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## THE ANALYTIC DOMAIN IN THE IMPLICIT FUNCTION THEOREM

H.C. CHANG, W. HE AND N. PRABHU

School of Industrial Engineering  
Purdue University  
West Lafayette, IN 47907  
EMail: [prabhu@ecn.purdue.edu](mailto:prabhu@ecn.purdue.edu)

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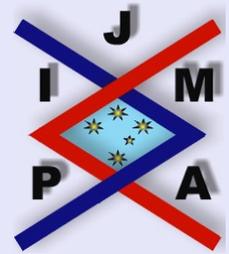


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## Abstract

The Implicit Function Theorem asserts that there exists a ball of nonzero radius within which one can express a certain subset of variables, in a system of analytic equations, as analytic functions of the remaining variables. We derive a nontrivial lower bound on the radius of such a ball. To the best of our knowledge, our result is the first bound on the domain of validity of the Implicit Function Theorem.

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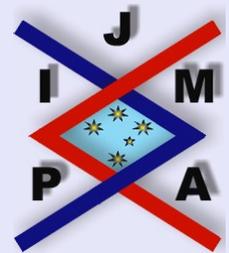
# 1. The Size of the Analytic Domain

The *Implicit Function Theorem* is one of the fundamental theorems in multi-variable analysis [1, 4, 5, 6, 7]. It asserts that if  $\varphi_i(x, z) = 0$ ,  $i = 1, \dots, m$ ,  $x \in \mathbf{C}^n$ ,  $z \in \mathbf{C}^m$  are complex analytic functions in a neighborhood of a point  $(x^{(0)}, z^{(0)})$  and  $\mathbf{J} \left( \begin{smallmatrix} \varphi_1, \dots, \varphi_m \\ z_1, \dots, z_m \end{smallmatrix} \right) \Big|_{(x^{(0)}, z^{(0)})} \neq 0$ , where  $\mathbf{J}$  is the Jacobian determinant, then there exists an  $\epsilon > 0$  and analytic functions  $g_1(x), \dots, g_m(x)$  defined in the domain  $\mathbf{D} = \{x \mid \|x - x^{(0)}\| < \epsilon\}$  such that  $\varphi_i(x, g_1(x), \dots, g_m(x)) = 0$ , for  $i = 1, \dots, m$  in  $\mathbf{D}$ . Besides its central role in analysis the theorem also plays an important role in multi-dimensional nonlinear optimization algorithms [2, 3, 8, 9]. Surprisingly, despite its overarching importance and widespread use, a nontrivial lower bound on the size of the domain  $\mathbf{D}$  has not been reported in the literature and in this note, we present the first lower bound on the size of  $\mathbf{D}$ . The bound is derived in two steps. First we use Roche's Theorem to derive a lower bound for the case of one dependent variable – i.e.,  $m = 1$  – and then extend the result to the general case.

**Theorem 1.1.** *Let  $\varphi(x, z)$  be an analytic function of  $n + 1$  complex variables,  $x \in \mathbf{C}^n$ ,  $z \in \mathbf{C}$  at  $(0, 0)$ . Let  $\frac{\partial \varphi(0, 0)}{\partial z} = a \neq 0$ , and let  $|\varphi(0, z)| \leq M$  on  $B$  where  $B = \{(x, z) \mid \|(x, z)\| \leq R\}$ . Then  $z = g(x)$  is an analytic function of  $x$  in the ball*

$$(1.1) \quad \|x\| \leq \Theta_1(M, a, R; \varphi) := \frac{1}{M} \left( |a| r - \frac{Mr^2}{R^2 - rR} \right),$$

$$\text{where } r = \min \left( \frac{R}{2}, \frac{|a|R^2}{2M} \right).$$



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*Proof.* Since  $\varphi(x, z)$  is an analytic function of complex variables, by the Implicit Function Theorem  $z = g(x)$  is an analytic function in a neighborhood  $U$  of  $(0, 0)$ . To find the domain of analyticity of  $g$  we first find a number  $r > 0$  such that  $\varphi(0, z)$  has  $(0, 0)$  as its unique zero in the disc  $\{(0, z) : |z| \leq r\}$ . Then we will find a number  $s > 0$  so that  $\varphi(x, z)$  has a unique zero  $(x, g(x))$  in the disc  $\{(x, z) : |z| \leq r\}$  for  $|x| \leq s$  with the help of Roche's theorem. Then we show that in the domain  $\{x : \|x\| \leq s\}$  the implicit function  $z = g(x)$  is well defined and analytic.

