

Journal of Inequalities in Pure and Applied Mathematics

BEURLING VECTORS OF QUASIELLIPTIC SYSTEMS OF DIFFERENTIAL OPERATORS

RACHID CHAILI

Département de Mathématiques,
U.S.T.O. Oran, Algérie.

EMail: chaili@univ-usto.dz

©2000 Victoria University
ISSN (electronic): 1443-5756
061-03



volume 4, issue 5, article 85,
2003.

*Received 12 May, 2003;
accepted 27 June, 2003.*

Communicated by: D. Hinton

[Abstract](#)

[Contents](#)



[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)

Abstract

We show the iterate property in Beurling classes for quasielliptic systems of differential operators.

2000 Mathematics Subject Classification: 35H30, 35H10

Key words: Beurling vectors, Quasielliptic systems, Differential operators

Contents

1	Introduction	3
2	Preliminary Lemmas	6
3	Local Estimates	9
4	The Main Result	14
	References	



Beurling Vectors of Quasielliptic Systems of Differential Operators

Rachid Chaili

Title Page

Contents



Go Back

Close

Quit

Page 2 of 19

1. Introduction

The aim of this work is to show the iterate property in Beurling classes for quasielliptic systems of differential operators. This property is proved for elliptic systems in [2]. A synthesis of results on the iterate problem is given in [1].

Let $(m_1, \dots, m_n) \in \mathbb{Z}_+^n, m_j \geq 1, 1 \leq j \leq n$, we set $\mu = \prod_{j=1}^n m_j, m = \max\{m_j\}, q_j = \frac{m}{m_j}$ and $q = (q_1, \dots, q_n)$. If $\alpha \in \mathbb{Z}_+^n$ and $\beta \in \mathbb{Z}_+^n$, we denote $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n, D^\alpha = D_1^{\alpha_1} \circ \dots \circ D_n^{\alpha_n}$, where $D_j = \frac{1}{i} \cdot \frac{\partial}{\partial x_j}$, $\langle \alpha, q \rangle = \sum_{j=1}^n \alpha_j q_j$ and $\binom{\alpha}{\beta} = \prod_{j=1}^n \binom{\alpha_j}{\beta_j}$.

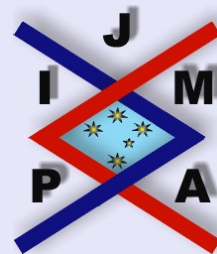
Let $(M_p)_{p=0}^{+\infty}$ be a sequence of real positive numbers such that

$$(1.1) \quad M_0 = 1, \exists a > 0, 1 \leq \frac{M_p}{M_{p-1}} \leq \frac{M_{p+1}}{M_p} \leq a^p, p \in \mathbb{Z}_+^*,$$

$$(1.2) \quad \exists b > 0, \exists c > 0, \quad c \binom{p}{j} M_{p-j} M_j \\ \leq M_p \leq b^p M_{p-j} M_j, \quad p, j \in \mathbb{Z}_+, j \leq p,$$

$$(1.3) \quad \forall m \geq 2, \exists d > 0, \forall p, h \in \mathbb{Z}_+, h \leq m; \\ (M_{pm})^{m-h} (M_{pm+m})^h \leq d (M_{pm+h})^m,$$

$$(1.4) \quad \forall m \geq 2, \exists H > 0, \forall p, h \in \mathbb{Z}_+, h \leq p; \frac{M_{pm}}{M_{hm}} \leq H^{p-h} \left(\frac{M_p}{M_h} \right)^m.$$



Let $(P_j(x, D))_{j=1}^N$ be q -quasihomogeneous differential operators of order m with C^∞ coefficients in an open subset Ω of \mathbb{R}^n , i.e.

$$P_j(x, D) = \sum_{\langle \alpha, q \rangle \leq m} a_{j\alpha}(x) D^\alpha.$$

We define the quasiprincipal symbol of the operator $P_j(x, D)$ by

$$P_{jm}(x, \xi) = \sum_{\langle \alpha, q \rangle = m} a_{j\alpha}(x) \xi^\alpha.$$

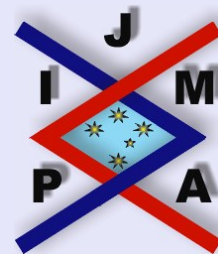
Definition 1.1. The system $(P_j)_{j=1}^N$ is said q -quasielliptic in Ω if for each $x_0 \in \Omega$ we have

$$(1.5) \quad \sum_{j=1}^N |P_{jm}(x_0, \xi)| \neq 0, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

