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A TWO-SIDED ESTIMATE OF $e^x - (1 + x/n)^n$

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Abstract

In this paper we refine an old inequality of G. N. Watson related to the formula $e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$.

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The exponential function can be defined by the formula

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n,$$

the convergence being uniform on compact subsets of $\mathbb{R}$. The "speed" of convergence is discussed in many places, including the classical book of D. S. Mitrinović [2], where the following formulae are presented:

$$0 \le e^x - \left(1+\frac{x}{n}\right)^n \le \frac{x^2 e^x}{n} \quad \text{for } |x| < n \text{ and } n \in \mathbb{N}^\star$$

$$0 \le e^{-x} - \left(1-\frac{x}{n}\right)^n \le \frac{x^2(1+x)e^{-x}}{2n} \quad \text{for } 0 \le x \le n,\ n \in \mathbb{N},\ n \ge 2$$

$$0 \le e^{-x} - \left(1-\frac{x}{n}\right)^n \le \frac{x^2}{2n} \quad \text{for } 0 \le x \le n,\ n \in \mathbb{N}^\star.$$

Here $\mathbb{N}^\star$ stands for the set of positive naturals.

See [3], [4], [8], [9], [10] for history, applications and related results. As noticed by G.N. Watson in [9], the first inequality yields a quick proof of the equivalence of two basic definitions of the Gamma function. In fact, for $x > 0$ it yields

$$\lim_{n\to\infty} \int_0^n s^{x-1} \left(1 - \frac{s}{n}\right)^n ds = \int_0^\infty s^{x-1} e^{-s} ds,$$

while a small computation shows that the integral on the left is equal to

$$\frac{n! n^x}{x(x+1)\cdots(x+n)}.$$

The aim of the present note is to prove stronger estimates.

Theorem 1.

$i)$ *If* $x > 0$, $t > 0$ *and* $t > \frac{1-x}{2}$, *then*

$$\frac{x^2 e^x}{2t + x + \max\{x, x^2\}} < e^x - \left(1 + \frac{x}{t}\right)^t < \frac{x^2 e^x}{2t + x}.$$

$ii)$ *If* $x > 0$, $t > 0$ *and* $t > \frac{x-1}{2}$, *then*

$$\frac{x^2 e^{-x}}{2t - x + x^2} < e^{-x} - \left(1 - \frac{x}{t}\right)^t < \frac{x^2 e^{-x}}{2t - 2x + \min\{x, x^2\}}.$$

For $x = 1$ and $t = n \in \mathbb{N}^\star$ the inequalities $i)$ yield,

$$\frac{e}{2n+2} < e - \left(1 + \frac{1}{n}\right)^n < \frac{e}{2n+1},$$

which constitutes Problem 170 in G. Pólya and G. Szegö [6].

For $x = 1$ and $t = n \in \mathbb{N}^\star$ the inequalities $ii)$ read as

$$\frac{1}{2n\,e} < \frac{1}{e} - \left(1 - \frac{1}{n}\right)^n < \frac{1}{(2n-1)e}$$

and this fact improves the result of Problem B3 given at the 63rd *Annual William Lowell Putnam Mathematical Competition*. See [5]. Needless to say, the solutions presented in [1] and [11] both missed the question of whether the original pair of inequalities are optimal or not.

The result of Theorem 1 above can be easily extended for positive elements in a $C^\star$-algebra (particularly in $M_n(\mathbb{R})$). This is important since the solution $u \in C^1([0,\infty), \mathbb{R}^n)$ of the differential system

$$\text{(1)} \qquad \begin{cases} \dfrac{du}{dt} + Au = 0 & \text{for } t \in [0,\infty) \\ u(0) = u_0 \end{cases}$$

for $A \in M_n(\mathbb{R})$, has an exponential representation,

$$\begin{aligned} u(t) &= e^{-tA}u_0 \\ &= \lim_{n\to\infty} \left(I - \frac{t}{n}A\right)^n u_0. \end{aligned}$$

Since $e^{-tA} = \left(e^{tA}\right)^{-1}$, we can rewrite $u(t)$ as

$$\text{(2)} \qquad u(t) = \lim_{n\to\infty} \left[\left(I + \frac{t}{n}A\right)^{-1}\right]^n u_0.$$

This led K. Yosida [7] to his semigroup approach of evolution equations:

Theorem 2. *Let E be a Banach space and let $A : \mathcal{D}(A) \subset E \to E$ be a densely defined linear operator such that for every $\lambda > 0$, the operator $I + \lambda A$ is a bijection between $\mathcal{D}(A)$ and E with $\left\|(I + \lambda A)^{-1}\right\| \leq 1$.*

Then for every $u_0 \in \mathcal{D}(A)$ the formula (2) provides the unique solution $u \in C^1([0,\infty), E) \cap C([0,\infty), \mathcal{D}(A))$ of the Cauchy problem (1).

