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ON GENERALIZED MONOTONE MULTIFUNCTIONS WITH APPLICATIONS TO OPTIMALITY CONDITIONS IN GENERALIZED CONVEX PROGRAMMING

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ABSTRACT. Characterization of quasiconvexity and pseudoconvexity of lower semicontinuous functions on Banach spaces are presented in terms of abstract subdifferentials relying on a Mean Value Theorem. We give some properties of the normal cone to the lower level set of f. We also obtain necessary and sufficient optimality conditions in quasiconvex and pseudoconvex programming via variational inequalities.

Key words and phrases: Generalized monotone multifunction, Generalized convex function, Quasiconvex, Pseudoconvex, Generalized subdifferentials, Normal cone, Level set, Local minimum, Global minimum, Variational inequalities.

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1. INTRODUCTION

It is natural in convex analysis to search for characterizations of generalized convex functions in terms of some kind of generalized derivatives or subdifferentials. Several contributions to this question has been made recently. The reader may consult for example [3, 5, 11, 13, 16, 20] for quasiconvex functions and [2, 8, 21, 23, 25] for pseudoconvex functions.

In this paper, we shall define an abstract subdifferential as in [1, 23] which allows us to extend some results in [1, 2, 8, 23] and to give some properties of the normal cone to lower level sets of a given function f.

Notice that the condition $0 \in \partial f(\bar{x})$ for $\bar{x} \in X$, is known to be a necessary but not a sufficient optimality condition in quasiconvex programming for some subdifferentials. We give, using some variational inequalities, a necessary and sufficient condition for a point to be either a local or a global minimum.

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After the introduction of some notations and definitions in Section 2, we present in Section 3 some properties of the abstract subdifferential and normal cone to lower level sets of quasiconvex and pseudoconvex functions. Then, in Section 4, we give some optimality conditions involving variational inequalities. This should extend our previous results stated for quasiconvex lower semicontinuous functions on Banach spaces with the Clarke-Rockafellar subdifferential in [13].

2. PRELIMINARIES

Let X be a real Banach space, X^* its dual and $\langle \cdot, \cdot \rangle$ the duality pairing between X^* and X. The segment $[a, b]$ is the set $\{a+t(b-a); \quad t \in [0, 1]\}$ while $[a, b[$ is the set $[a, b] \setminus \{b\}$. The open ball with center x and radius r in X is denoted by $B(x, r)$, and the polar cone of a nonempty subset A of X is

$$A^\circ = \{x^* \in X^*; \quad \langle x^*, a \rangle \leq 0, \quad \forall a \in A\}.$$

For an extended real valued function $f : X \mapsto \mathbb{R} \cup \{+\infty\}$, the effective domain is defined by

$$\mathrm{dom}(f) = \{x \in X; \quad f(x) < \infty\}.$$

We write l.s.c. for lower semicontinuous, and $x_n \to_f x$ when $x_n \to x$ and $f(x_n) \to f(x)$.

The abstract subdifferential we consider here is defined as follows:

Definition 2.1. An operator ∂ that associates to any l.s.c. function $f : X \mapsto \mathbb{R} \cup \{+\infty\}$ and a point $x \in X$ a subset $\partial f(x)$ of X^* is a subdifferential if the following assertions hold:

- **(P1)** $\partial f(x) = \{\, x^* \in X^*; \quad f(y) \geq f(x) + \langle x^*, y - x \rangle \quad \forall y \in X \,\}$ when f is convex.
- **(P2)** If $x \in \mathrm{dom}\, f$ is a local minimum of f, then $0 \in \partial f(x)$.
- **(P3)** $\partial f(x) = \partial g(x)$, for any $g : X \mapsto \mathbb{R} \cup \{+\infty\}$ such that $f - g$ is constant in a neighborhood of x.
- **(P4)** $\partial f(x) = \emptyset$, for any $x \in X$ such that $f(x) = +\infty$.

It is well known that the Clarke-Rockafellar subdifferential $\partial^{CR} f$ satisfies Zagrodny's Mean value theorem [27]. In order to extend this theorem to our subdifferential, we shall deal with a particular space associated with ∂ called ∂-reliable.

