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SOME INEQUALITIES FOR SPECTRAL VARIATIONS

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ABSTRACT. Over the last couple of decades, significant progress for the spectral variation of a matrix has been made in partially extending the classical Weyl and Lidskii theory [11, 7] to normal matrices and even to diagonalizable matrices for example. Recently these theories have been established for relative perturbations. In this paper, we shall establish relative perturbation theorems for generalized normal matrix. Some well-known perturbation theorems for normal matrix are extended. As applying, some perturbation theorems for positive definite matrix (possibly non-Hermitian) are established.

Key words and phrases: Spectral variation; Unitarily invariant norm; Hadamard product; Relative perturbation theorem.

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1. INTRODUCTION

The set of all $\lambda \in C$ that are eigenvalues of $A \in M_n(C)$ is called the spectrum of A and is denoted by $\sigma(A)$. The spectral radius of A is the nonnegative real number $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$. We shall use $|||\cdot|||$ to denote a unitarily invariant norm (see [5, 9, 13, 3, 20, 21]). $\|X\|_2$, the largest singular value of X, is a frequently used unitarily invariant norm. Let $X \circ Y = (x_{ij}y_{ij})$ be the Hadamard product of $X = (x_{ij})$ and $Y = (y_{ij})$. A matrix $A \in M_n(C)$ is said to be a generalized normal matrix with respect to H (It is called "generalized normal matrix" for short) or H^+-normal if there exists a positive definite Hermitian matrix H such that $A^*HA = AHA^*$, where "*" denotes the conjugate transpose. The definition was given first by [19, 18]. A generalized normal matrix is a very important kind of matrix which contains two subclasses of important matrices: normal matrices and positive definite matrices (possibly non-Hermitian), where a matrix A is called normal if $A^*A = AA^*$ and positive definite if $\mathrm{Re}(x^*Ax) > 0$ for any non-zero $x \in C^n$ (see [5, 6]). In recent years, the geometric significance, sixty-two equivalent conditions and many properties have been established for generalized normal matrices in [19, 17, 18]. We have

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Lemma 1.1 (see [19])**.** *Suppose $A \in M_n(C)$. Then*

(1) *A is a generalized normal matrix with respect to H if and only if $H^{1/2}AH^{1/2}$ is normal.*
(2) *A is a generalized normal matrix with respect to H if and only if there exists a nonsingular matrix P such that $H = (PP^*)^{-1}$ and*

$$A = P\Lambda P^*, \tag{1.1}$$

where $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$. Furthermore, $\lambda_1, \lambda_2, \dots, \lambda_n$ are n eigenvalues of HA.

Remark 1.2. (1.1) is equivalent to $HA = P^{-*}\Lambda P^*$ with $P^{-*} = (P^{-1})^*$, so we say that A has generalized eigen-decomposition (1.1), and $\lambda_1, \lambda_2, \dots, \lambda_n$ are the generalized eigenvalues of matrix A.

The spectral variation of a matrix has recently been a very active research subject in both matrix theory and numerical linear algebra. Over the last couple of decades significant progress has been made in partially extending the classical Weyl and Lidskii theory [11, 16] to normal matrices and even to diagonalizable matrices for example. This note will show how certain perturbation problems can be reformulated as simple matrix optimization problems involving Hadamard products. When A and $\tilde{A}$ are normal, we have shown one of many perturbation theorems that can be interpreted as bounding the norms of $Q \circ Z$ where Q is unitary and Z is a special matrix defined by the eigenalues (see [10]). In this paper, we shall extend the above result, and shall show how certain perturbation problems can be reformulated as generalized normal matrix optimization problems involving Hadamard products. Also, we study how generalized eigenvalues of a generalized normal matrix A change when it is perturbed to $\tilde{A} = D^*AD$, where D is a nonsingular matrix. As applications, some perturbation theorems for positive definite matrices (possibly non-Hermitian) are established.