Definition 1.2. Let $M = (M_p)$ be a sequence satisfying (1.1) – (1.4), the space of Beurling vectors of the system $(P_j(x, D))_{j=1}^N$ in Ω , denoted $B_M(\Omega, (P_j)_{j=1}^N)$, is the space of $u \in C^\infty(\Omega)$ such that $\forall K$ compact of $\Omega, \forall L > 0, \exists C > 0, \forall k \in \mathbb{Z}_+$,

$$(1.6) \quad \|P_{i_1} \dots, P_{i_k} u\|_{L^2(K)} \leq CL^{km} M_{km},$$

where $1 \leq i_l \leq N, l \leq k$.



**Beurling Vectors of
Quasielliptic Systems of
Differential Operators**

Rachid Chaili

Title Page

Contents



Go Back

Close

Quit

Page 4 of 19

Definition 1.3. Let $l = (l_1, \dots, l_n) \in \mathbb{R}_+^n$ and M be a sequence satisfying (1.1) – (1.4), we call anisotropic Beurling space in Ω , denoted $B_M^l(\Omega)$, the space of $u \in C^\infty(\Omega)$ such that $\forall K$ compact of $\Omega, \forall L > 0, \exists C > 0, \forall \alpha \in Z_+^n$,

$$(1.7) \quad \|D^\alpha u\|_{L^2(K)} \leq CL^{<\alpha, l>} \prod_{j=1}^n (M_{\alpha_j})^{l_j}.$$

Remark 1.1. If $l_j = 1, j = 1, \dots, n$, we obtain, thanks to (1.2) the definition of isotropic Beurling space $B_M(\Omega)$, (see [4]).

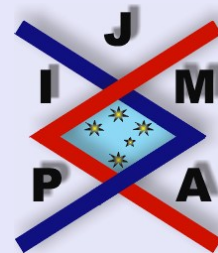
The principal result of this work is the following theorem:

Theorem 1.1. Let M and M' be two sequences satisfying (1.1) – (1.4) and

$$(1.8) \quad \lim_{p \rightarrow +\infty} \sum_{h=0}^p \frac{M'_{hm} M_{pm+m}}{M_{hm} M'_{pm+m}} = 0.$$

Let $(P_j)_{j=1}^N$ be q -quasielliptic system with $B_M^q(\Omega)$ coefficients, then

$$B_{M'}\left(\Omega, (P_j)_{j=1}^N\right) \subset B_{M'}^q(\Omega).$$



Beurling Vectors of
Quasielliptic Systems of
Differential Operators

Rachid Chaili

Title Page

Contents



Go Back

Close

Quit

Page 5 of 19

2. Preliminary Lemmas

Let ω be an open neighbourhood of the origin, we set $\mathcal{K} = \{k = \langle \alpha, q \rangle, \alpha \in \mathbb{Z}_+^n\}$ and we define

$$|u|_{k,\omega} = \sum_{\langle \alpha, q \rangle = k} \|D^\alpha u\|_{L^2(\omega)}, \quad u \in C^\infty(\omega), \quad k \in \mathcal{K}.$$

If $\rho > 0$ we set

$$B_\rho = \left\{ x \in \mathbb{R}^n, \left(\sum_{j=1}^n (x_j)^{\frac{2}{q_j}} \right)^{\frac{1}{2}} < \rho \right\}.$$

The two following lemmas are in [6].

Lemma 2.1. *Let $u \in C^\infty(\Omega)$, $r \in \mathcal{K}$ and $p \in \mathbb{Z}_+$, then*

$$(2.1) \quad |u|_{pm+r,\omega} \leq \sum_{\langle \alpha, q \rangle = pm} |D^\alpha u|_{r,\omega}.$$

Lemma 2.2. *Let $k = pm + r < pm + jm$, where $k, r \in \mathcal{K}$ and $p, j \in \mathbb{Z}_+^*$, then $\exists c(j) > 0, \forall B_\rho \subset \omega, \forall \varepsilon \in]0, 1[, \forall u \in C^\infty(\omega)$,*

$$(2.2) \quad |u|_{k,B_\rho} \leq \varepsilon |u|_{(p+j)m,B_\rho} + c(j) \varepsilon^{-\frac{r}{jm-r}} |u|_{pm,B_\rho}.$$

If $a \in C^\infty(\omega)$, we denote $[a, D^\alpha]u = D^\alpha(au) - aD^\alpha u$ and if P is a differential operator, we define $[P, D^\alpha]u = D^\alpha(Pu) - P(D^\alpha u)$.