Definition 2.2. [23]. A Banach space X is ∂-reliable if for each l.s.c. function $f : X \mapsto \mathbb{R} \cup \{+\infty\}$, for any Lipschitz convex function g and any $x \in \mathrm{dom}\, f$ such that $f + g$ achieves its minimum in X and each $\varepsilon > 0$ we have:

$$0 \in \partial f(u) + \partial g(v) + \varepsilon B_1^*(0),$$

for some $u, v \in B_\varepsilon(x)$ such that $|f(u) - f(v)| < \varepsilon$.

In the case of the Clarke-Rockafellar subdifferential ∂^{CR} [26] or Iofee subdifferential ∂^I [7], any Banach space is ∂-reliable.

In the sequel, we will restrict ourselves to subdifferentials that are included in the dag subdifferential

$$\partial^\dagger f(x) = \{\, x^* \in X^*; \quad \langle x^*, v \rangle \leq f^\dagger(x, v) \quad \forall v \in X\},$$

where

$$f^\dagger(x, v) = \limsup_{(t,y)\to(0_+,x)} t^{-1}(f(y + t(v + x - y) - f(y)).$$

This subdifferential was introduced by Penot (see [22]), it is large enough to contain the Clarke-Rockafellar ∂^{CR} and Upper Dini ∂^{D+} subdifferentials and still has good properties.

Our results rely on the following mean value theorem.

Theorem 2.1. [23]. *Let X be a ∂-reliable space and $f : X \mapsto \mathbb{R} \cup \{+\infty\}$ a l.s.c. function. For any $a \in \operatorname{dom} f$, $b \in X \setminus \{a\}$, $\beta \leq b$, there exists a sequence c_n in X converging to some $c \in [a, b)$ and a sequence $c_n^* \in \partial f(c_n)$ such that for any $b' = c + t(b - a)$, with $t > 0$ we have:*

i) $\liminf_n \langle c_n^*, b - a \rangle \geq \beta - f(a)$,
ii) $\liminf_n \langle c_n^*, c - c_n \rangle \geq 0$,
iii) $\liminf_n \left\langle c_n^*, \frac{||b-a||}{||b'-c||}(b' - c_n) \right\rangle \geq \beta - f(a)$.

Following the methods of [1, 16, 20], we get a similar lemma for our abstract subdifferential, which is immediate by Theorem 2.1.

Lemma 2.2. *Let X be a Banach ∂-reliable space, f a l.s.c. function. Let $a, b \in X$ with $f(a) < f(b)$ then there exists $c \in [a, b[$ and two sequences $c_n \to c$, $c_n^* \in \partial f(c_n)$ with*

$$\langle c_n^*, x - c_n \rangle > 0,$$

for any $x = c + t(b - a)$ with $t > 0$.

Proof. Let $a, b \in X$ with $f(a) < f(b)$, then we can find by Theorem 2.1, $c \in [a, b[$ and two sequences $c_n \to c, \quad c_n^* \in \partial f(c_n)$ with

$$\liminf_n \langle c_n^*, c - c_n \rangle \geq 0,$$

and

$$\liminf_n \langle c_n^*, b - a \rangle \geq f(b) - f(a) > 0.$$

For $x = c + t(b - a)$ with $t > 0$, we have

$$\langle c_n^*, x - c_n \rangle = \langle c_n^*, c - c_n \rangle + t \langle c_n^*, b - a \rangle.$$

It follows that

$$\liminf_n \langle c_n^*, x - c_n \rangle > 0.$$

Hence, for n large enough, we have that

$$\langle c_n^*, x - c_n \rangle > 0.$$

□

3. GENERALIZED CONVEX FUNCTIONS AND GENERALIZED MONOTONE MULTIFUNCTIONS

3.1. **Quasiconvex Functions and Quasimonotone Multifunctions.** We recall the characterization of quasiconvex functions of [22, 23]. It will allow us to extend and generalize some properties of the normal cone to the lower level set given in [12, 13] to a more general setting.

