

HADAMARD TYPE INEQUALITIES FOR m -CONVEX AND (α, m) -CONVEX FUNCTIONS

M. KLARIČIĆ BAKULA

Department of Mathematics
Faculty of Science
University of Split
Teslina 12, 21000 Split, Croatia
EMail: milica@pmfst.hr

M. E. ÖZDEMİR

Atatürk University
K. K. Education Faculty
Department of Mathematics
25240 Kampüs, Erzurum, Turkey
EMail: emos@atauni.edu.tr

J. PEČARIĆ

Faculty of Textile Technology
University of Zagreb
Pierottijeva 6, 10000 Zagreb, Croatia
EMail: pecaric@hazu.hr

Received: 03 March, 2008

Accepted: 31 July, 2008

Communicated by: E. Neuman

2000 AMS Sub. Class.: 26D15, 26A51.

Key words: m -convex functions, (α, m) -convex functions, Hadamard's inequalities

**Inequalities for m -Convex and
 (α, m) -Convex Functions**

M. Klaričić Bakula, M. E. Özdemir
and J. Pečarić

vol. 9, iss. 4, art. 96, 2008

[Title Page](#)

[Contents](#)



Page 1 of 25

[Go Back](#)

[Full Screen](#)

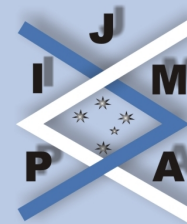
[Close](#)

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

Abstract:

In this paper we establish several Hadamard type inequalities for differentiable m -convex and (α, m) -convex functions. We also establish Hadamard type inequalities for products of two m -convex or (α, m) -convex functions. Our results generalize some results of B.G. Pachpatte as well as some results of C.E.M. Pearce and J. Pečarić.



Inequalities for m -Convex and (α, m) -Convex Functions

M. Klaričić Bakula, M. E. Özdemir
and J. Pečarić

vol. 9, iss. 4, art. 96, 2008

[Title Page](#)

[Contents](#)



Page 2 of 25

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

Contents

1	Introduction	4
2	Inequalities for m -Convex Functions	9
3	Inequalities for (α, m) -Convex Functions	18



**Inequalities for m -Convex and
 (α, m) -Convex Functions**

M. Klaričić Bakula, M. E. Özdemir
and J. Pečarić

vol. 9, iss. 4, art. 96, 2008

Title Page

Contents



Page 3 of 25

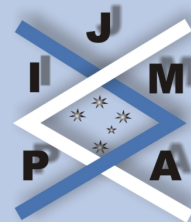
Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



[Title Page](#)

[Contents](#)

[◀◀](#) [▶▶](#)

[◀](#) [▶](#)

Page 4 of 25

[Go Back](#)

[Full Screen](#)

[Close](#)

1. Introduction

The following definitions are well known in literature.

Let $[0, b]$, where b is greater than 0, be an interval of the real line \mathbb{R} , and let $K(b)$ denote the class of all functions $f : [0, b] \rightarrow \mathbb{R}$ which are continuous and nonnegative on $[0, b]$ and such that $f(0) = 0$.

We say that the function f is *convex* on $[0, b]$ if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in [0, b]$ and $t \in [0, 1]$. Let $K_C(b)$ denote the class of all functions $f \in K(b)$ convex on $[0, b]$, and let $K_F(b)$ be the class of all functions $f \in K(b)$ *convex in mean* on $[0, b]$, that is, the class of all functions $f \in K(b)$ for which $F \in K_C(b)$, where the mean function F of the function $f \in K(b)$ is defined by

$$F(x) = \begin{cases} \frac{1}{x} \int_0^x f(t) dt, & x \in (0, b]; \\ 0, & x = 0. \end{cases}$$

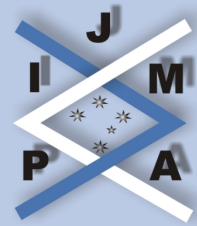
Let $K_S(b)$ denote the class of all functions $f \in K(b)$ which are *starshaped* with respect to the origin on $[0, b]$, that is, the class of all functions f with the property that

$$f(tx) \leq tf(x)$$

holds for all $x \in [0, b]$ and $t \in [0, 1]$. In [1] Bruckner and Ostrow, among others, proved that

$$K_C(b) \subset K_F(b) \subset K_S(b).$$

In [9] G. Toader defined *m-convexity*: another intermediate between the usual convexity and starshaped convexity.