Beurling Vectors of
Quasielliptic Systems of
Differential Operators

Rachid Chaili

Title Page

Contents



Go Back

Close

Quit

Page 6 of 19

Lemma 2.3. Let B be a bounded subset of \mathbb{R}^n and $a \in B_M^q(\overline{B})$, then $\forall L > 0, \exists C > 0, \forall u \in C^\infty(\overline{B}), \forall p \in \mathbb{Z}_+^*$,

$$(2.3) \quad \sum_{\langle \alpha, q \rangle = pm} |[a, D^\alpha] u|_{0, B} \leq C \sum_{\substack{k \leq pm-1 \\ k \in \mathcal{K}}} L^{pm-k} \left(\frac{M_{pm\mu}}{M_{k\mu}} \right)^{\frac{1}{\mu}} |u|_{k, B}.$$

Proof. Let $L > 0$, as $a \in B_M^q(\overline{B})$, there exists $C_1 > 0$ such that

$$|D^\alpha a| \leq C_1 L^{\langle \alpha, q \rangle} \prod_{j=1}^n (M_{\alpha_j})^{q_j}, \quad \forall \alpha \in \mathbb{Z}_+^n,$$

therefore, with the Leibniz formula, we get

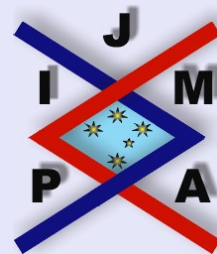
$$(2.4) \quad |[a, D^\alpha] u|_{0, B} \leq \sum_{\beta < \alpha} \binom{\alpha}{\beta} |D^\beta u|_{0, B} C_1 L^{\langle \alpha - \beta, q \rangle} \prod_{j=1}^n (M_{\alpha_j - \beta_j})^{q_j}.$$

We need the following easy inequality

$$(2.5) \quad \binom{\alpha}{\beta} \leq \left(\prod_{j=1}^n \binom{\alpha_j}{\beta_j} \right)^{q_j \mu} \leq \left(\left(\begin{array}{c} \langle \alpha, q \rangle \mu \\ \langle \beta, q \rangle \mu \end{array} \right) \right)^{\frac{1}{\mu}}.$$

It is easy to check that from condition (1.2) we have

$$(2.6) \quad c^{l-1} \prod_{j=1}^l M_{h_j} \leq M_{\sum_{j=1}^l h_j} \leq b^{(l-1)\sum_{j=1}^l h_j} \prod_{j=1}^l M_{h_j},$$



**Beurling Vectors of
Quasielliptic Systems of
Differential Operators**

Rachid Chaili

Title Page

Contents



Go Back

Close

Quit

Page 7 of 19

hence

$$\prod_{j=1}^n (M_{\alpha_j - \beta_j})^{q_j \mu} \leq \left(\frac{1}{c}\right)^{\sum_{j=1}^n q_j \mu - 1} M.$$

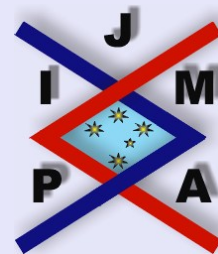
This inequality with (1.2) and (2.5) imply

$$(2.7) \quad \binom{\alpha}{\beta} \prod_{j=1}^n (M_{\alpha_j - \beta_j})^{q_j} \leq \left(\frac{1}{c}\right)^{\sum_{j=1}^l q_j} \left(\frac{M_{\langle \alpha, q \rangle \mu}}{M_{\langle \beta, q \rangle \mu}}\right)^{\frac{1}{\mu}} |u|_{k, B}.$$

As the number of $\alpha \in \mathbb{Z}_+^*$ satisfying $\langle \alpha, q \rangle = pm$ and $\alpha > \beta$, is limited by $C_2^{pm - \langle \beta, q \rangle}$, where C_2 depends only of n , then (2.4) and (2.7) give

$$\sum_{\langle \alpha, q \rangle = pm} |[a, D^\alpha] u|_{0, B} \leq \sum_{\substack{k \leq pm - 1 \\ k \in \mathcal{K}}} C_1 \left(\frac{1}{c}\right)^{\sum_{j=1}^l q_j} (C_2 L)^{pm - k} \left(\frac{M_{pm\mu}}{M_{k\mu}}\right)^{\frac{1}{\mu}} |u|_{k, B},$$

from which the desired estimate is obtained. □



Beurling Vectors of Quasielliptic Systems of Differential Operators

Rachid Chaili

Title Page

Contents



Go Back

Close

Quit

Page 8 of 19

3. Local Estimates

Let $(P_j)_{j=1}^N$ be a q -quasielliptic system with coefficients in $B_M^q(\overline{B})$, where B is a neighbourhood of the origin. The following lemma is a light modification of an analogous lemma in [6, Lemma 2.3].