Indeed, for $f : X \mapsto \mathbb{R} \cup \{+\infty\}$ a l.s.c. function, f is said to be quasiconvex if for every $x, y \in X$ and $\lambda \in [0, 1]$ one has

$$f(\lambda x + (1 - \lambda) y) \leq \max\{f(x), f(y)\}.$$

And denoting by

$$S_f(\lambda) = \{ x \in X; \quad f(x) \leq \lambda \}.$$

Quasiconvexity is geometrically equivalent to the fact that $S_f(\lambda)$ is a convex set for all $\lambda \in \mathbb{R}$. In the above one could use the strict level sets as well.

Recall that a multifunction $A : X \to X^*$ is said to be quasimonotone if for every pair of distinct points $x, y \in X$:

$$\exists x^* \in A(x), \quad \text{such that} \quad \langle x^*, y - x \rangle > 0 \quad then, \quad \forall y^* \in A(y), \quad \langle y^*, y - x \rangle \geq 0.$$

Theorem 3.1. [22, 23] *Let X be a Banach space and $f : X \mapsto \mathbb{R} \cup \{+\infty\}$ a l.s.c. function. And consider the following assertions*

i) *f is quasiconvex.*
ii) *∂f is quasimonotone.*

*Then **i)** implies **ii)** if $\partial f \subset \partial^\dagger f$. And **ii)** implies **i)** if X is ∂-reliable.*

For $x_0 \in X$, set

$$L(x_0) = \{x \in X; \quad f(x) = f(x_0)\}.$$

Then we have

Proposition 3.2. *Let X be a Banach ∂-reliable space, and f a l.s.c. quasiconvex function such that $\partial f \subset \partial^\dagger f$. If for $x_0 \in X$ there exists $r > 0$ with*

$$0 \notin \partial f(x), \quad \text{for all } x \in B(x_0, r) \cap L(x_0),$$

then we have

$$[\partial f(x_0)]^{\circ\circ} \subset N(S_f(f(x_0)); x_0),$$

where $N(S_f(f(x_0)); x_0)$ is the normal cone to the lower level set $S_f(f(x_0))$ at the point x_0.

Proof. Suppose by contradiction that there exists v such that

$$v \in [\partial f(x_0)]^{\circ\circ} \quad \text{and} \quad v \notin N(S_f(f(x_0)); x_0).$$

We can check that

$$Cl(\mathbb{R}_+ co(\partial f(x_0))) = [\partial f(x_0)]^{\circ\circ}.$$

So, we can suppose without loss of generality that $v = x_0^* \in \partial f(x_0)$. Then, we can find some $x_1 \in S_f(x_0)$ such that

$$\langle x_0^*, x_1 - x_0 \rangle > 0. \tag{3.1}$$

We claim that $f(x_0) = f(x_1)$. Otherwise by Lemma 2.2, there exists $c \in [x_1, x_0[$ and two sequences $c_n \to_f c$ and $c_n^* \in \partial f(c_n)$ with

$$\langle c_n^*, x_0 - c_n \rangle > 0.$$

By using the quasimonotonicity of ∂f we have:

$$\langle x_0^*, x_0 - c_n \rangle \geq 0.$$

Then, letting $n \to +\infty$ we get

$$\langle x_0^*, x_0 - c \rangle \geq 0.$$

It follows that

$$\langle x_0^*, x_0 - x_1 \rangle \geq 0.$$

A contradiction with (3.1), thus $f(x_0) = f(x_1)$.

Now, set $\quad V_{x_1} = \{x \in X : \langle x_0^*, x - x_0 \rangle > 0\}$.

V_{x_1} is an open neighborhood of x_1 and using the same argument as above we can check that x_1 is a minimum of f on V_{x_1}, and that

$$x_\lambda = x_0 + \lambda(x_1 - x_0) \in V_{x_1} \text{ and } f(x_\lambda) = f(x_0) \text{ for any } \lambda \in]0, 1[.$$

Then there exists $r > 0$ and $\bar{\lambda} \in]0, 1[$ such that $x_{\bar{\lambda}}$ is a global minimum of f on $B(x_0, r) \cap V_{x_1}$. Therefore $0 \in \partial f(x_{\bar{\lambda}})$, which is impossible. ☐

