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## THE ANALYTIC DOMAIN IN THE IMPLICIT FUNCTION THEOREM


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## Abstract

The Implicit Function Theorem asserts that there exists a ball of nonzero radius within which one can express a certain subset of variables, in a system of analytic equations, as analytic functions of the remaining variables. We derive a nontrivial lower bound on the radius of such a ball. To the best of our knowledge, our result is the first bound on the domain of validity of the Implicit Function Theorem.

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# Contents

# 1. The Size of the Analytic Domain

The *Implicit Function Theorem* is one of the fundamental theorems in multi-variable analysis [1, 4, 5, 6, 7]. It asserts that *if* $\varphi_i(x,z) = 0$, $i = 1,\dots,m$, $x \in \mathbf{C}^n$, $z \in \mathbf{C}^m$ *are complex analytic functions in a neighborhood of a point* $(x^{(0)}, z^{(0)})$ *and* $\mathbf{J}\left(\frac{\varphi_1,\dots,\varphi_m}{z_1,\dots,z_m}\right)\Big|_{(x^{(0)},z^{(0)})} \neq 0$, *where* $\mathbf{J}$ *is the Jacobian determinant, then there exists an* $\epsilon > 0$ *and analytic functions* $g_1(x),\dots,g_m(x)$ *defined in the domain* $\mathbf{D} = \{x \mid \|x - x^{(0)}\| < \epsilon\}$ *such that* $\varphi_i(x, g_1(x),\dots,g_m(x)) = 0$, *for* $i = 1,\dots,m$ *in* $\mathbf{D}$. Besides its central role in analysis the theorem also plays an important role in multi-dimensional nonlinear optimization algorithms [2, 3, 8, 9]. Surprisingly, despite its overarching importance and widespread use, a nontrivial lower bound on the size of the domain $\mathbf{D}$ has not been reported in the literature and in this note, we present the first lower bound on the size of $\mathbf{D}$. The bound is derived in two steps. First we use Roche's Theorem to derive a lower bound for the case of one dependent variable – *i.e.*, $m = 1$ – and then extend the result to the general case.

**Theorem 1.1.** *Let* $\varphi(x,z)$ *be an analytic function of* $n+1$ *complex variables,* $x \in \mathbf{C}^n$, $z \in \mathbf{C}$ *at* $(0,0)$. *Let* $\frac{\partial\varphi(0,0)}{\partial z} = a \neq 0$, *and let* $|\varphi(0,z)| \leq M$ *on* $B$ *where* $B = \{(x,z) | \, \|(x,z)\| \leq R\}$. *Then* $z = g(x)$ *is an analytic function of* $x$ *in the ball*

$$\|x\| \leq \mathbf{\Theta}_1(M,a,R;\varphi) := \frac{1}{M}\left(|a|\, r - \frac{Mr^2}{R^2 - rR}\right), \tag{1.1}$$

$$\text{where } r = \min\left(\frac{R}{2}, \frac{|a|\, R^2}{2M}\right).$$

*Proof.* Since $\varphi(x,z)$ is an analytic function of complex variables, by the Implicit Function Theorem $z = g(x)$ is an analytic function in a neighborhood $U$ of $(0,0)$. To find the domain of analyticity of $g$ we first find a number $r > 0$ such that $\varphi(0,z)$ has (0,0) as its unique zero in the disc $\{(0,z) : |z| \le r\}$. Then we will find a number $s > 0$ so that $\varphi(x,z)$ has a unique zero $(x, g(x))$ in the disc $\{(x,z) : |z| \le r\}$ for $|x| \le s$ with the help of Roche's theorem. Then we show that in the domain $\{x : \|x\| \le s\}$ the implicit function $z = g(x)$ is well defined and analytic.