Lemma 3.1. *Let ω be a small neighbourhood of the origin, $\rho > 0$ and $\delta \in]0, 1[$, such that $\overline{B}_{\rho+\delta} \subset \omega$. Then there exists $C > 0$, not depending on ρ and δ , such that for any $u \in C^\infty(\overline{\omega})$,*

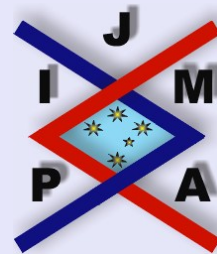
$$(3.1) \quad |u|_{m, B_\rho} \leq C \left(\sum_{j=1}^N |P_j u|_{0, B_{\rho+\delta}} + \sum_{\substack{k \leq m-1 \\ k \in \mathcal{K}}} \delta^{-m+k} |u|_{k, B_{\rho+\delta}} \right).$$

Lemma 3.2. *Let ω , ρ and δ be as in Lemma 3.1, then $\exists C > 0, \forall L > 0, \exists A > 0, \forall p \in \mathbb{Z}_+^*, \forall u \in C^\infty(\overline{\omega})$*

$$(3.2) \quad |u|_{(p+1)m, B_\rho} \leq C \left(\sum_{j=1}^N |P_j u|_{pm, B_{\rho+\delta}} + \delta^{-m} |u|_{pm, B_{\rho+\delta}} + \frac{1}{(4e)^m} |u|_{(p+1)m, B_{\rho+\delta}} + A \sum_{h=0}^p L^{(p+1-h)m} \frac{M_{pm+m}}{M_{hm}} |u|_{hm, B_{\rho+\delta}} \right),$$

and

$$(3.3) \quad |u|_{m, B_\rho} \leq C \left(\sum_{j=1}^N |P_j u|_{0, B_{\rho+\delta}} + \delta^{-m} |u|_{0, B_{\rho+\delta}} + \frac{1}{(4e)^m} |u|_{m, B_{\rho+\delta}} \right).$$



Proof. From (2.1) and (3.1) we obtain

$$(3.4) \quad |u|_{(p+1)m, B_\rho} \leq C \left(\sum_{j=1}^N |P_j u|_{pm, B_{\rho+\delta}} + \sum_{j=1}^N \sum_{\langle \alpha, q \rangle = pm} |[P_j, D^\alpha] u|_{0, B_{\rho+\delta}} + \sum_{\substack{k \leq m-1 \\ k \in \mathcal{K}}} \delta^{-m+k} |u|_{pm+k, B_{\rho+\delta}} \right),$$

Following the same idea as in the proof of Lemma 2.2 of [2], we get

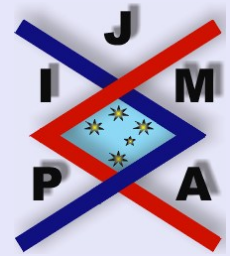
$$(3.5) \quad \sum_{\langle \alpha, q \rangle = pm} |[P_j, D^\alpha] u|_{0, B_{\rho+\delta}} \leq C' \sum_{\substack{s \leq pm+m-1 \\ s \in \mathcal{K}}} L^{pm+m-s} \left(\frac{M_{(pm+m)\mu}}{M_{s\mu}} \right)^{\frac{1}{\mu}} |u|_{s, B_{\rho+\delta}}.$$

On the other hand, there exists $h \in \mathbb{Z}_+$ and $r \in \mathcal{K}$ such that $s = hm + r$, $r < nm - n$, (see [6, (1.3)]). As $s \leq pm + m - 1$, then $h \leq p$. From (2.2) we have

$$(3.6) \quad |u|_{s, B_{\rho+\delta}} \leq \varepsilon |u|_{(h+n)m, B_{\rho+\delta}} + C_2 \varepsilon^{-\frac{r}{nm-r}} |u|_{hm, B_{\rho+\delta}}$$

if $s = hm + r$, where $0 \leq h \leq p - n + 1$ and $0 \leq r < nm - n$, and

$$(3.7) \quad |u|_{s, B_{\rho+\delta}} \leq \varepsilon |u|_{pm+m, B_{\rho+\delta}} + C_2 \varepsilon^{-\frac{r}{jm-r}} |u|_{hm, B_{\rho+\delta}}$$



**Beurling Vectors of
Quasielliptic Systems of
Differential Operators**

Rachid Chaili

Title Page

Contents



Go Back

Close

Quit

Page 10 of 19

if $s = hm + r$ where $h = p + 1 - j$, $1 \leq j \leq n - 1$ and $0 \leq r \leq jm - 1$.