The former proposition extends some already known results for differentiable functions (see for instance [5]). If we denote by $T(S_f(f(x);x)$, the tangent cone of the lower level convex set $S_f(f(x))$ at the point $x \in X$, then

$$T(S_f(f(x));x) = [N(S_f(f(x));x)]^\circ.$$

A sufficient condition that allows us to obtain the equality in Proposition 3.2 is stated in the following proposition

Proposition 3.3. *Under the hypothesis of Proposition 3.2 and if*

$$[\partial f(x)]^\circ \subset T(S_f(f(x));x).$$

Then we have

$$N(S_f(f(x));x) = [\partial f(x)]^{\circ\circ}.$$

Proof. By the bipolar theorem [4] one has

$$[\partial f(x)]^{\circ\circ} \supset N(S_f(f(x));x).$$

And from Proposition 3.2, the equality immediately holds. □

The following condition

$$N(S_f(f(x));x) = [\partial f(x)]^{\circ\circ},$$

is in fact a certain kind of regularity condition, which holds only for a subclass of quasiconvex functions. Another abstract aproach was developed in [15] based on Crouzeix's representation theorem [6] who obtained a similar equality for his quasi-subdifferential.

Consider the multifunction Γ from X to X^* defined by

$$\Gamma(x) = N(S_f(f(x));x), \quad \text{for } x \in X.$$

Then by using Proposition 3.3, we obtain

Proposition 3.4. *Let X be a Banach ∂-reliable space, f a l.s.c. quasiconvex function. If for any $x \in X$, $\partial f(x)$ is nonempty such that*

$$(\partial f(x))^\circ \subset T(S_f(f(x));x).$$

Then, the multifunction Γ is quasimonotone.

Proof. Since f is quasiconvex, by Theorem 3.1 ∂f is quasimonotone. Using Proposition 2.8 of [12], it follows easily that the multifunction $x \mapsto [\partial f(x)]^{\circ\circ}$ is quasimonotone. Then by Proposition 3.3, Γ is also quasimonotone.

It follows that Γ is quasimonotone. □

A particular case of this proposition when ∂ coincides with the Clarke-Rockafellar subdifferential ∂^{CR}, was treated in [13], whose exact statement is the following.

Proposition 3.5. *Let X be a Banach space, f a l.s.c. function from X to $\mathbb{R} \cup \{+\infty\}$ such that $\partial^{CR} f(x)$ is nonempty and $0 \notin \partial^{CR} f(x)$ for all $x \in X$.*

If f is quasiconvex then the multifunction Γ is quasimonotone.

3.2. **Pseudoconvexity and Subdifferential Properties.** The original definition of pseudoconvexity was introduced by Mangazarian in [21] for differentiable functions. This concept was exended later by many authors (see for instance [17, 22, 24]) for arbitrary functions. We will here use the following form:

A function f is said to be pseudoconvex for the subdifferential ∂ if for any $x, y \in X$:

$$\exists x^* \in \partial f(x): \quad \langle x^*, y - x\rangle \geq 0 \implies f(x) \leq f(y).$$

A multifunction $A : X \to X^*$ is said to be pseudomonotone if for every pair of distinct points $x, y \in X$

$$\exists x^* \in A(x) : \langle x^*, y - x\rangle > 0 \quad \text{then,} \quad \forall y^* \in A(y),\ \langle y^*, y - x\rangle > 0.$$

As in the differentiable case, every pseudoconvex function satisfies the fundamental properties:

- every local minimum of f is global.
- $0 \in \partial f(x)$ implies that x is a global minimum of f.

Another interesting property extending a result of [8] where it was stated for the Clarke-Rockafellar subdifferential is the following.

Proposition 3.6. *Let X be a Banach ∂-reliable and $f : X \mapsto \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function and pseudoconvex function such that $\partial f \subset \partial^\dagger f$, let $x, y \in X$. Then the existence of $x^* \in \partial f(x)$ verifying $\langle x^*, y - x\rangle > 0$ implies $f(x) < f(y)$.*

Proof. Let $x, y \in X$ such that $\langle x^*, y - x\rangle > 0$ for some $x^* \in \partial f(x)$, then there exists $\varepsilon > 0$ such that

$$\langle x^*, y' - x\rangle > 0, \quad \forall y' \in B(y, \varepsilon).$$

By the pseudoconvexity of f, we have $f(y') \geq f(x)$.