[Title Page](#)

[Contents](#)

[◀◀](#) [▶▶](#)

[◀](#) [▶](#)

Page 5 of 25

[Go Back](#)

[Full Screen](#)

[Close](#)

Definition 1.1. The function $f : [0, b] \rightarrow \mathbb{R}$, $b > 0$, is said to be m -convex, where $m \in [0, 1]$, if we have

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$. We say that f is m -concave if $-f$ is m -convex. Denote by $K_m(b)$ the class of all m -convex functions on $[0, b]$ for which $f(0) \leq 0$.

Obviously, for $m = 1$ Definition 1.1 recaptures the concept of standard convex functions on $[0, b]$, and for $m = 0$ the concept of starshaped functions.

The following lemmas hold (see [10]).

Lemma A. If f is in the class $K_m(b)$, then it is starshaped.

Lemma B. If f is in the class $K_m(b)$ and $0 < n < m \leq 1$, then f is in the class $K_n(b)$.

From Lemma A and Lemma B it follows that

$$K_1(b) \subset K_m(b) \subset K_0(b),$$

whenever $m \in (0, 1)$. Note that in the class $K_1(b)$ are only convex functions $f : [0, b] \rightarrow \mathbb{R}$ for which $f(0) \leq 0$, that is, $K_1(b)$ is a proper subclass of the class of convex functions on $[0, b]$.

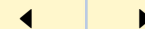
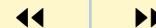
It is interesting to point out that for any $m \in (0, 1)$ there are continuous and differentiable functions which are m -convex, but which are not convex in the standard sense (see [11]).

In [3] S.S. Dragomir and G. Toader proved the following Hadamard type inequality for m -convex functions.



Title Page

Contents



Page 6 of 25

Go Back

Full Screen

Close

Theorem A. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be an m -convex function with $m \in (0, 1]$. If $0 \leq a < b < \infty$ and $f \in L^1([a, b])$ then

$$(1.1) \quad \frac{1}{b-a} \int_a^b f(x) dx \leq \min \left\{ \frac{f(a) + mf\left(\frac{b}{m}\right)}{2}, \frac{f(b) + mf\left(\frac{a}{m}\right)}{2} \right\}.$$

Some generalizations of this result can be found in [4].

The notion of m -convexity has been further generalized in [5] as it is stated in the following definition:

Definition 1.2. The function $f : [0, b] \rightarrow \mathbb{R}$, $b > 0$, is said to be (α, m) -convex, where $(\alpha, m) \in [0, 1]^2$, if we have

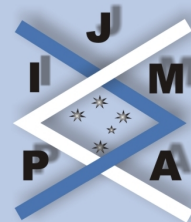
$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha) f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

Denote by $K_m^\alpha(b)$ the class of all (α, m) -convex functions on $[0, b]$ for which $f(0) \leq 0$.

It can be easily seen that for $(\alpha, m) \in \{(0, 0), (\alpha, 0), (1, 0), (1, m), (1, 1), (\alpha, 1)\}$ one obtains the following classes of functions: increasing, α -starshaped, starshaped, m -convex, convex and α -convex functions respectively. Note that in the class $K_1^1(b)$ are only convex functions $f : [0, b] \rightarrow \mathbb{R}$ for which $f(0) \leq 0$, that is $K_1^1(b)$ is a proper subclass of the class of all convex functions on $[0, b]$. The interested reader can find more about partial ordering of convexity in [8, p. 8, 280].

In [2] in order to prove some inequalities related to Hadamard's inequality S. S. Dragomir and R. P. Agarwal used the following lemma.



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 7 of 25

Go Back

Full Screen

Close

Lemma C. Let $f : I \rightarrow \mathbb{R}$, $I \subset \mathbb{R}$, be a differentiable mapping on $\overset{\circ}{I}$, and $a, b \in I$, where $a < b$. If $f' \in L^1([a, b])$, then

$$(1.2) \quad \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx = \frac{b-a}{2} \int_0^1 (1-2t) f'(ta + (1-t)b) dt.$$

Here $\overset{\circ}{I}$ denotes the interior of I .