Note that if we expand the Taylor series of the function $\varphi$ with respect to the variable $z$, we get

$$\varphi(0,z) = \frac{\partial \varphi(0,0)}{\partial z} z + \sum_{j=2}^{\infty} \frac{\frac{\partial^j \varphi(0,0)}{\partial z^j} z^j}{j!}.$$

Let us assume that $|\frac{\partial \varphi(0,0)}{\partial z}| = a > 0$. $|\varphi(0,z)| \le M$ on $B$ where $B = \{(x,z) : \|(x,z)\| \le R\}$ . Then by Cauchy's estimate, we have

$$\left| \frac{\frac{\partial^j \varphi(0,0)}{\partial z^j} z^j}{j!} \right| \le \frac{|z|^j}{R^j} M.$$

This implies that

$$\begin{aligned} |\varphi(0,z)| &\ge |a| \cdot |z| - \sum_{j=2}^{\infty} M \left( \frac{|z|}{R} \right)^j \\ &= |a| \cdot |z| - \frac{M|z|^2}{R^2 - |z|R}. \end{aligned} \tag{1.2}$$

Since our goal is to have $|\varphi(0,z)| > 0$, it is sufficient to have $|a|\cdot|z| - \frac{M|z|^2}{R^2-|z|R} > 0$. Dividing both sides by $|z|$ we get

$$|a| > \frac{M|z|}{R^2-|z|R} \iff |a|(R^2-|z|R) - M|z| > 0 \iff |z|(|a|R+M) < |a|R^2$$
$$\iff |z| < \frac{|a|R^2}{|a|R+M} = \frac{R}{1+\frac{M}{|a|R}}.$$

We next choose

$$\begin{aligned} r &= \min\left(\frac{R}{1+1}, \frac{R}{\frac{M}{|a|R}+\frac{M}{|a|R}}\right) \\ &= \min\left(\frac{R}{2}, \frac{|a|R^2}{2M}\right). \end{aligned}$$

To compute $s$ we need Roche's Theorem.

**Theorem 1.2 (Roche's Theorem).** *[1] Let $h_1$ and $h_2$ be analytic on the open set $U \subset C$, with neither $h_1$ nor $h_2$ identically $0$ on any component of $U$. Let $\gamma$ be a closed path in $U$ such that the winding number $n(\gamma,z) = 0$, $\forall z \notin U$. Suppose that*

$$|h_1(z) - h_2(z)| < |h_2(z)|, \qquad \forall z \in \gamma.$$

*Then $n(h_1\circ\gamma, 0) = n(h_1\circ\gamma, 0)$. Thus $h_1$ and $h_2$ have the same number of zeros inside $\gamma$, counting multiplicity and index.*

Let $h_1(z) := \varphi(0,z)$, and $h_2 := \varphi(x,z)$. We can treat $x$ as a parameter, so our goal is to find $s > 0$ such that if $|x| < s$, then

$$|\varphi(0,z) - \varphi(x,z)| < |\varphi(0,z)|, \qquad \forall z \in \gamma,$$

where $\gamma = \{z : |z| = r\}$. We know $|\varphi(0,z) - \varphi(x,z)| < Ms$ if $\gamma \subset B$ and we also have $|\varphi(0,z)| > |a| \cdot |z| - \frac{M|z|^2}{R^2 - |z|R}$ from (1.2). It is sufficient to have

$$Ms < |a| \cdot |z| - \frac{M|z|^2}{R^2 - |z|R}.$$

On $\gamma$, we know $|z| = r$, and therefore as long as

$$s < \frac{1}{M}\left(|a|r - \frac{Mr^2}{R^2 - rR}\right),$$

we can apply the Roche's theorem and guarantee that the function $\varphi(x,z)$ has a unique zero, call it $g(x)$. That is, $\varphi(x, g(x)) = 0$ and $g(x)$ is hence a well defined function of $x$.