Note that if we expand the Taylor series of the function  $\varphi$  with respect to the variable  $z$ , we get

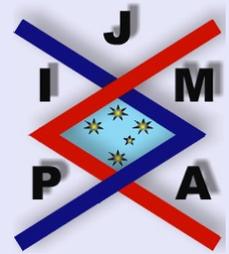
$$\varphi(0, z) = \frac{\partial\varphi(0, 0)}{\partial z}z + \sum_{j=2}^{\infty} \frac{\partial^j\varphi(0, 0)}{j!}z^j.$$

Let us assume that  $|\frac{\partial\varphi(0, 0)}{\partial z}| = a > 0$ .  $|\varphi(0, z)| \leq M$  on  $B$  where  $B = \{(x, z) : \|(x, z)\| \leq R\}$ . Then by Cauchy's estimate, we have

$$\left| \frac{\partial^j\varphi(0, 0)}{j!}z^j \right| \leq \frac{|z|^j}{R^j}M.$$

This implies that

$$\begin{aligned} |\varphi(0, z)| &\geq |a| \cdot |z| - \sum_{j=2}^{\infty} M \left(\frac{|z|}{R}\right)^j \\ (1.2) \qquad &= |a| \cdot |z| - \frac{M|z|^2}{R^2 - |z|R}. \end{aligned}$$




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Since our goal is to have  $|\varphi(0, z)| > 0$ , it is sufficient to have  $|a| \cdot |z| - \frac{M|z|^2}{R^2 - |z|R} > 0$ . Dividing both sides by  $|z|$  we get

$$|a| > \frac{M|z|}{R^2 - |z|R} \iff |a|(R^2 - |z|R) - M|z| > 0 \iff |z|(|a|R + M) < |a|R^2$$

$$\iff |z| < \frac{|a|R^2}{|a|R + M} = \frac{R}{1 + \frac{M}{|a|R}}.$$

We next choose

$$r = \min \left( \frac{R}{1+1}, \frac{R}{\frac{M}{|a|R} + \frac{M}{|a|R}} \right)$$

$$= \min \left( \frac{R}{2}, \frac{|a|R^2}{2M} \right).$$

To compute  $s$  we need Roche's Theorem.

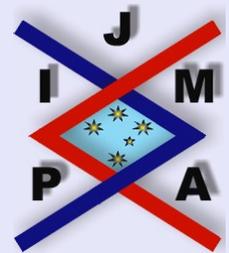
**Theorem 1.2 (Roche's Theorem).** [1] *Let  $h_1$  and  $h_2$  be analytic on the open set  $U \subset \mathbb{C}$ , with neither  $h_1$  nor  $h_2$  identically 0 on any component of  $U$ . Let  $\gamma$  be a closed path in  $U$  such that the winding number  $n(\gamma, z) = 0, \forall z \notin U$ . Suppose that*

$$|h_1(z) - h_2(z)| < |h_2(z)|, \quad \forall z \in \gamma.$$

*Then  $n(h_1 \circ \gamma, 0) = n(h_2 \circ \gamma, 0)$ . Thus  $h_1$  and  $h_2$  have the same number of zeros inside  $\gamma$ , counting multiplicity and index.*

Let  $h_1(z) := \varphi(0, z)$ , and  $h_2 := \varphi(x, z)$ . We can treat  $x$  as a parameter, so our goal is to find  $s > 0$  such that if  $|x| < s$ , then

$$|\varphi(0, z) - \varphi(x, z)| < |\varphi(0, z)|, \quad \forall z \in \gamma,$$