In [7], using the same Lemma C, C.E.M. Pearce and J. Pečarić proved the following theorem.

Theorem B. Let $f : I \rightarrow \mathbb{R}$, $I \subset \mathbb{R}$, be a differentiable mapping on I^0 , and $a, b \in I$, where $a < b$. If $|f'|^q$ is convex on $[a, b]$ for some $q \geq 1$, then

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}$$

and

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}.$$

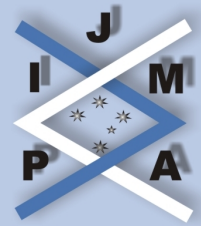
In [6] B. G. Pachpatte established two new Hadamard type inequalities for products of convex functions. They are given in the next theorem.

Theorem C. Let $f, g : [a, b] \rightarrow [0, \infty)$ be convex functions on $[a, b] \subset \mathbb{R}$, where $a < b$. Then

$$(1.3) \quad \frac{1}{b-a} \int_a^b f(x) g(x) dx \leq \frac{1}{3} M(a, b) + \frac{1}{6} N(a, b),$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$ and $N(a, b) = f(a)g(b) + f(b)g(a)$.

The main purpose of this paper is to establish new inequalities like those given in Theorems **A**, **B** and **C**, but now for the classes of m -convex functions (Section 2) and (α, m) -convex functions (Section 3).



**Inequalities for m -Convex and
 (α, m) -Convex Functions**

M. Klaričić Bakula, M. E. Özdemir
and J. Pečarić

vol. 9, iss. 4, art. 96, 2008

[Title Page](#)

[Contents](#)



Page 8 of 25

[Go Back](#)

[Full Screen](#)

[Close](#)

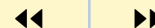
journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



Title Page

Contents



Page 9 of 25

Go Back

Full Screen

Close

2. Inequalities for m -Convex Functions

Theorem 2.1. Let I be an open real interval such that $[0, \infty) \subset I$. Let $f : I \rightarrow \mathbb{R}$ be a differentiable function on I such that $f' \in L^1([a, b])$, where $0 \leq a < b < \infty$. If $|f'|^q$ is m -convex on $[a, b]$ for some fixed $m \in (0, 1]$ and $q \in [1, \infty)$, then

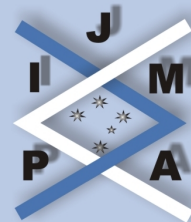
$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \min \left\{ \left(\frac{|f'(a)|^q + m |f'(\frac{b}{m})|^q}{2} \right)^{\frac{1}{q}}, \left(\frac{m |f'(\frac{a}{m})|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}} \right\}.$$

Proof. Suppose that $q = 1$. From Lemma C we have

$$(2.1) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2} \int_0^1 |1-2t| |f'(ta + (1-t)b)| dt.$$

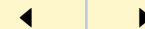
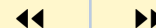
Since $|f'|$ is m -convex on $[a, b]$ we know that for any $t \in [0, 1]$

$$\begin{aligned} |f'(ta + (1-t)b)| &= \left| f' \left(ta + m(1-t) \frac{b}{m} \right) \right| \\ &\leq t |f'(a)| + m(1-t) \left| f' \left(\frac{b}{m} \right) \right|, \end{aligned}$$



Title Page

Contents



Page 10 of 25

Go Back

Full Screen

Close

journal of **inequalities**
 in pure and applied
 mathematics

issn: 1443-5756

hence

$$\begin{aligned}
 & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
 & \leq \frac{b-a}{2} \int_0^1 |1-2t| \left[t |f'(a)| + m(1-t) \left| f' \left(\frac{b}{m} \right) \right| \right] dt \\
 & = \frac{b-a}{2} \int_0^1 \left[t |1-2t| |f'(a)| + m(1-t) |1-2t| \left| f' \left(\frac{b}{m} \right) \right| \right] dt \\
 & = \frac{b-a}{2} \left\{ \int_0^{\frac{1}{2}} \left[t(1-2t) |f'(a)| + m(1-t)(1-2t) \left| f' \left(\frac{b}{m} \right) \right| \right] dt \right. \\
 & \quad \left. + \int_{\frac{1}{2}}^1 \left[t(2t-1) |f'(a)| + m(1-t)(2t-1) \left| f' \left(\frac{b}{m} \right) \right| \right] dt \right\} \\
 & = \frac{b-a}{8} \left(|f'(a)| + m \left| f' \left(\frac{b}{m} \right) \right| \right).
 \end{aligned}$$