Note that in Roche's theorem, the number of zeros includes the multiplicity and index. Therefore all the zeros we get are simple zeros since $(0,0)$ is a simple zero for $\varphi(0,z)$. This is because $\varphi(0,0) = 0$ and $\varphi_z(0,0) \neq 0$. Hence we can conclude that for any $x$ such that $|x| < s$, we can find a unique $g(x)$ so that $\varphi(x, g(x)) = 0$ and $\varphi_z(x, g(x)) \neq 0$. □

We use Theorem 1.1 to derive a lower bound for $m \geq 1$ below. Let $\varphi_i(x,z) = 0$, $i = 1, \dots, m$, $x \in \mathbf{C}^n$, $z \in \mathbf{C}^m$ be analytic functions at $(x^{(0)}, z^{(0)})$. Let

$$J_m(x^{(0)}, z^{(0)}) := \begin{vmatrix} \frac{\partial \varphi_1(x^{(0)}, z^{(0)})}{\partial z_1} & \cdots & \frac{\partial \varphi_1(x^{(0)}, z^{(0)})}{\partial z_m} \\ \vdots & & \vdots \\ \frac{\partial \varphi_m(x^{(0)}, z^{(0)})}{\partial z_1} & \cdots & \frac{\partial \varphi_m(x^{(0)}, z^{(0)})}{\partial z_m} \end{vmatrix} = a_m \neq 0 \tag{1.3}$$

and let

$$\left|\varphi_i(x^{(0)}, z_1, \ldots, z_m)\right| \leq M, \text{ for } i = 1, \ldots, m \tag{1.4}$$

on

$$B = \{(x, z_1, \ldots, z_m) \mid \|(x, z) - (x^{(0)}, z^{(0)})\| \leq R\}. \tag{1.5}$$

Since $J_m(x^{(0)}, z^{(0)}) \neq 0$, some $(m-1) \times (m-1)$ sub-determinant in the matrix corresponding to $J_m(x^{(0)}, z^{(0)})$ must be nonzero. Without loss of generality, we may assume that

$$J_{m-1}(x^{(0)}, z^{(0)}) := \begin{vmatrix} \frac{\partial \varphi_2(x^{(0)}, z^{(0)})}{\partial z_2} & \cdots & \frac{\partial \varphi_2(x^{(0)}, z^{(0)})}{\partial z_m} \\ \vdots & & \vdots \\ \frac{\partial \varphi_m(x^{(0)}, z^{(0)})}{\partial z_2} & \cdots & \frac{\partial \varphi_m(x^{(0)}, z^{(0)})}{\partial z_m} \end{vmatrix} = a_{m-1} \neq 0. \tag{1.6}$$

By induction we conclude that there exist analytic functions $\psi_2(x, z_1), \ldots, \psi_m(x, z_1)$ and that we can compute a $\Theta_{m-1}(x^{(0)}, z_1^{(0)}; \varphi_2, \ldots, \varphi_m) > 0$ such that

$$\varphi_i(x, z_1, \psi_2(x, z_1), \ldots, \psi_m(x, z_1)) = 0, \quad i = 2, \ldots, m$$

in

$$\mathbf{D}_{n+1} := \{(x, z_1) \mid \|(x, z_1) - (x^{(0)}, z_1^{(0)})\| \leq \Theta_{m-1}(x^{(0)}, z_1^{(0)}; \varphi_2, \ldots, \varphi_m)\}.$$

Define

$$\Gamma(x, z_1) := \varphi_1(x, z_1, \psi_2(x, z_1), \ldots, \psi_m(x, z_1)). \tag{1.7}$$

Then we have

$$\frac{\partial \Gamma}{\partial z_1} = \frac{\partial \varphi_1}{\partial z_1} + \sum_{i=2}^{m} \frac{\partial \varphi_1}{\partial z_i} \cdot \frac{\partial \psi_i}{\partial z_1}. \tag{1.8}$$

Since $\varphi_2(x, z_1, \psi_2, \dots, \psi_m) = 0, \dots, \varphi_m(x, z_1, \psi_2, \dots, \psi_m) = 0$ in $\mathbf{D}_{n+1}$, differentiating with respect to $z_1$ we have

$$\frac{\partial \varphi_i}{\partial z_1} = \frac{\partial \varphi_i}{\partial z_1} + \sum_{j=2}^{m} \frac{\partial \varphi_i}{\partial z_j} \cdot \frac{\partial \psi_j}{\partial z_1} = 0; \quad i = 2, \dots, m$$