2. MAIN RESULT

Suppose that A and $\tilde{A}$ are generalized normal matrices with respect to a common positive definite matrix H, and have generalized eigen-decompositions

$$A = P\Lambda P^* \quad \text{and} \quad \tilde{A} = \tilde{P}\tilde{\Lambda}\tilde{P}^*, \tag{2.1}$$

where

$$\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) \quad \text{and} \quad \tilde{\Lambda} = \operatorname{diag}(\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_n) \tag{2.2}$$

and λ_i are the generalized eigenvalues of A**,** and $\tilde{\lambda}_i$ are the generalized eigenvalues of $\tilde{A}$ $(i = 1, 2, \dots, n)$.

Notice $H = (PP^*)^{-1}$ and $H = (\tilde{P}\tilde{P}^*)^{-1}$, so $(P^{-1}\tilde{P})^*(P^{-1}\tilde{P}) = \tilde{P}^*H\tilde{P} = I$, then $Q = P^{-1}\tilde{P}$ is unitary and

$$\tilde{P} = PQ \tag{2.3}$$

Define

$$Z_1 = \left(\lambda_i - \tilde{\lambda}_j\right)_{i,j=1}^{n}. \tag{2.4}$$

We have the following result.

Theorem 2.1. *Suppose A and $\tilde{A}$ are H^+-normal with generalized eigen-decomposition (2.1), Then*

$$\rho(H)^{-1} \,|||Q \circ Z_1||| \le \left|\left|\left|A - \tilde{A}\right|\right|\right| \le \rho(H^{-1}) \,|||Q \circ Z_1|||\,, \tag{2.5}$$

where $Q = P^{-1}\tilde{P}$ is unitary and Z_1 is defined in Eq.(2.4).

Proof. For A and $\tilde{A}$ having generalized eigen-decomposition (2.1), noticing that $\tilde{P} = PQ$, where $Q = P^{-1}\tilde{P}$ is unitary, $|||WY||| \le \|W\|_2 \, |||Y|||$ and $|||YZ||| \le |||Y||| \, \|Z\|_2$ (see [9, p. 961]), we have

$$\left|\left|\left|P\Lambda P^* - PQ\tilde{\Lambda}Q^*P^*\right|\right|\right| \le \|P\|_2 \left|\left|\left|\Lambda - Q\tilde{\Lambda}Q^*\right|\right|\right| \|P^*\|_2 ,$$

then

$$\left|\left|\left|A - \tilde{A}\right|\right|\right| \le \left\|H^{-1}\right\|_2 \left|\left|\left|\Lambda - Q\tilde{\Lambda}Q^*\right|\right|\right| .$$

Since

$$\left|\left|\left|\Lambda - Q\tilde{\Lambda}Q^*\right|\right|\right| = \left|\left|\left|\Lambda Q - Q\tilde{\Lambda}\right|\right|\right| = |||Q \circ Z_1|||$$

and $\|H^{-1}\|_2 = \rho(H^{-1})$,

$$\left|\left|\left|A - \tilde{A}\right|\right|\right| \le \rho(H^{-1}) \, |||Q \circ Z_1||| . \tag{2.6}$$

On the other hand, we have

$$\begin{aligned}\left\|P^{-1}\right\|_2 \left|\left|\left|P\Lambda P^* - PQ\tilde{\Lambda}Q^*P^*\right|\right|\right| \left\|P^{-*}\right\|_2 &\ge \left|\left|\left|\Lambda - Q\tilde{\Lambda}Q^*\right|\right|\right| \\ &= |||Q \circ Z_1||| .\end{aligned}$$

Similarly for $H = (PP^*)^{-1}$ and $\|P^{-1}\|_2 = \|P^{-*}\|_2 = \sqrt{\rho(H)}$, we obtain

$$\rho(H) \left|\left|\left|A - \tilde{A}\right|\right|\right| \ge |||Q \circ Z_1||| ,$$

hence

$$\left|\left|\left|A - \tilde{A}\right|\right|\right| \ge \rho(H)^{-1} \, |||Q \circ Z_1||| . \tag{2.7}$$