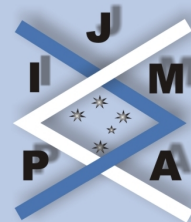
Analogously we obtain

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{8} \left(m \left| f' \left(\frac{a}{m} \right) \right| + |f'(b)| \right),$$

which completes the proof for this case.

Suppose now that $q > 1$. Using the well known Hölder inequality for q and $p = q/(q-1)$ we obtain

$$\begin{aligned}
 & \int_0^1 |1-2t| |f'(ta + (1-t)b)| dt \\
 & = \int_0^1 |1-2t|^{1-\frac{1}{q}} |1-2t|^{\frac{1}{q}} |f'(ta + (1-t)b)| dt
 \end{aligned}$$



[Title Page](#)

[Contents](#)

◀◀ ▶▶

◀ ▶

Page 11 of 25

[Go Back](#)

[Full Screen](#)

[Close](#)

$$(2.2) \quad \leq \left(\int_0^1 |1 - 2t| dt \right)^{\frac{q-1}{q}} \left(\int_0^1 |1 - 2t| |f'(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}}.$$

Since $|f'|^q$ is m -convex on $[a, b]$ we know that for every $t \in [0, 1]$

$$(2.3) \quad |f'(ta + (1-t)b)|^q \leq t |f'(a)|^q + m(1-t) \left| f' \left(\frac{b}{m} \right) \right|^q,$$

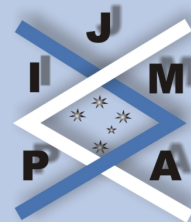
hence from (2.1), (2.2) and (2.3) we have

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{2} \left(\int_0^1 |1 - 2t| dt \right)^{\frac{q-1}{q}} \left(\int_0^1 |1 - 2t| \left| f' \left(ta + m(1-t) \frac{b}{m} \right) \right|^q dt \right)^{\frac{1}{q}} \\ & \leq \frac{b-a}{2} \left(\int_0^1 |1 - 2t| dt \right)^{\frac{q-1}{q}} \left[\frac{1}{4} \left(|f'(a)|^q + m \left| f' \left(\frac{b}{m} \right) \right|^q \right) \right]^{\frac{1}{q}} \\ & = \frac{b-a}{4} \left(\frac{m |f'(a)|^q + m \left| f' \left(\frac{b}{m} \right) \right|^q}{2} \right)^{\frac{1}{q}} \end{aligned}$$

and analogously

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \left(\frac{m \left| f' \left(\frac{a}{m} \right) \right|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}},$$

which completes the proof. □



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 12 of 25

Go Back

Full Screen

Close

Theorem 2.2. Suppose that all the assumptions of Theorem 2.1 are satisfied. Then

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \min \left\{ \left(\frac{|f'(a)|^q + m |f'\left(\frac{b}{m}\right)|^q}{2} \right)^{\frac{1}{q}}, \left(\frac{m |f'\left(\frac{a}{m}\right)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}} \right\}.$$

Proof. Our starting point here is the identity (see [7, Theorem 2])

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx = \frac{1}{b-a} \int_a^b S(x) f'(x) dx,$$

where

$$S(x) = \begin{cases} x - a, & x \in \left[a, \frac{a+b}{2}\right); \\ x - b, & x \in \left[\frac{a+b}{2}, b\right]. \end{cases}$$

