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INEQUALITIES INVOLVING THE INNER PRODUCT

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ABSTRACT. The paper contains inequalities related to generalizations of Schwarz's inequality.

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1. INTRODUCTION

Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space over the field $K = \mathbb{R}$ or $K = \mathbb{C}$. Then for all $x, a, b \in H$ the following inequality holds:

$$\left|\langle a, x\rangle\langle x, b\rangle - \frac{1}{2}\langle a, b\rangle \|x\|^2\right| \leq \frac{1}{2}\|a\|\,\|b\|\,\|x\|^2. \tag{1.1}$$

In particular, for $a = b$ (1.1) reduces to Schwarz's inequality.

For historical remarks, proofs, extensions, generalizations and applications of (1.1), see [1] – [5] and the references given therein.

In this paper we consider a suitable quadratic form and derive inequalities related to (1.1). More precisely, we obtain estimates involving the real and the imaginary part of the expression whose absolute value is contained in the left hand of (1.1).

2. THE RESULTS

Let $v_1, \dots, v_n$ $(n \geq 2)$ be linearly independent vectors in H, and $v := v_1 + \cdots + v_n$.

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Consider the matrix

$$A := \begin{pmatrix} \langle v_1, v - v_1\rangle & \langle v_2, v - v_1\rangle & \cdots & \langle v_n, v - v_1\rangle \\ \langle v_1, v - v_2\rangle & \langle v_2, v - v_2\rangle & \cdots & \langle v_n, v - v_2\rangle \\ \vdots & \vdots & \cdots & \vdots \\ \langle v_1, v - v_n\rangle & \langle v_2, v - v_n\rangle & \cdots & \langle v_n, v - v_n\rangle \end{pmatrix}$$

Theorem 2.1.

(i) *The matrix A has real eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$; moreover, $\lambda_1 \leq 0$ and $\lambda_n \geq 0$.*
(ii) *The following inequalities hold:*

$$\lambda_1 \|x\|^2 \leq \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \langle v_i, x\rangle \langle x, v_j\rangle \leq \lambda_n \|x\|^2, \quad x \in H. \tag{2.1}$$

Proof. Let H_n denote the linear subspace of H generated by $v_1, \ldots, v_n$. Consider the linear operator $T : H_n \to H_n$ defined by

$$Tx = \langle x, v\rangle v - \sum_{i=1}^{n} \langle x, v_i\rangle v_i, \quad x \in H_n.$$

Then for all $x, y \in H_n$ we have

$$\begin{aligned} \langle Tx, y\rangle &= \langle x, v\rangle\langle v, y\rangle - \sum_{i=1}^{n} \langle x, v_i\rangle\langle v_i, y\rangle \\ &= \overline{\langle y, v\rangle}\langle x, v\rangle - \sum_{i=1}^{n} \overline{\langle y, v_i\rangle}\langle x, v_i\rangle \\ &= \left\langle x, \langle y, v\rangle v - \sum_{i=1}^{n} \langle y, v_i\rangle v_i \right\rangle \\ &= \langle x, Ty\rangle. \end{aligned}$$

We conclude that T is self-adjoint, hence it has real eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ and:

$$\lambda_1 \|x\|^2 \leq \langle Tx, x\rangle \leq \lambda_n \|x\|^2, \quad x \in H_n. \tag{2.2}$$

On the other hand,

$$\begin{aligned} Tv_j &= \langle v_j, v\rangle v - \sum_{i=1}^{n} \langle v_j, v_i\rangle v_i \\ &= \sum_{i=1}^{n} (\langle v_j, v\rangle - \langle v_j, v_i\rangle) v_i = \sum_{i=1}^{n} \langle v_j, v - v_i\rangle v_i \end{aligned}$$

for all $j = 1, 2, \ldots, n$.

This means that A is the matrix of T with respect to the basis $\{v_1, v_2, \ldots, v_n\}$ of H_n, and so $\lambda_1 \leq \cdots \leq \lambda_n$ are the eigenvalues of the matrix A.

Now remark that

$$\begin{aligned}\langle Tx, x\rangle &= \langle x, v\rangle\langle v, x\rangle - \sum_{i=1}^{n}\langle x, v_i\rangle\langle v_i, x\rangle \\ &= \sum_{i=1}^{n}\langle v_i, x\rangle\sum_{i=1}^{n}\langle x, v_i\rangle - \sum_{i=1}^{n}\langle v_i, x\rangle\langle x, v_i\rangle \\ &= \sum_{\substack{i,j=1\\ i\neq j}}^{n}\langle v_i, x\rangle\langle x, v_j\rangle, \quad x \in H_n.\end{aligned}$$

Combined with (2.2), this gives

$$\lambda_1\|x\|^2 \le \sum_{\substack{i,j=1\\ i\neq j}}^{n}\langle v_i, x\rangle\langle x, v_j\rangle \le \lambda_n\|x\|^2, \quad x \in H_n. \tag{2.3}$$

Let $x \in H_n$, $x \neq 0$, $\langle x, v_i\rangle = 0$, $i = 1, 2, \ldots, n-1$.

From (2.3) we infer that

$$\lambda_1 \le 0 \le \lambda_n. \tag{2.4}$$

Let $y \in H$. Then $y = x + z$, $x \in H_n$, $z \in H_n^{\perp}$ and $\|y\|^2 = \|x\|^2 + \|z\|^2$, so that $\|y\|^2 \ge \|x\|^2$. Moreover,

$$\langle v_i, y\rangle = \langle v_i, x + z\rangle = \langle v_i, x\rangle, \quad i = 1, \ldots, n.$$

Using (2.3) and (2.4), we get

$$\lambda_1\|y\|^2 \le \lambda_1\|x\|^2 \le \sum_{\substack{i,j=1\\ i\neq j}}^{n}\langle v_i, y\rangle\langle y, v_j\rangle \le \lambda_n\|x\|^2 \le \lambda_n\|y\|^2$$

and this concludes the proof. □

Corollary 2.2. *Let* $a, b, x \in H$. *Then*

$$\left|\operatorname{Re}\left(\langle a, x\rangle\langle x, b\rangle - \frac{1}{2}\|x\|^2\langle a, b\rangle\right)\right| \le \frac{1}{2}\|x\|^2\sqrt{\|a\|^2\|b\|^2 - (\operatorname{Im}\langle a, b\rangle)^2}. \tag{2.5}$$

Proof. If a and b are linearly dependent, (2.5) can be verified directly. Otherwise it is a consequence of Theorem 2.1.

Indeed for $n = 2$, $v_1 = a$, $v_2 = b$, the eigenvalues of the matrix A are

$$\lambda_{1,2} = \operatorname{Re}\langle a, b\rangle \pm \sqrt{\|a\|^2\|b\|^2 - (\operatorname{Im}\langle a, b\rangle)^2}$$

and

$$\langle Tx, x\rangle = 2\operatorname{Re}\langle x, a\rangle\langle b, x\rangle, \quad x \in H.$$

□

Remark 2.3.

(i) When $K = \mathbb{R}$, (2.5) coincides with (1.1).

(ii) Let $K = \mathbb{C}$. Applying Corollary 2.2 to the vectors ia, b, x we get

$$\left|\operatorname{Im}\left(\langle a, x\rangle\langle x, b\rangle - \frac{1}{2}\|x\|^2\langle a, b\rangle\right)\right| \le \frac{1}{2}\|x\|^2\sqrt{\|a\|^2\|b\|^2 - (\operatorname{Re}\langle a, b\rangle)^2}. \tag{2.6}$$

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