in (5.1), G/J is the graph obtained from G by contracting all edges in J, and $(G/J)[K]$ is the edge-subgraph induced by K in G/J. If $\varrho(U)$ is even, then the expression in (6.2) is at least zero and hence, Theorem 5.1 can be applied. On the other hand, if $\varrho(U)$ is odd, then the expression in (6.2) is at most zero, whence the dual version of Theorem 5.1 can be applied. In either case we obtain

$$(-1)^r \sum_{\substack{I \subseteq U \\ |I| \leq r}} (-1)^{|I|} \lambda^{\varrho(U)-\varrho(I)} - \frac{r+1}{|U|} \sum_{\substack{I \subseteq U \\ |I| = r+1}} \lambda^{\varrho(U)-\varrho(I)}.$$

as an upper bound for $(-1)^{\varrho(U)}(-1)^r R(M; -\lambda, -1)$. By this and the definition of the chromatic polynomial,

$$(-1)^r P_G(\lambda) \leq (-1)^r \sum_{\substack{I \subseteq U \\ |I| \leq r}} (-1)^{|I|} \lambda^{\varrho(U)-\varrho(I)+c(G)} - \frac{r+1}{|U|} \sum_{\substack{I \subseteq U \\ |I| = r+1}} \lambda^{\varrho(U)-\varrho(I)+c(G)},$$

from which the result follows since $\varrho(U) - \varrho(I) + c(G) = c(V, I)$. □

The following result appears in [2]. Recall that the *girth* of G, $g(G)$, is the length of a smallest cycle in G if G is not cycle-free; otherwise, $g(G) := +\infty$.

Corollary 6.2. *Under the requirements of Theorem 6.1 for* $0 \leq r < g(G)$,

$$(-1)^r P_G(\lambda) \leq (-1)^r \sum_{k=0}^{r} (-1)^k \binom{|U|}{k} \lambda^{|V|-k} - \binom{|U|-1}{r} \lambda^{|V|-r-1}. \tag{6.3}$$

Proof. Note that for any $I \subseteq U$, $c(V, I) = |V| - |I|$ if $|I| \leq g(G) - 1$, and $c(V, I) \geq |V| - |I|$ if $|I| \geq g(G)$. Thus, for $0 \leq r < g(G)$, Theorem 6.1 gives

$$(-1)^r P_G(\lambda) \leq (-1)^r \sum_{\substack{I \subseteq U \\ |I| \leq r}} (-1)^{|I|} \lambda^{|V|-|I|} - \frac{r+1}{|U|} \sum_{\substack{I \subseteq U \\ |I| = r+1}} \lambda^{|V|-|I|},$$

which simplifies to (6.3). $\square$

Remark 6.3. Corollary 6.2 can also be deduced from Corollary 5.3 and its dual in the same way as Theorem 6.1 is deduced from Theorem 5.1 and its dual.

REFERENCES

[1] K. DOHMEN, *Improved Bonferroni Inequalities via Abstract Tubes, Inequalities and Identities of Inclusion-Exclusion Type*, Lecture Notes in Mathematics, No. 1826, Springer-Verlag, Berlin Heidelberg, 2003.

[2] K. DOHMEN, Bounds to the chromatic polynomial of a graph, *Result. Math.*, **33** (1998), 87–88.

[3] J. GALAMBOS, Methods for proving Bonferroni type inequalities, *J. London Math. Soc.*, **2** (1975), 561–564.

[4] J. GALAMBOS AND I. SIMONELLI, *Bonferroni-type Inequalities with Applications*, Springer Series in Statistics, Probability and Its Applications, Springer-Verlag, New York, 1996.

[5] D.J.A. WELSH, *Complexity: Knots, Colourings and Counting*, Cambridge University Press, 1993.