We have

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{1}{b-a} \left[\int_a^{\frac{a+b}{2}} (x-a) |f'(x)| dx + \int_{\frac{a+b}{2}}^b (b-x) |f'(x)| dx \right] \\ & = (b-a) \left[\int_0^{\frac{1}{2}} t |f'(ta + (1-t)b)| dt + \int_{\frac{1}{2}}^1 (1-t) |f'(ta + (1-t)b)| dt \right] \end{aligned}$$

$$\begin{aligned} &\leq (b-a) \left[\int_0^{\frac{1}{2}} t \left(t |f'(a)| + m(1-t) \left| f' \left(\frac{b}{m} \right) \right| \right) dt \right. \\ &\quad \left. + \int_{\frac{1}{2}}^1 (1-t) \left(t |f'(a)| + m(1-t) \left| f' \left(\frac{b}{m} \right) \right| \right) dt \right] \\ &= \frac{b-a}{8} \left(|f'(a)| + m \left| f' \left(\frac{b}{m} \right) \right| \right), \end{aligned}$$

and analogously

$$\left| f \left(\frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{8} \left(m \left| f' \left(\frac{a}{m} \right) \right| + |f'(b)| \right).$$

This completes the proof for the case $q = 1$.

An argument similar to the one used in the proof of Theorem 2.1 gives the proof for the case $q \in (1, \infty)$. \square

As a special case of Theorem 2.1 for $m = 1$, that is for $|f'|^q$ convex on $[a, b]$, we obtain the first inequality in Theorem B. Similarly, as a special case of Theorem 2.2 we obtain the second inequality in Theorem B.

Theorem 2.3. *Let I be an open real interval such that $[0, \infty) \subset I$. Let $f : I \rightarrow \mathbb{R}$ be a differentiable function on I such that $f' \in L^1([a, b])$, where $0 \leq a < b < \infty$. If $|f'|^q$ is m -convex on $[a, b]$ for some fixed $m \in (0, 1]$ and $q \in (1, \infty)$, then*

$$\begin{aligned} (2.4) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| &\leq \frac{b-a}{4} \left(\frac{q-1}{2q-1} \right)^{\frac{q-1}{q}} \left(\mu_1^{\frac{1}{q}} + \mu_2^{\frac{1}{q}} \right) \\ &\leq \frac{b-a}{4} \left(\mu_1^{\frac{1}{q}} + \mu_2^{\frac{1}{q}} \right), \end{aligned}$$



[Title Page](#)

[Contents](#)

[◀◀](#) [▶▶](#)

[◀](#) [▶](#)

Page 13 of 25

[Go Back](#)

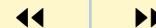
[Full Screen](#)

[Close](#)



Title Page

Contents



Page 14 of 25

Go Back

Full Screen

Close

where

$$\mu_1 = \min \left\{ \frac{|f'(a)|^q + m \left| f' \left(\frac{a+b}{2m} \right) \right|^q}{2}, \frac{\left| f' \left(\frac{a+b}{2} \right) \right|^q + m \left| f' \left(\frac{a}{m} \right) \right|^q}{2} \right\},$$

$$\mu_2 = \min \left\{ \frac{|f'(b)|^q + m \left| f' \left(\frac{a+b}{2m} \right) \right|^q}{2}, \frac{\left| f' \left(\frac{a+b}{2} \right) \right|^q + m \left| f' \left(\frac{b}{m} \right) \right|^q}{2} \right\}.$$

Proof. If $|f'|^q$ is m -convex from Theorem A we have

$$2 \int_{\frac{1}{2}}^1 |f'(ta + (1-t)b)|^q dt \leq \mu_1,$$

$$2 \int_0^{\frac{1}{2}} |f'(ta + (1-t)b)|^q dt \leq \mu_2,$$

hence

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{2} \int_0^1 |1-2t| |f'(ta + (1-t)b)| dt \\ & = \frac{b-a}{2} \left[\int_0^{\frac{1}{2}} (1-2t) |f'(ta + (1-t)b)| dt + \int_{\frac{1}{2}}^1 (2t-1) |f'(ta + (1-t)b)| dt \right]. \end{aligned}$$