In particular, $f(y) \geq f(x)$. If we suppose by contradiction that $f(y) = f(x)$, then y must be a global minimum. On the other hand, since $f^\dagger(x, y - x) > 0$ then, there exist two sequences $x_n \to x, t_n \to 0^+$ such that

$$t_n{}^{-1}\big[f(x_n + t_n(y - x_n) - f(x_n))\big] > 0.$$

By the quasiconvexity of the function f (see for instance the proof of Proposition 2.2 in [8]), we get $f(y) > f(x_n)$ which is impossible. $\square$

We use this proposition to prove a known result for the Clarke-Rockafellar subdifferential for bigger subdifferentials

Theorem 3.7. *Let X be a ∂-reliable space and $f : X \mapsto \mathbb{R} \cup \{+\infty\}$ a l.s.c. function such that $\partial f \subset \partial^\dagger f$. And consider the following assertions*

i) *f is pseudoconvex.*

ii) *∂f is pseudomonotone.*

Then, ***i)*** *implies* ***ii)****. And* ***ii)*** *implies* ***i)*** *if f is radially continuous.*

Proof. The implication **ii)** $\Longrightarrow$ **i)** is in [23]. For **i)** $\Longrightarrow$ **ii)**, suppose by contradiction that there exist $x, y \in X$, such that there exist $x^* \in \partial f(x)$ and $y^* \in \partial f(y)$ verifying

$$\langle x^*, y - x\rangle > 0 \quad \text{and} \quad \langle y^*, y - x\rangle \leq 0.$$

Then, from Proposition 3.5 and the pseudoconvexity of f we have

$$f(x) < f(y) \quad \text{and} \quad f(y) \leq f(x).$$

A contradiction. $\square$

Now, we state a similar result to Proposition 3.2 for pseudoconvex functions.

Proposition 3.8. *Let X be a Banach ∂-reliable space with $\partial \subset \partial^\dagger$, f a l.s.c. and pseudoconvex function from X to $\mathbb{R} \cup \{+\infty\}$. Then we have*

$$[\partial f(x)]^{\circ\circ} \subset N(S_f(f(x)); x).$$

Proof. Let $x^* \in \partial f(x)$ and suppose by contradiction that $x^* \notin N(S_f(f(x)); x)$. Then, there exists $y \in S_f(f(x))$ such that $\langle x^*, y - x\rangle > 0$ for some $x^* \in \partial f(x)$. It follows then by Proposition 3.6 that $f(y) > f(x)$, which is impossible. □

4. OPTIMALITY CONDITIONS AND VARIATIONAL INEQUALITIES

4.1. **Quasiconvex Programming.** We recall the Minty variational inequality (we use the terminology of Giannessi [9]) that we shall use for our subdifferential. It will be exploited to give some conditions of optimality in nonlinear programming and necessary and sufficient conditions for optimality in quasiconvex programming.

Let Γ be a multifunction from X to X^*, $S \subset X$ and $\bar{x} \in S$.

A point $\bar{x}$ is a Minty equilibrium of Γ if the following variational inequality holds

$$\text{(D)} \qquad \forall x \in S, \quad \langle \gamma(x), x - \bar{x}\rangle \geq 0, \qquad \forall \gamma(x) \in \Gamma(x).$$

Suppose that f is a l.s.c. function from X to $\mathbb{R} \cup \{+\infty\}$ and consider the following minimisation problem

$$\text{minimize } f(x), \text{ subject to } x \in C. \tag{4.1}$$

Then we have

Proposition 4.1. *Let X be a Banach ∂-reliable space. If $\bar{x}$ is a Minty equilibrium point of ∂f, then we have*

- **i)** *If $S = X$, then $\bar{x}$ is a global minimum of f.*
- **ii)** *If $S = N$, where N is a convex open neighborhood of $\bar{x}$ then $\bar{x}$ is a local minimum of f.*

Proof. It is enough to prove (ii). Suppose by contradiction that $\bar{x}$ is not a solution of the program (4.1), then there exists $x \in S$ such that $f(x) < f(\bar{x})$. By Lemma 2.2, there exists $c \in [x, \bar{x}[$ and two sequences $c_n \to_f c$, $c_n^* \in \partial f(c_n)$ with

$$\langle c_n^*, d - c_n\rangle > 0,$$

for any $d = c + t(\bar{x} - x)$ where $t > 0$.