Let $\varepsilon' \in]0, 1[$ and put

$$\varepsilon = \varepsilon' \left(\frac{M_{s\mu}}{M_{(h+n)m\mu}} \right)^{\frac{1}{\mu}} L^{-nm+r} \text{ in (3.6)}$$

and

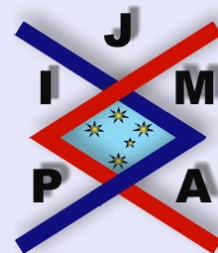
$$\varepsilon = \varepsilon' \left(\frac{M_{s\mu}}{M_{(p+1)m\mu}} \right)^{\frac{1}{\mu}} L^{-jm+r} \text{ in (3.7).}$$

According to (1.3) we obtain for any s satisfying (3.6),

$$\begin{aligned} \frac{L^{-s}}{(M_{s\mu})^{\frac{1}{\mu}}} |u|_{s, B_{\rho+\delta}} &\leq \varepsilon' \frac{L^{-(h+n)m}}{(M_{(h+n)m\mu})^{\frac{1}{\mu}}} |u|_{(h+n)m, B_{\rho+\delta}} \\ &+ C_2 d' \varepsilon'^{-m} \frac{L^{-hm}}{(M_{hm\mu})^{\frac{1}{\mu}}} |u|_{hm, B_{\rho+\delta}} \end{aligned}$$

and for any s satisfying (3.7),

$$\begin{aligned} \frac{L^{-s}}{(M_{s\mu})^{\frac{1}{\mu}}} |u|_{s, B_{\rho+\delta}} &\leq \varepsilon' \frac{L^{-(p+1)m}}{(M_{(p+1)m\mu})^{\frac{1}{\mu}}} |u|_{(p+1)m, B_{\rho+\delta}} \\ &+ C_2 d'' \varepsilon'^{-nm} \frac{L^{-hm}}{(M_{hm\mu})^{\frac{1}{\mu}}} |u|_{hm, B_{\rho+\delta}}. \end{aligned}$$



**Beurling Vectors of
Quasielliptic Systems of
Differential Operators**

Rachid Chaili

Title Page

Contents

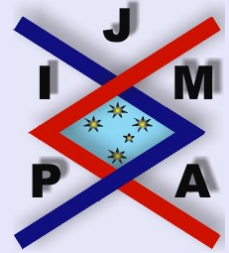


Go Back

Close

Quit

Page 11 of 19



These inequalities and (3.5) give

$$\sum_{\langle \alpha, q \rangle = pm} |[P_j, D^\alpha] u|_{0, B_{\rho+\delta}} \leq C' \left(n \varepsilon' |u|_{(p+1)m, B_{\rho+\delta}} + c (\varepsilon') \sum_{h=0}^p L^{(p+1-h)m} \left(\frac{M_{(pm+m)\mu}}{M_{hm\mu}} \right)^{\frac{1}{\mu}} |u|_{hm, B_{\rho+\delta}} \right).$$

Choosing $\varepsilon' = (2CC'Nn(4e)^m)^{-1}$, then we obtain, with (1.4),

$$\sum_{J=1}^N \sum_{\langle \alpha, q \rangle = pm} |[P_j, D^\alpha] u|_{0, B_{\rho+\delta}} \leq \frac{1}{2C} \frac{1}{(4e)^m} |u|_{(p+1)m, B_{\rho+\delta}} + A \sum_{h=0}^p (HL)^{(p+1-h)m} \frac{M_{pm+m}}{M_{hm}} |u|_{hm, B_{\rho+\delta}}.$$