Using Hölder's inequality for $q \in (1, \infty)$ and $p = q/(q-1)$ we obtain

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right|$$

$$\begin{aligned} &\leq \frac{b-a}{2} \left[\left(\int_0^{\frac{1}{2}} (1-2t)^{\frac{q}{q-1}} dt \right)^{\frac{q-1}{q}} \left(\int_0^{\frac{1}{2}} |f'(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}} \right. \\ &\quad \left. + \left(\int_{\frac{1}{2}}^1 (2t-1)^{\frac{q}{q-1}} dt \right)^{\frac{q-1}{q}} \left(\int_{\frac{1}{2}}^1 |f'(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}} \right] \\ &\leq \frac{b-a}{4} \left(\frac{q-1}{2q-1} \right)^{\frac{q-1}{q}} \left(\mu_1^{\frac{1}{q}} + \mu_2^{\frac{1}{q}} \right), \end{aligned}$$

since

$$\int_0^{\frac{1}{2}} (1-2t)^{\frac{q}{q-1}} dt = \int_{\frac{1}{2}}^1 (2t-1)^{\frac{q}{q-1}} dt = \frac{q-1}{2(2q-1)}.$$

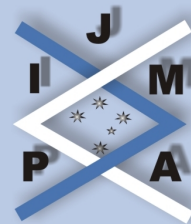
This completes the proof of the first inequality in (2.4). The second inequality in (2.4) follows from the fact

$$\frac{1}{2} < \left(\frac{q-1}{2q-1} \right)^{\frac{q-1}{q}} < 1, \quad q \in (1, \infty).$$

□

Theorem 2.4. *Let $f, g : [0, \infty) \rightarrow [0, \infty)$ be such that fg is in $L^1([a, b])$, where $0 \leq a < b < \infty$. If f is m_1 -convex and g is m_2 -convex on $[a, b]$ for some fixed $m_1, m_2 \in (0, 1]$, then*

$$\frac{1}{b-a} \int_a^b f(x)g(x) dx \leq \min \{M_1, M_2\},$$



Title Page

Contents

◀◀ ▶▶

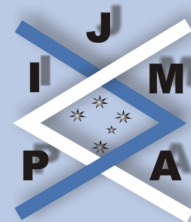
◀ ▶

Page 15 of 25

Go Back

Full Screen

Close



[Title Page](#)

[Contents](#)

[◀◀](#) [▶▶](#)

[◀](#) [▶](#)

Page 16 of 25

[Go Back](#)

[Full Screen](#)

[Close](#)

where

$$M_1 = \frac{1}{3} \left[f(a)g(a) + m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right) \right] \\ + \frac{1}{6} \left[m_2f(a)g\left(\frac{b}{m_2}\right) + m_1f\left(\frac{b}{m_1}\right)g(a) \right],$$

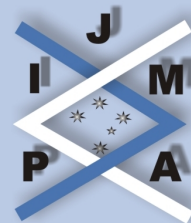
$$M_2 = \frac{1}{3} \left[f(b)g(b) + m_1m_2f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right) \right] \\ + \frac{1}{6} \left[m_2f(b)g\left(\frac{a}{m_2}\right) + m_1f\left(\frac{a}{m_1}\right)g(b) \right].$$

Proof. We have

$$f\left(ta + m_1(1-t)\frac{b}{m_1}\right) \leq tf(a) + m_1(1-t)f\left(\frac{b}{m_1}\right), \\ g\left(ta + m_2(1-t)\frac{b}{m_2}\right) \leq tg(a) + m_2(1-t)g\left(\frac{b}{m_2}\right),$$

for all $t \in [0, 1]$. f and g are nonnegative, hence

$$f\left(ta + m_1(1-t)\frac{b}{m_1}\right)g\left(ta + m_2(1-t)\frac{b}{m_2}\right) \\ \leq t^2f(a)g(a) + m_2t(1-t)f(a)g\left(\frac{b}{m_2}\right) + m_1t(1-t)f\left(\frac{b}{m_1}\right)g(a) \\ + m_1m_2(1-t)^2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right).$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 17 of 25

Go Back

Full Screen

Close

Integrating both sides of the above inequality over $[0, 1]$ we obtain

$$\begin{aligned} & \int_0^1 f(ta + (1-t)b)g(ta + (1-t)b) dt \\ &= \frac{1}{b-a} \int_a^b f(x)g(x) dx \\ &\leq \frac{1}{3} \left(f(a)g(a) + m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right) \right) \\ &\quad + \frac{1}{6} \left(m_2f(a)g\left(\frac{b}{m_2}\right) + m_1f\left(\frac{b}{m_1}\right)g(a) \right). \end{aligned}$$