Since S is a convex open neighborhood of $\bar{x}$ then $[x, \bar{x}] \subset S$. Furthermore, for n large enough $c_n \in S$.

In the particular case where $d = \bar{x}$, we have:

$$\langle c_n^*, \bar{x} - c_n\rangle > 0.$$

A contradiction with the variational inequality (D), thus $\bar{x}$ is a local minimum of f. □

This proposition extends Theorem 2.2 of [18] for nondifferentiable optimization problems.

If in the problem (4.1), the function f to be minimized is l.s.c. and quasiconvex, then we have

Theorem 4.2. *Let X be a Banach ∂-reliable, and f be a l.s.c. and quasiconvex function such that $\partial f \subset \partial^\dagger f$, and $\bar{x} \in S$. If $S = N$, where N is an open and convex neighborhood of $\bar{x}$ or $S = X$, then the following assertions are equivalent*

- **i)** *$\bar{x}$ is an optimal solution of (4.1).*
- **ii)** *$\bar{x}$ is a Minty equilibrium point of ∂f.*

Proof. **ii)** $\Longrightarrow$**i)** is obtained from Proposition 4.1. Let us show that
i) $\Longrightarrow$ **ii)**. Assume that $\bar{x}$ is a strict minimum of (4.1), then for all $x \in S$ such that $x \neq \bar{x}$ we have $f(x) > f(\bar{x})$.

According to Lemma 2.2, there exist $c \in [\bar{x}, x[$, $c_n \to_f c$ and $c_n^* \in \partial f(c_n)$ such that

$$\langle c_n^*, d - c_n \rangle > 0,$$

for all $d = c + t(x - \bar{x})$ where $t > 0$.

When $d = x$, we obtain that

$$\langle c_n^*, x - c_n \rangle > 0.$$

f being quasiconvex, by Theorem 2.1, ∂f is quasimonotone. It follows then that

$$\text{for all } x^* \in \partial f(x), \qquad \langle x^*, x - \bar{x} \rangle \geq 0.$$

Hence, ∂f satisfies the variational inequality (D).

Suppose that we are in the case where $\bar{x}$ is not a strict minimum of (4.1) and let us consider the function $f_{\bar{x}}$ defined by

$$f_{\bar{x}}(x) = \max\{f(x), f(\bar{x})\},$$

and define h by

$$h(x) = \begin{cases} f_{\bar{x}}(x) & \text{for } x \neq \bar{x} \\ \nu & \text{for } x = \bar{x} \end{cases} \tag{4.2}$$

where $\nu < f(\bar{x})$. We see easily that h is l.s.c. and quasiconvex and that $\bar{x}$ is a strict local minimum of h. Then, we have

$$\forall x \neq \bar{x} \quad \langle x^*, x - \bar{x} \rangle \geq 0, \quad \forall x^* \in \partial h(x).$$

From $(P3)$, we get $\partial f(x) = \partial h(x)$. □

In the case when 0 is in the interior of $\partial f(\bar{x})$, i.e. $0 \in \operatorname{int}(\partial f(\bar{x}))$, we have the more precise result

Proposition 4.3. *Let X be a ∂-reliable space and $f : X \mapsto \mathbb{R} \cup \{+\infty\}$ a l.s.c. and quasiconvex function. If $0 \in \operatorname{int}(\partial f(\bar{x}))$ then $\bar{x}$ is a Minty equilibrium point of ∂f. Moreover $\bar{x}$ is a global minimum of f.*

Proof. Assume that $0 \in \operatorname{int}(\partial f(x))$ then

$$\text{there exists } \varepsilon > 0 \text{ such that } B_{X^*}(0, \varepsilon) \subset \partial f(x),$$

where

$$B_{X^*}(0, \varepsilon) = \{\, x^* \in X^* : \quad \|x^*\| < \varepsilon\}.$$

Let $d \in X$ such that $d \neq 0$ and consider the linear mapping ℓ_d defined by

$$\ell_d(x^*) = \langle x^*, d \rangle, \qquad \text{for } x^* \in X^*.$$

By the open mapping Theorem [4] one has

$$\langle B_{X^*}(0, \varepsilon), d \rangle \subset \langle \partial f(x), d \rangle.$$

Since f is quasiconvex, then ∂f is quasimonotone.