It is unclear up to what extent an analogue of Theorem 1 is valid in the context of unbounded generators A.

Proof of Theorem 1. We shall detail here only the case $i)$. The case $ii)$ can be treated in a similar way.

We shall need the Harmonic, Logarithmic and Arithmetic Mean Inequality,

$$\frac{2uv}{v+u} < \frac{v-u}{\ln v - \ln u} < \frac{u+v}{2}, \quad \text{for every } u, v > 0,\ u < v,$$

from which we get the following two-sided estimate

$$\frac{2x}{2t+x} < \ln(t+x) - \ln t < \frac{(2t+x)x}{2t(t+x)}, \quad \text{for every } t, x > 0. \tag{3}$$

The left-hand side inequality in $i)$ is equivalent to

$$u(t) := \frac{2t + x + \max\{x, x^2\}}{2t + x + \max\{x, x^2\} - x^2}\left(1 + \frac{x}{t}\right)^t < \mathrm{e}^x \tag{4}$$

for $t > \max\left\{0, \frac{1-x}{2}\right\}$.

If the parameter x belongs to $(0,1]$, then

$$u(t)=\frac{2t+2x}{2t+2x-x^2}\left(1+\frac{x}{t}\right)^t,$$

so that

$$\begin{aligned}u'(t)&=\left[\left(\ln(t+x)-\ln t-\frac{x}{t+x}\right)\frac{2t+2x}{2t+2x-x^2}-\frac{2x^2}{(2t+2x-x^2)^2}\right]\left(1+\frac{x}{t}\right)^t\\&>\left[\left(\frac{2x}{2t+x}-\frac{x}{t+x}\right)\frac{2t+2x}{2t+2x-x^2}-\frac{2x^2}{(2t+2x-x^2)^2}\right]\left(1+\frac{x}{t}\right)^t\\&=\frac{2x^3(1-x)}{(2t+2x-x^2)^2(2t+x)}\left(1+\frac{x}{t}\right)^t\geq 0\end{aligned}$$

by the left-hand side inequality in (3). Therefore the function $u(t)$ is increasing. As $\lim_{t\to\infty}u(t)=e^x$, this proves (4) for $x\in(0,1]$.

For $x\geq 1$, the inequality (4) reads

$$u(t)=\frac{2t+x+x^2}{2t+x}\left(1+\frac{x}{t}\right)^t<e^x \quad \text{for every } t>0.$$

In this case,

$$u'(t)=\left[\left(\ln(t+x)-\ln t-\frac{x}{t+x}\right)\frac{2t+x+x^2}{2t+x}-\frac{2x^2}{(2t+x)^2}\right]\left(1+\frac{x}{t}\right)^t$$

and the left part of (3) yields

$$\begin{aligned}u'(t)&>\left[\left(\frac{2x}{2t+x}-\frac{x}{t+x}\right)\frac{2t+x+x^2}{2t+x}-\frac{2x^2}{(2t+x)^2}\right]\left(1+\frac{x}{t}\right)^t\\&=\frac{x^3(x-1)}{(2t+x)^2(t+x)}\left(1+\frac{x}{t}\right)^t\geq 0\end{aligned}$$

since $t > 0$ and $x \geq 1$. Then $u(t)$ is increasing and thus

$$u(t) < \lim_{t\to\infty} u(t) = e^x$$

for every $t > 0$ and every $x \geq 1$. Hence (4), and this shows that the left-hand side inequality in $i)$ holds for every $x > 0$.

The right-hand side inequality in Theorem 1 $i)$ is equivalent to

$$e^x < \frac{2t+x}{2t+x-x^2}\left(1+\frac{x}{t}\right)^t = v(t)$$

for every $x > 0$ and every $t > \max\left\{0, \frac{1-x}{2}\right\}$. Again, we shall use a monotonicity argument. According to the right-hand side inequality in (3) we have

$$\begin{aligned} v'(t) &= \left[\left(\ln(t+x) - \ln t - \frac{x}{t+x}\right)\frac{2t+x}{2t+x-x^2} - \frac{2x^2}{(2t+x-x^2)^2}\right]\left(1+\frac{x}{t}\right)^t \\ &< \left[\left(\frac{(2t+x)x}{2t(t+x)} - \frac{x}{t+x}\right)\frac{2t+x}{2t+x-x^2} - \frac{2x^2}{(2t+x-x^2)^2}\right]\left(1+\frac{x}{t}\right)^t \\ &= \frac{x^4(-2t+(1-x))}{2t(t+x)(2t+x-x^2)^2} < 0 \end{aligned}$$

from which we infer that $v(t)$ is decreasing. Consequently,

$$v(t) > \lim_{t\to\infty} v(t) = e^x$$

for every $x > 0$ and every $t > \max\left\{0, \frac{1-x}{2}\right\}$. Thus also the right-hand side inequality in $i)$ holds and the proof is complete. □

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