Analogously we obtain

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(x)g(x) dx &\leq \frac{1}{3} \left(f(b)g(b) + m_1m_2f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right) \right) \\ &\quad + \frac{1}{6} \left(m_2f(b)g\left(\frac{a}{m_2}\right) + m_1f\left(\frac{a}{m_1}\right)g(b) \right), \end{aligned}$$

hence

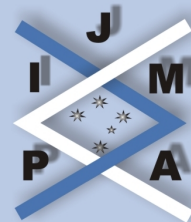
$$\frac{1}{b-a} \int_a^b f(x)g(x) dx \leq \min \{M_1, M_2\}.$$

□

Remark 1. If in Theorem 2.4 we choose a 1-convex (convex) function $g : [0, \infty) \rightarrow [0, \infty)$ defined by $g(x) = 1$ for all $x \in [0, \infty)$, we obtain

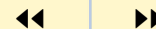
$$\frac{1}{b-a} \int_a^b f(x) dx \leq \min \left\{ \frac{f(a) + mf\left(\frac{b}{m}\right)}{2}, \frac{f(b) + mf\left(\frac{a}{m}\right)}{2} \right\},$$

which is (1.1). If the functions f and g are 1-convex we obtain (1.3).



Title Page

Contents



Page 18 of 25

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

3. Inequalities for (α, m) -Convex Functions

In this section on two examples we illustrate how the same inequalities as in Section 2 can be obtained for the class of (α, m) -convex functions.

Theorem 3.1. *Let I be an open real interval such that $[0, \infty) \subset I$. Let $f : I \rightarrow \mathbb{R}$ be a differentiable function on I such that $f' \in L^1([a, b])$, where $0 \leq a < b < \infty$. If $|f'|^q$ is (α, m) -convex on $[a, b]$ for some fixed $\alpha, m \in (0, 1]$ and $q \in [1, \infty)$, then*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2} \left(\frac{1}{2} \right)^{\frac{q-1}{q}} \cdot \min \left\{ \left(\nu_1 |f'(a)|^q + \nu_2 m \left| f' \left(\frac{b}{m} \right) \right|^q \right)^{\frac{1}{q}}, \left(\nu_1 |f'(b)|^q + \nu_2 m \left| f' \left(\frac{a}{m} \right) \right|^q \right)^{\frac{1}{q}} \right\},$$

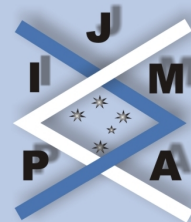
where

$$\nu_1 = \frac{1}{(\alpha+1)(\alpha+2)} \left[\alpha + \left(\frac{1}{2} \right)^\alpha \right],$$

$$\nu_2 = \frac{1}{(\alpha+1)(\alpha+2)} \left[\frac{\alpha^2 + \alpha + 2}{2} - \left(\frac{1}{2} \right)^\alpha \right].$$

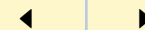
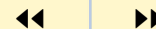
Proof. Suppose that $q = 1$. From Lemma A we have

$$(3.1) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2} \int_0^1 |1-2t| |f'(ta + (1-t)b)| dt.$$



Title Page

Contents



Page 19 of 25

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

© 2007 Victoria University. All rights reserved.

Since $|f'|$ is (α, m) -convex on $[a, b]$ we know that for any $t \in [0, 1]$

$$\left| f' \left(ta + m(1-t) \frac{b}{m} \right) \right| \leq t^\alpha |f'(a)| + m(1-t^\alpha) \left| f' \left(\frac{b}{m} \right) \right|,$$

thus we have

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{2} \int_0^1 |1-2t| \left[t^\alpha |f'(a)| + m(1-t^\alpha) \left| f' \left(\frac{b}{m} \right) \right| \right] dt \\ & = \frac{b-a}{2} \int_0^1 \left[t^\alpha |1-2t| |f'(a)| + m(1-t^\alpha) |1-2t| \left| f' \left(\frac{b}{m} \right) \right| \right] dt. \end{aligned}$$