It follows from this inequality: $\forall L > 0, \exists A > 0,$

$$(3.8) \quad \sum_{J=1}^N \sum_{\langle \alpha, q \rangle = pm} |[P_j, D^\alpha] u|_{0, B_{\rho+\delta}} \leq \frac{1}{2C} \frac{1}{(4e)^m} |u|_{(p+1)m, B_{\rho+\delta}} + A \sum_{h=0}^p L^{(p+1-h)m} \frac{M_{pm+m}}{M_{hm}} |u|_{hm, B_{\rho+\delta}}.$$

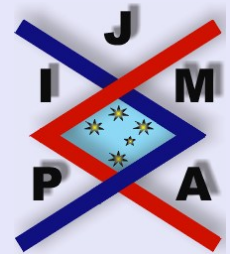
It remains the estimate of the third term of the right-hand side of (3.4). From (2.2), we have

$$|u|_{pm+k, B_{\rho+\delta}} \leq \varepsilon |u|_{pm+m, B_{\rho+\delta}} + C_2 \varepsilon^{-\frac{k}{m-k}} |u|_{pm, B_{\rho+\delta}}.$$

Setting $\varepsilon = \varepsilon' \delta^{m-k}$ and choosing $\varepsilon' = (2C_1 C (4e)^m)^{-1}$, then we obtain

$$(3.9) \quad \sum_{\substack{k \leq m-1 \\ k \in \mathcal{K}}} \delta^{-m+k} |u|_{pm+k, B_{\rho+\delta}} \leq \frac{1}{2C} \frac{1}{(4e)^m} |u|_{(p+1)m, B_{\rho+\delta}} + C'_2 \delta^{-m} |u|_{pm, B_{\rho+\delta}}.$$

The estimates (3.4), (3.8) and (3.9) imply (3.2). The estimate (3.3) is obtained from (3.1) and (3.9) with $p = 0$. \square



**Beurling Vectors of
Quasielliptic Systems of
Differential Operators**

Rachid Chaili

Title Page

Contents



Go Back

Close

Quit

Page 13 of 19

4. The Main Result

Let $R > 0$, to every sequence M satisfying (1.1) – (1.4) we define

$$\sigma_M^p(u) = \frac{1}{M_{pm}} \sup_{R/2 \leq \rho < R} (R - \rho)^{pm} |u|_{pm, B_\rho}.$$

The following lemma is in [2].

Lemma 4.1. *Let ω be as in Lemma 3.1, $R \in]0, 1[$ such that $\overline{B}_R \subset \omega$, M, M' two sequences satisfying (1.1) – (1.4) and $u \in B_{M'}(\overline{\omega}, (P_j)_{j=1}^N)$, then for any $L > 0$, there exists an increasing positive sequence $(C_p)_{p=0}^{+\infty}$ such that $\forall p, l \in \mathbb{Z}_+$,*

$$(4.1) \quad \sigma_{M'}^p(P_{i_0} \cdots P_{i_l} u) \leq C_p \frac{M'_{pm+lm}}{M'_{pm}} L^{pm+lm}.$$

where the sequence (C_p) is constructed by recurrence,

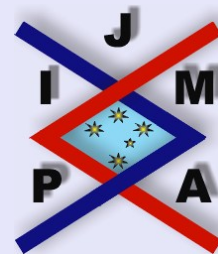
$$C_{p+1} = C_p \left(NC + A \sum_{h=0}^p \frac{M'_{hm}}{M_{hm}} \frac{M_{pm+m}}{M'_{pm+m}} \right),$$

where C and A are the constants of Lemma 3.2 and C_0 is the constant satisfying

$$\|P_{i_0} \cdots P_{i_l} u\|_{L^2(B_R)} \leq C_0 L^{lm} M'_{lm}.$$

Theorem 4.2. *Let M and M' be two sequences satisfying (1.1) – (1.4) and*

$$(4.2) \quad \lim_{p \rightarrow +\infty} \sum_{h=0}^p \frac{M'_{hm}}{M_{hm}} \frac{M_{pm+m}}{M'_{pm+m}} = 0.$$



Let $(P_j)_{j=1}^N$ be q -quasielliptic system with coefficients in $B_M^q(\Omega)$, then

$$B_{M'}\left(\Omega, (P_j)_{j=1}^N\right) \subset B_{M'}^q(\Omega).$$

Proof. We must verify (1.7) near every point x of Ω . By a translation of x at the origin, there exists a neighbourhood ω of the origin for which the precedent lemmas are true. Let $L > 0$ and let $(C_p)_{p=0}^{+\infty}$ be as in Lemma 4.1, then from (4.2) there exists $p_0 \in \mathbb{Z}_+$ such that $C_{p+1} \leq 2NCC_p$, $p \geq p_0$, hence

$$C_p \leq C_{p_0} (2NC)^{p-p_0} \leq C_{p_0} (2NC)^{pm+lm}, \quad \forall l \in \mathbb{Z}_+.$$