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where  $\gamma = \{z : |z| = r\}$ . We know  $|\varphi(0, z) - \varphi(x, z)| < Ms$  if  $\gamma \subset B$  and we also have  $|\varphi(0, z)| > |a| \cdot |z| - \frac{M|z|^2}{R^2 - |z|R}$  from (1.2). It is sufficient to have

$$Ms < |a| \cdot |z| - \frac{M|z|^2}{R^2 - |z|R}.$$

On  $\gamma$ , we know  $|z| = r$ , and therefore as long as

$$s < \frac{1}{M} \left( |a|r - \frac{Mr^2}{R^2 - rR} \right),$$

we can apply the Roche's theorem and guarantee that the function  $\varphi(x, z)$  has a unique zero, call it  $g(x)$ . That is,  $\varphi(x, g(x)) = 0$  and  $g(x)$  is hence a well defined function of  $x$ .

Note that in Roche's theorem, the number of zeros includes the multiplicity and index. Therefore all the zeros we get are simple zeros since  $(0, 0)$  is a simple zero for  $\varphi(0, z)$ . This is because  $\varphi(0, 0) = 0$  and  $\varphi_z(0, 0) \neq 0$ . Hence we can conclude that for any  $x$  such that  $|x| < s$ , we can find a unique  $g(x)$  so that  $\varphi(x, g(x)) = 0$  and  $\varphi_z(x, g(x)) \neq 0$ .  $\square$

We use Theorem 1.1 to derive a lower bound for  $m \geq 1$  below. Let  $\varphi_i(x, z) = 0, i = 1, \dots, m, x \in \mathbf{C}^n, z \in \mathbf{C}^m$  be analytic functions at  $(x^{(0)}, z^{(0)})$ . Let

$$(1.3) \quad J_m(x^{(0)}, z^{(0)}) := \begin{vmatrix} \frac{\partial \varphi_1(x^{(0)}, z^{(0)})}{\partial z_1} & \dots & \frac{\partial \varphi_1(x^{(0)}, z^{(0)})}{\partial z_m} \\ \vdots & & \vdots \\ \frac{\partial \varphi_m(x^{(0)}, z^{(0)})}{\partial z_1} & \dots & \frac{\partial \varphi_m(x^{(0)}, z^{(0)})}{\partial z_m} \end{vmatrix} = a_m \neq 0$$




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and let

$$(1.4) \quad |\varphi_i(x^{(0)}, z_1, \dots, z_m)| \leq M, \text{ for } i = 1, \dots, m$$

on

$$(1.5) \quad B = \{(x, z_1, \dots, z_m) \mid \|(x, z) - (x^{(0)}, z^{(0)})\| \leq R\}.$$

Since  $J_m(x^{(0)}, z^{(0)}) \neq 0$ , some  $(m-1) \times (m-1)$  sub-determinant in the matrix corresponding to  $J_m(x^{(0)}, z^{(0)})$  must be nonzero. Without loss of generality, we may assume that

$$(1.6) \quad J_{m-1}(x^{(0)}, z^{(0)}) := \begin{vmatrix} \frac{\partial \varphi_2(x^{(0)}, z^{(0)})}{\partial z_2} & \dots & \frac{\partial \varphi_2(x^{(0)}, z^{(0)})}{\partial z_m} \\ \vdots & & \vdots \\ \frac{\partial \varphi_m(x^{(0)}, z^{(0)})}{\partial z_2} & \dots & \frac{\partial \varphi_m(x^{(0)}, z^{(0)})}{\partial z_m} \end{vmatrix} \\ = a_{m-1} \neq 0.$$

By induction we conclude that there exist analytic functions  $\psi_2(x, z_1), \dots, \psi_m(x, z_1)$  and that we can compute a  $\Theta_{m-1}(x^{(0)}, z_1^{(0)}; \varphi_2, \dots, \varphi_m) > 0$  such that