or in other words

$$\begin{bmatrix} \frac{\partial \varphi_2}{\partial z_2} & \cdots & \frac{\partial \varphi_2}{\partial z_m} \\ \vdots & & \vdots \\ \frac{\partial \varphi_m}{\partial z_2} & \cdots & \frac{\partial \varphi_m}{\partial z_m} \end{bmatrix} \begin{bmatrix} \frac{\partial \psi_2}{\partial z_1} \\ \vdots \\ \frac{\partial \psi_m}{\partial z_1} \end{bmatrix} = - \begin{bmatrix} \frac{\partial \varphi_2}{\partial z_1} \\ \vdots \\ \frac{\partial \varphi_m}{\partial z_1} \end{bmatrix}. \tag{1.9}$$

Using Cramer's rule and (1.9) we have

$$\frac{\partial \psi_i}{\partial z_1} = - \frac{\begin{vmatrix} \frac{\partial \varphi_2}{\partial z_2} & \cdots & \frac{\partial \varphi_2}{\partial z_{i-1}} & \frac{\partial \varphi_2}{\partial z_1} & \frac{\partial \varphi_2}{\partial z_{i+1}} & \cdots & \frac{\partial \varphi_2}{\partial z_m} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial \varphi_m}{\partial z_2} & \cdots & \frac{\partial \varphi_m}{\partial z_{i-1}} & \frac{\partial \varphi_m}{\partial z_1} & \frac{\partial \varphi_m}{\partial z_{i+1}} & \cdots & \frac{\partial \varphi_m}{\partial z_m} \end{vmatrix}}{J_{m-1}}; \quad i = 2, \dots, m. \tag{1.10}$$

Substituting (1.10) into (1.8) and simplifying we get

$$\frac{\partial\Gamma(x^{(0)},z_1^{(0)})}{\partial z_1} = \frac{\begin{vmatrix} \frac{\partial\varphi_1(x^{(0)},z^{(0)})}{\partial z_1} & \cdots & \frac{\partial\varphi_1(x^{(0)},z^{(0)})}{\partial z_m} \\ \vdots & & \vdots \\ \frac{\partial\varphi_m(x^{(0)},z^{(0)})}{\partial z_1} & \cdots & \frac{\partial\varphi_m(x^{(0)},z^{(0)})}{\partial z_m} \end{vmatrix}}{J_{m-1}(x^{(0)},z^{(0)})}$$
$$= \frac{J_m(x^{(0)},z^{(0)})}{J_{m-1}(x^{(0)},z^{(0)})} = \frac{a_m}{a_{m-1}} \neq 0.$$

Therefore we can apply Theorem 1.1 to $\Gamma(x,z_1)$ and conclude that there exists an implicit function $z_1 = g_1(x)$ in

$$\mathbf{D}_n := \Bigg\{ x \in \mathbf{C}^n \Big| \|x - x^{(0)}\| \leq \Theta_1\left(M, \frac{a_m}{a_{m-1}}, \min\left(R, \Theta_{m-1}(x^{(0)},z_1^{(0)};\varphi_2,\dots,\varphi_m)\right);\varphi_1\right)\Bigg\}$$

such that in $\mathbf{D}_n$, $\varphi_i(x,g_1(x),g_2(x),\dots,g_m(x)) = 0$, $i = 1,\dots,m$ where $g_j(x) := \psi_j(x,g_1(x))$, $j = 2,\dots,m$.

In summary, the sought lower bound on the size of the analytic domain of implicit functions is expressed recursively as

$$\text{(1.11)}\quad \Theta_m(x^{(0)},z^{(0)};\varphi_1,\dots,\varphi_m) = \Theta_1\left(M, \frac{a_m}{a_{m-1}}, \min(R, \Theta_{m-1}(x^{(0)},z_1^{(0)};\varphi_2,\dots,\varphi_m));\varphi_1\right)$$

using the definition of $\Theta_1$ in Theorem 1.1 and of $M, a_m, a_{m-1}$ and $R$ in equations (1.4), (1.3), (1.6) and (1.5) respectively.

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