We already know by Definition 2.1 of [12] that the multifunction $\partial f_{x,d}$ defined by

$$\partial f_{x,d}(\lambda) = \langle \partial f(x + \lambda d), d \rangle,$$

is quasimonotone, and we can see easily that

$$\langle \lambda d, \partial f(x + \lambda d) \rangle \subset \mathbb{R}_+,$$

for all $\lambda \in \mathbb{R}$ and $d \in X \setminus \{0\}$, thus (D) holds for ∂f. □

4.2. **Pseudoconvex Programming.** For the pseudoconvex function f, we shall get necessary and sufficient conditions for a point $\bar{x}$ to be a global extremum of f over a convex set C.

First consider the problem (4.1), with f is pseudoconvex, l.s.c. and radially continuous, then we have

Theorem 4.4. *Let X be a Banach space ∂-reliable, and f a pseudoconvex l.s.c. such that $\partial f \subset \partial^\dagger f$, and let $\bar{x} \in C$. Then the following assertions are equivalent*

- **i)** *$\bar{x}$ is an optimal solution of (4.1).*
- **ii)** *(D) holds.*

Proof. Suppose that $\bar{x}$ is a solution of (4.1), then by Proposition 3.6, if $f(\bar{x}) \leq f(x)$, then we must have

$$\forall x^* \in \partial f(x), \quad \langle x^*, \bar{x} - x \rangle \leq 0.$$

This means that the variational inequality (D) holds.

Converesly, let $x \in C$ such that $x \neq \bar{x}$ then for some $y \in (\bar{x}, x)$, we have

$$\forall y^* \in \partial f(y), \quad \langle y^*, \bar{x} - y \rangle \leq 0.$$

It follows that

$$\forall y^* \in \partial f(y), \quad \langle y^*, x - y \rangle \leq 0.$$

Since $\partial f(y)$ is nonempty and from the pseudoconvexity of f we have

$$f(y) \leq f(x), \quad \forall y \in (\bar{x}, x).$$

But since f is s.c.i., then $f(\bar{x}) \leq f(x)$. □

We now proceed to the maximisation problem

$$\text{maximize } f(x), \text{ subject to } x \in C. \tag{4.3}$$

For $z \in C$, we denote by

$$C_z = \{x \in C; \quad f(x) = f(z)\}.$$

Then we have

Theorem 4.5. *Let X be a ∂-reliable space and f a pseudoconvex, l.s.c. and radially continuous such that for any x in C, $\partial f(x)$ is nonempty and $\partial f(x) \subset \partial^\dagger f(x)$. Let $\bar{x} \in C$ such that*

$$-\infty \leq \inf_C f < f(\bar{x}).$$

Then $\bar{x}$ is a maximum of f on C if and only if

$$\textit{for all } x \in C_{\bar{x}}, \quad \partial f(x) \subset N(C, x).$$

Proof. Suppose that

$$f(y) \leq f(\bar{x}); \quad \forall y \in C.$$

By Proposition 3.6 we have:

$$\text{for all } x \in C_{\bar{x}}, \quad \partial f(x) \subset N(C, x).$$

Conversely, by contradiction assume that there exists $\bar{z} \in C$ such that

$$f(\bar{z}) > f(\bar{x}).$$

Since by hypothesis, we can find some $z \in C$ with $f(z) < f(\bar{x})$.

By the radial continuity of f, there exists some $x_0 \in (z, \bar{z})$ such that

$$f(x_0) = f(\bar{x}).$$

It follows then that

$$\text{for all } x_0^* \in \partial f(x_0), \quad \langle x_0^*, z - x_0 \rangle = 0.$$

Since f is pseudoconvex then, $f(x_0) \leq f(z)$, a contradiction. $\square$

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