For $p \leq p_0$, this inequality is true because the sequence $(C_p)_{p=0}^{+\infty}$ is increasing. Let $R \in]0, 1[$ such that $\overline{B}_R \subset \omega$, from (4.1) we obtain

$$\sigma_{M'}^p(P_{i_0} \cdots P_{i_l} u) \leq C_{p_0} \frac{M'_{pm+lm}}{M'_{pm}} (2NCL)^{pm+lm}, \quad \forall p, l \in \mathbb{Z}_+.$$

In particular for $l = 0$,

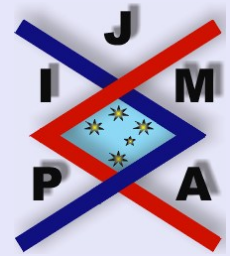
$$\left(\frac{R}{2}\right)^{pm} \frac{1}{M'_{pm}} |u|_{pm, B_{R/2}} \leq \sigma_{M'}^p(u) \leq C_{p_0} (2NCL)^{pm},$$

hence

$$|u|_{pm, B_{R/2}} \leq C_{p_0} \left(\frac{4NC}{R} L\right)^{pm} M'_{pm},$$

which can be rewritten as

$$(4.3) \quad \forall L > 0, \exists C > 0, |u|_{pm, B_{R/2}} \leq CL^{pm} M'_{pm}.$$



**Beurling Vectors of
Quasielliptic Systems of
Differential Operators**

Rachid Chaili

Title Page

Contents



Go Back

Close

Quit

Page 15 of 19

The last inequality will allow us to conclude. In fact let $k \in \mathcal{K}$, then there exists $p \in \mathbb{Z}_+$ and $r \in \mathcal{K}$, $r < nm - n$, such that $k = pm + r$. From (2.2), (4.3) and (2.6), we obtain

$$\begin{aligned} |u|_{k, B_{R/2}} &\leq \varepsilon C' L^{(p+n)m} M'_{(p+n)m} + C' C'' \varepsilon^{-\frac{r}{nm-r}} L^{pm} M'_{pm} \\ &\leq \varepsilon C' L^{(p+n)m} \frac{1}{c} (M'_{(p+n)m\mu})^{\frac{1}{\mu}} + C' C'' \varepsilon^{-\frac{r}{nm-r}} L^{pm} \frac{1}{c} (M'_{pm\mu})^{\frac{1}{\mu}}. \end{aligned}$$

Setting

$$\varepsilon = \left(\frac{M'_{(pm+r)\mu}}{M'_{(p+n)m\mu}} \right)^{\frac{1}{\mu}} L^{-nm+r},$$

then from (1.3) we get

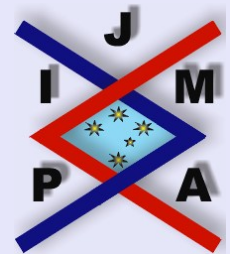
$$(4.4) \quad |u|_{k, B_{R/2}} \leq C_1 L^k (M'_{k\mu})^{\frac{1}{\mu}}.$$

By an imbedding theorem of anisotropic Sobolev spaces (see [5]), from (4.4) and (1.2) we obtain

$$\sup_{B_{R/2}} |D^\alpha u(x)| \leq C_2 (bL)^{\langle \alpha, q \rangle} (M'_{\langle \alpha, q \rangle \mu})^{\frac{1}{\mu}}.$$

The last estimate, with (2.6) gives

$$\sup_{B_{R/2}} |D^\alpha u(x)| \leq C_3 (bL)^{\langle \alpha, q \rangle} \left(b^{\langle \alpha, q \rangle n \mu} \prod_{j=1}^n M'_{\alpha_j q_j \mu} \right)^{\frac{1}{\mu}}$$



**Beurling Vectors of
Quasielliptic Systems of
Differential Operators**

Rachid Chaili

Title Page

Contents



Go Back

Close

Quit

Page 16 of 19

$$\begin{aligned} &\leq C_3 (bL)^{\langle\alpha,q\rangle} \left(b^{\langle\alpha,q\rangle n\mu} \prod_{j=1}^n b^{q_j\mu\langle\alpha_j,q_j\rangle} \left(M'_{\alpha_j} \right)^{q_j\mu} \right)^{\frac{1}{\mu}} \\ &\leq C_3 \left(b^{(1+n+m\mu)} L \right)^{\langle\alpha,q\rangle} \prod_{j=1}^n \left(M'_{\alpha_j} \right)^{q_j}, \end{aligned}$$

from there $u \in B_{M'}^q (B_{R/2})$. □

As a corollary we obtain from Theorem 1.1, the principal result of [2]. Theorem 1.1 also gives a result of regularity of solutions of differential equations in Beurling classes.