The inequality (2.5) completes the proof by inequalities (2.6) and (2.7). □

In particular, if $H = I$ is the identity matrix, then H^+-normal matrices A and $\tilde{A}$ are normal matrices, hence A and $\tilde{A}$ have eigen-decomposition

$$A = U\Lambda U^* \quad \text{and} \quad \tilde{A} = \tilde{U}\tilde{\Lambda}\tilde{U}^*, \tag{2.8}$$

where U and $\tilde{U}$ are unitary, and

$$\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n), \quad \tilde{\Lambda} = \operatorname{diag}(\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_n).$$

By Theorem 2.1, we have

Corollary 2.2 (see [10])**.** *If A and $\tilde{A}$ are normal matrices, then*

$$\left|\left|\left|A - \tilde{A}\right|\right|\right| = |||Q \circ Z_1||| , \tag{2.9}$$

where $Q = U^*\tilde{U}$ and $Z_1 = \left(\lambda_i - \tilde{\lambda}_j\right)_{i,j=1}^{n}$.

We denote the Cartesian decomposition $X = H(X) + K(X)$, where $H(X) = \frac{1}{2}(X + X^*)$, and $K(X) = \frac{1}{2}(X - X^*)$. Let $\sigma(H(A)) = \{h_1, h_2, \ldots, h_n\}$ be ordered so that $h_1 \ge h_2 \ge \cdots \ge h_n$. Then we have some perturbation theorems for positie definite matrices which are discussed as follows.

Corollary 2.3. *If $A = H(A) + K(A)$ and $\tilde{A} = H(\tilde{A}) + K(\tilde{A})$ are positive definite with generalized eigen-decomposition (2.1), and $Q = P^{-1}\tilde{P}$ is unitary, then*

$$h_n \, |||Q \circ Z_1||| \le \left|\left|\left|A - \tilde{A}\right|\right|\right| \le h_1 \, |||Q \circ Z_1||| , \tag{2.10}$$

where Z_1 is defined in Eq.(2.4).

Proof. Since $Q = P^{-1}\tilde{P}$ is unitary, $H(A) = H(\tilde{A})$. It is easy to see that

$$A^* H(A)^{-1} A = AH(A)^{-1} A^*$$

and

$$\tilde{A}^* H(\tilde{A})^{-1} \tilde{A} = \tilde{A} H(\tilde{A})^{-1} \tilde{A}^*.$$

So A and $\tilde{A}$ are generalized normal matrices with respect to $H(A)^{-1}$. It is easy to see that $\rho(H(A)^{-1})^{-1} = h_n$, $\rho(H(A)) = h_1$. Applying Theorem 2.1, inequality (2.10) completes the proof. □

Let $B, C \in M_n(C)$. Then $[B, C] = BC - CB$ is called a commutator and $[B, C]_H = BHC - CHB$ is called a commutator with respect to H. The matrices B and C are said to commute with respect to H iff $[B, C]_H = 0$. $\|X\|_F$ is the Frobenius norm.

Corollary 2.4. *Let A and $\tilde{A}$ be H^+-normal matrices. If A and $\tilde{A}$ commute with respect to H, then*

$$\rho(H)^{-1} \left|\left|\left| I \circ Z_1 \right|\right|\right| \le \left|\left|\left| A - \tilde{A} \right|\right|\right| \le \rho(H^{-1}) \left|\left|\left| I \circ Z_1 \right|\right|\right|, \tag{2.11}$$

where I is the identity matrix, and Z_1 is defined in Eq.(2.4).