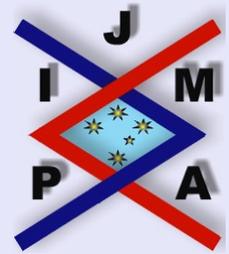
$$\varphi_i(x, z_1, \psi_2(x, z_1), \dots, \psi_m(x, z_1)) = 0, \quad i = 2, \dots, m$$

in

$$\mathbf{D}_{n+1} := \{(x, z_1) \mid \|(x, z_1) - (x^{(0)}, z_1^{(0)})\| \leq \Theta_{m-1}(x^{(0)}, z_1^{(0)}; \varphi_2, \dots, \varphi_m)\}.$$

Define

$$(1.7) \quad \Gamma(x, z_1) := \varphi_1(x, z_1, \psi_2(x, z_1), \dots, \psi_m(x, z_1)).$$



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Then we have

$$(1.8) \quad \frac{\partial \Gamma}{\partial z_1} = \frac{\partial \varphi_1}{\partial z_1} + \sum_{i=2}^m \frac{\partial \varphi_1}{\partial z_i} \cdot \frac{\partial \psi_i}{\partial z_1}.$$

Since  $\varphi_2(x, z_1, \psi_2, \dots, \psi_m) = 0, \dots, \varphi_m(x, z_1, \psi_2, \dots, \psi_m) = 0$  in  $D_{n+1}$ , differentiating with respect to  $z_1$  we have

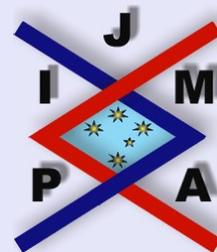
$$\frac{\partial \varphi_i}{\partial z_1} = \frac{\partial \varphi_i}{\partial z_1} + \sum_{j=2}^m \frac{\partial \varphi_i}{\partial z_j} \cdot \frac{\partial \psi_j}{\partial z_1} = 0; \quad i = 2, \dots, m$$

or in other words

$$(1.9) \quad \begin{bmatrix} \frac{\partial \varphi_2}{\partial z_2} & \cdots & \frac{\partial \varphi_2}{\partial z_m} \\ \vdots & & \vdots \\ \frac{\partial \varphi_m}{\partial z_2} & \cdots & \frac{\partial \varphi_m}{\partial z_m} \end{bmatrix} \begin{bmatrix} \frac{\partial \psi_2}{\partial z_1} \\ \vdots \\ \frac{\partial \psi_m}{\partial z_1} \end{bmatrix} = - \begin{bmatrix} \frac{\partial \varphi_2}{\partial z_1} \\ \vdots \\ \frac{\partial \varphi_m}{\partial z_1} \end{bmatrix}.$$

Using Cramer's rule and (1.9) we have

$$(1.10) \quad \frac{\partial \psi_i}{\partial z_1} = - \frac{\begin{vmatrix} \frac{\partial \varphi_2}{\partial z_2} & \cdots & \frac{\partial \varphi_2}{\partial z_{i-1}} & \frac{\partial \varphi_2}{\partial z_1} & \frac{\partial \varphi_2}{\partial z_{i+1}} & \cdots & \frac{\partial \varphi_2}{\partial z_m} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial \varphi_m}{\partial z_2} & \cdots & \frac{\partial \varphi_m}{\partial z_{i-1}} & \frac{\partial \varphi_m}{\partial z_1} & \frac{\partial \varphi_m}{\partial z_{i+1}} & \cdots & \frac{\partial \varphi_m}{\partial z_m} \end{vmatrix}}{J_{m-1}}; \quad i = 2, \dots, m.$$