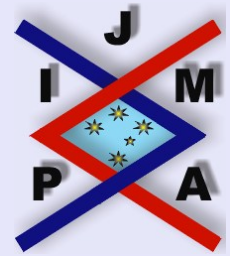
Corollary 4.3. *Under the assumptions of Theorem 1.1, the following assertions are equivalent:*

- i) $u \in \mathcal{D}'(\Omega)$ and $P_j u \in B_{M'}^q(\Omega)$,
- ii) $u \in B_{M'}^q(\Omega)$.

For anisotropic projective Gevrey classes $G^{\{s\},q}(\Omega) = B_M^q(\Omega)$, $M_p = (p!)^s$, $s \geq 1$, we have the same result.

Corollary 4.4. *Let s, s' be such that $s' > s \geq 1$ and $(P_j)_{j=1}^N$ q -quasielliptic system with coefficients in $G^{\{s\},q}(\Omega)$, then*

$$G^{\{s\}} \left(\Omega, (P_j)_{j=1}^N \right) \subset G^{\{s'\},q}(\Omega).$$



Beurling Vectors of Quasielliptic Systems of Differential Operators

Rachid Chaili

Title Page

Contents



Go Back

Close

Quit

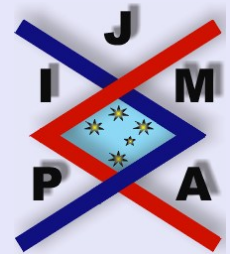
Page 17 of 19

Corollary 4.5. Let M and M' be two sequences satisfying (1.1) – (1.4) and

$$(4.5) \quad \lim_{p \rightarrow +\infty} \sum_{h=0}^p \frac{M'_{hm}}{M_{hm}} \frac{M_{pm+m}}{M'_{pm+m}} = 0,$$

and let $(P_j)_{j=1}^N$ be an elliptic system with coefficients in $B_M(\Omega)$, then

$$B_{M'}\left(\Omega, (P_j)_{j=1}^N\right) \subset B_{M'}(\Omega).$$



**Beurling Vectors of
Quasielliptic Systems of
Differential Operators**

Rachid Chaili

Title Page

Contents



Go Back

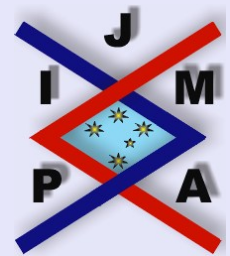
Close

Quit

Page 18 of 19

References

- [1] P. BOLLEY, J. CAMUS AND L. RODINO, Hypocoellipticité analytique-Gevrey et itérés d'opérateurs, *Ren. Sem. Mat. Univers. Politec. Torino*, **45**(3) (1989), 1–61.
- [2] C. BOUZAR AND R. CHAILI, Régularité des vecteurs de Beurling de systèmes elliptiques, *Maghreb Math. Rev.*, **9**(1-2) (2000), 43–53.
- [3] C. BOUZAR AND R. CHAILI, Vecteurs Gevrey d'opérateurs différentiels qusihomogènes, *Bull. Belg. Math. Soc.*, **9** (2002), 299–310.
- [4] J.L. LIONS AND E. MAGENES, *Problèmes aux Limites Non Homogènes et Applications*, t. 3, Dunod Paris, 1970.
- [5] P.I. LIZORKIN, Nonisotropic Bessel potentials. Imbedding theorems for Sobolev spaces $L_p^{(r_1, \dots, r_n)}$ with fractional derivatives, *Soviet Math. Dokl.*, **7**(5) (1966), 1222–1226.
- [6] L. ZANGHIRATI, Iterati di operatori quasi-ellittici e classi di Gevrey, *Boll. U.M.I.*, **5**(18-B) (1981), 411–428.



Beurling Vectors of
Quasielliptic Systems of
Differential Operators

Rachid Chaili

Title Page

Contents



Go Back

Close

Quit

Page 19 of 19