Proof. $[A, \tilde{A}]_H = 0$ if and only if there exists a nonsingular matrix P, such that $A = P\Lambda P^*$ and $\tilde{A} = P\tilde{\Lambda}P^*$, where $Q = P^{-1}P = I$ (see [17, Theorem 3] and Theorem 2.1). So Q is taken as the identity matrix I in Theorem 2.1, hence Eq. (2.11) holds. □

Applying Corollary 2.3 and Corollary 2.4, we have

Corollary 2.5. *Let the hypotheses of Corollary 2.3 hold. Moreover if matrices A and $\tilde{A}$ commute with respect to $H(A)^{-1}$, then*

$$h_n \left|\left|\left| I \circ Z_1 \right|\right|\right| \le \left|\left|\left| A - \tilde{A} \right|\right|\right| \le h_1 \left|\left|\left| I \circ Z_1 \right|\right|\right|, \tag{2.12}$$

where $h_1 = \max_{1\le i\le n} \lambda_i(H(A))$, $h_n = \min_{1\le i\le n} \lambda_i(H(A))$ and Z_1 is defined in Eq.(2.4).

In the following, we shall study how generalized eigenvalues of a generalized normal matrix A change when it is perturbed to $\tilde{A} = D^*AD$, where D is a nonsingular matrix. The $p-$relative distance between $\alpha, \tilde{\alpha} \in C$ is defined as

$$\varrho_p(\alpha, \tilde{\alpha}) = \frac{|\alpha - \tilde{\alpha}|}{\sqrt[p]{|\alpha|^p + |\tilde{\alpha}|^p}} \text{ for } 1 \le p \le \infty. \tag{2.13}$$

Theorem 2.6. *Suppose A and $\tilde{A}$ are H^+-normal matrices and $\tilde{A} = D^*AD$, where D is nonsingular. Let A and $\tilde{A}$ have generalized eigen-decomposition (2.1). Then there is a permutation τ of $\{1, 2, \ldots, n\}$ such that*

$$\sum_{i=1}^{n} [\varrho_2(\lambda_i, \tilde{\lambda}_{\tau(i)})]^2 \le c(\|I - D\|_F^2 + \left\|D^{-*} - I\right\|_F^2) \tag{2.14}$$

where $c = \max_{1\le i\le n} \lambda_i(H) / \min_{1\le i\le n} \lambda_i(H)$.

Proof. Notice that

$$A - \tilde{A} = A - D^*AD = A(I - D) + (D^{-*} - I)\tilde{A}.$$

Pre- and postmultiply the equations by $P^{-1}$and $\tilde{P}^{-*}$ respectively, to get

$$\Lambda P^* \tilde{P}^{-*} - P^{-1}\tilde{P}\tilde{\Lambda} = \Lambda P^*(I - D)\tilde{P}^{-*} + P^{-1}(D^{-*} - I)\tilde{P}\tilde{\Lambda}. \tag{2.15}$$

Set $Q = P^{-1}\tilde{P} = (q_{ij})$, then Q is unitary and $Q = P^*\tilde{P}^{-*}$. Let

$$E = P^*(I-D)\tilde{P}^{-*} = (e_{ij}), \tilde{E} = P^{-1}(D^{-*}-I)\tilde{P} = (\tilde{e}_{ij}). \tag{2.16}$$

Then (2.15) implies that $\Lambda Q - Q\tilde{\Lambda} = \Lambda E + \tilde{E}\tilde{\Lambda}$ or componentwise $\lambda_i q_{ij} - q_{ij}\tilde{\lambda}_j = \lambda_i e_{ij} + \tilde{e}_{ij}\tilde{\lambda}_j$, so

$$\left|(\lambda_i - \tilde{\lambda}_j)q_{ij}\right|^2 = \left|\lambda_i e_{ij} + \tilde{e}_{ij}\tilde{\lambda}_j\right|^2 \le (|\lambda_i|^2 + \left|\tilde{\lambda}_j\right|^2)(|e_{ij}|^2 + |\tilde{e}_{ij}|^2),$$

which yields $[\varrho_2(\lambda_i, \tilde{\lambda}_j)]^2 |q_{ij}|^2 \le |e_{ij}|^2 + |\tilde{e}_{ij}|^2$. Hence