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Substituting (1.10) into (1.8) and simplifying we get

$$\begin{aligned} \frac{\partial \Gamma(x^{(0)}, z_1^{(0)})}{\partial z_1} &= \frac{\begin{vmatrix} \frac{\partial \varphi_1(x^{(0)}, z^{(0)})}{\partial z_1} & \dots & \frac{\partial \varphi_1(x^{(0)}, z^{(0)})}{\partial z_m} \\ \vdots & & \vdots \\ \frac{\partial \varphi_m(x^{(0)}, z^{(0)})}{\partial z_1} & \dots & \frac{\partial \varphi_m(x^{(0)}, z^{(0)})}{\partial z_m} \end{vmatrix}}{J_{m-1}(x^{(0)}, z^{(0)})} \\ &= \frac{J_m(x^{(0)}, z^{(0)})}{J_{m-1}(x^{(0)}, z^{(0)})} = \frac{a_m}{a_{m-1}} \neq 0. \end{aligned}$$

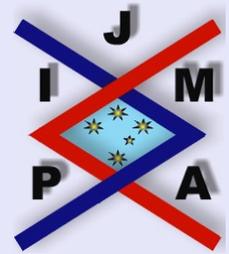
Therefore we can apply Theorem 1.1 to  $\Gamma(x, z_1)$  and conclude that there exists an implicit function  $z_1 = g_1(x)$  in

$$\begin{aligned} \mathbf{D}_n &:= \left\{ x \in \mathbf{C}^n \mid \|x - x^{(0)}\| \right. \\ &\quad \left. \leq \Theta_1 \left( M, \frac{a_m}{a_{m-1}}, \min(R, \Theta_{m-1}(x^{(0)}, z_1^{(0)}; \varphi_2, \dots, \varphi_m)); \varphi_1 \right) \right\} \end{aligned}$$

such that in  $\mathbf{D}_n$ ,  $\varphi_i(x, g_1(x), g_2(x), \dots, g_m(x)) = 0$ ,  $i = 1, \dots, m$  where  $g_j(x) := \psi_j(x, g_1(x))$ ,  $j = 2, \dots, m$ .

In summary, the sought lower bound on the size of the analytic domain of implicit functions is expressed recursively as

$$\begin{aligned} (1.11) \quad \Theta_m(x^{(0)}, z^{(0)}; \varphi_1, \dots, \varphi_m) \\ = \Theta_1 \left( M, \frac{a_m}{a_{m-1}}, \min(R, \Theta_{m-1}(x^{(0)}, z_1^{(0)}; \varphi_2, \dots, \varphi_m)); \varphi_1 \right) \end{aligned}$$



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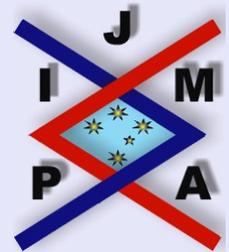
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using the definition of  $\Theta_1$  in Theorem 1.1 and of  $M, a_m, a_{m-1}$  and  $R$  in equations (1.4), (1.3), (1.6) and (1.5) respectively.



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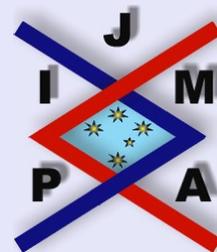
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## References

- [1] R.B. ASH, *Complex Variables*, Academic Press, 1971.
- [2] D.P. BERTSEKAS, *Nonlinear Programming*, Athena Scientific Press, 1999.
- [3] R. FLETCHER, *Practical Methods of Optimization*, John Wiley and Sons, 2000.
- [4] R.C. GUNNING, *Introduction to Holomorphic Functions of Several Variables: Function Theory*, CRC Press, 1990.
- [5] L. HORMANDER, *Introduction to Complex Analysis in Several Variables*, Elsevier Science Ltd., 1973.
- [6] S.G. KRANTZ, *Function Theory of Several Complex Variables*, Wiley-Interscience, 1982.
- [7] R. NARASIMHAN, *Several Complex Variables*, University of Chicago Press, 1974.
- [8] S. NASH AND A. SOFER, *Linear and Nonlinear Programming*, McGraw-Hill, 1995.
- [9] J. NOCEDAL AND S.J. WRIGHT, *Numerical Optimization*, Springer Verlag, 1999.



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