$$\begin{aligned} &\sum_{i,j=1}^{n} [\varrho_2(\lambda_i, \tilde{\lambda}_j)]^2 |q_{ij}|^2 \\ &\le \left\|P^*(I-D)\tilde{P}^{-*}\right\|_F^2 + \left\|P^{-1}(D^{-*}-I)\tilde{P}\right\|_F^2 \\ &\le \|P^*\|_2^2 \|I-D\|_F^2 \left\|\tilde{P}^{-*}\right\|_2^2 + \left\|P^{-1}\right\|_2^2 \left\|D^{-*}-I\right\|_F^2 \left\|\tilde{P}\right\|_2^2. \end{aligned}$$

Notice that

$$\|P^*\|_2^2 = \max_{1\le i\le n} \lambda_i(H) \quad \text{and} \quad \left\|P^{-1}\right\|_2^2 = \left(\min_{1\le i\le n} \lambda_i(H)\right)^{-1}$$

by $\sigma(PP^*) = \sigma(P^*P) = \sigma(H)$. Similarly, we have $\left\|\tilde{P}\right\|_2^2 = \max_{1\le i\le n} \lambda_i(H)$ and

$$\left\|\tilde{P}^{-*}\right\|_2^2 = \max_{1\le i\le n}\lambda_i(H^{-1}) = \left(\min_{1\le i\le n} \lambda_i(H)\right)^{-1},$$

so

$$\sum_{i,j=1}^{n} \left[\varrho_2\left(\lambda_i, \tilde{\lambda}_j\right)\right]^2 |q_{ij}|^2 \le c\left(\|I-D\|_F^2 + \left\|D^{-*}-I\right\|_F^2\right),$$

where $c = \max_{1\le i\le n} \lambda_i(H)/\min_{1\le i\le n}\lambda_i(H)$.

The matrix $\left(|q_{ij}|^2\right)_{n\times n}$ is a doubly stochastic matrix. The above inequality and [9, Lemma 5.1] imply inequality (2.14). □

If A and $\tilde{A}$ are normal matrices, then they are generalized normal matrices with respect to H and $H = I$. Applying Theorem 2.6, it is easy to get

Corollary 2.7. *If $A, \tilde{A} \in M_n(C)$ are normal matrices with $A = U\Lambda U^*$ and $\tilde{A} = \tilde{U}\tilde{\Lambda}\tilde{U}^*$ where both U and $\tilde{U}$ are unitary, and $\tilde{A} = D^*AD$, where D is nonsingular, then*

$$\sum_{i=1}^{n} [\varrho_2(\lambda_i, \tilde{\lambda}_{\tau(i)})]^2 \le \|I-D\|_F^2 + \left\|D^{-*}-I\right\|_F^2. \tag{2.17}$$

Corollary 2.8. *Let $A = H(A) + K(A)$ and $\tilde{A} = H(\tilde{A}) + K(\tilde{A})$ be positive definite matrices with generalized eigen-decomposition (2.1), and $\tilde{A} = D^*AD$, where D is nonsingular. If $Q = P^{-1}\tilde{P}$ is unitary, then*

$$\sum_{i,=1}^{n} \left[\varrho_2\left(\lambda_i, \tilde{\lambda}_{\tau(i)}\right)\right]^2 \le c\left(\|I-D\|_F^2 + \left\|D^{-*}-I\right\|_F^2\right), \tag{2.18}$$

where $c = \max_{1\le i\le n}\lambda_i(H(A))/\min_{1\le i\le n}\lambda_i(H(A))$.

Proof. By the proof of Corollary 2.3, A and $\tilde{A}$ are generalized normal matrices with respect to $H(A)^{-1}$, and

$$\max_{1\le i\le n}\lambda_i(H(A)^{-1})/\min_{1\le i\le n}\lambda_i(H(A)^{-1}) = \max_{1\le i\le n}\lambda_i(H(A))/\min_{1\le i\le n}\lambda_i(H(A)).$$

Inequality (2.18) is proved by Theorem 2.6. □

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