We have

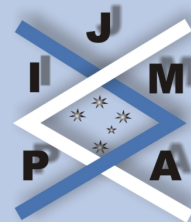
$$\begin{aligned} \int_0^1 t^\alpha |1-2t| dt &= \frac{1}{(\alpha+1)(\alpha+2)} \left[\alpha + \left(\frac{1}{2} \right)^\alpha \right] = \nu_1, \\ \int_0^1 (1-t^\alpha) |1-2t| dt &= \frac{1}{(\alpha+1)(\alpha+2)} \left[\frac{\alpha^2 + \alpha + 2}{2} - \left(\frac{1}{2} \right)^\alpha \right] = \nu_2, \end{aligned}$$

hence

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2} \left(\nu_1 |f'(a)| + \nu_2 m \left| f' \left(\frac{b}{m} \right) \right| \right).$$

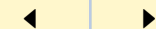
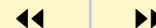
Analogously we obtain

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2} \left(\nu_1 |f'(b)| + \nu_2 m \left| f' \left(\frac{a}{m} \right) \right| \right),$$



[Title Page](#)

[Contents](#)



Page 20 of 25

[Go Back](#)

[Full Screen](#)

[Close](#)

which completes the proof for this case.

Suppose now that $q \in (1, \infty)$. Similarly to Theorem 2.1 we have

$$(3.2) \quad \int_0^1 |1 - 2t| |f'(ta + (1 - t)b)| dt \\ \leq \left(\int_0^1 |1 - 2t| dt \right)^{\frac{q-1}{q}} \left(\int_0^1 |1 - 2t| |f'(ta + (1 - t)b)|^q dt \right)^{\frac{1}{q}}.$$

Since $|f'|^q$ is (α, m) -convex on $[a, b]$ we know that for every $t \in [0, 1]$

$$(3.3) \quad \left| f' \left(ta + m(1 - t) \frac{b}{m} \right) \right|^q \leq t^\alpha |f'(a)|^q + m(1 - t^\alpha) \left| f' \left(\frac{b}{m} \right) \right|^q,$$

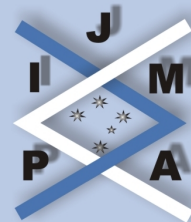
hence from (3.1), (3.2) and (3.3) we obtain

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) dx \right| \\ \leq \frac{b - a}{2} \left(\int_0^1 |1 - 2t| dt \right)^{\frac{q-1}{q}} \left(\int_0^1 |1 - 2t| \left| f' \left(ta + m(1 - t) \frac{b}{m} \right) \right|^q dt \right)^{\frac{1}{q}} \\ \leq \frac{b - a}{2} \left(\frac{1}{2} \right)^{\frac{q-1}{q}} \left(\nu_1 |f'(a)|^q + \nu_2 m \left| f' \left(\frac{b}{m} \right) \right|^q \right)^{\frac{1}{q}}$$

and analogously

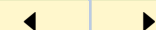
$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) dx \right| \leq \frac{b - a}{2} \left(\frac{1}{2} \right)^{\frac{q-1}{q}} \left(\nu_1 |f'(b)|^q + \nu_2 m \left| f' \left(\frac{a}{m} \right) \right|^q \right)^{\frac{1}{q}},$$

which completes the proof. \square



Title Page

Contents



Page 21 of 25

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

Observe that if in Theorem 3.1 we have $\alpha = 1$ the statement of Theorem 3.1 becomes the statement of Theorem 2.1.

Theorem 3.2. Let $f, g : [0, \infty) \rightarrow [0, \infty)$ be such that fg is in $L^1([a, b])$, where $0 \leq a < b < \infty$. If f is (α_1, m_1) -convex and g is (α_2, m_2) -convex on $[a, b]$ for some fixed $\alpha_1, m_1, \alpha_2, m_2 \in (0, 1]$, then

$$\frac{1}{b-a} \int_a^b f(x)g(x) dx \leq \min \{N_1, N_2\},$$

where

$$\begin{aligned} N_1 = & \frac{f(a)g(a)}{\alpha_1 + \alpha_2 + 1} + m_2 \left[\frac{1}{\alpha_1 + 1} - \frac{1}{\alpha_1 + \alpha_2 + 1} \right] f(a)g\left(\frac{b}{m_2}\right) \\ & + m_1 \left[\frac{1}{\alpha_2 + 1} - \frac{1}{\alpha_1 + \alpha_2 + 1} \right] f\left(\frac{b}{m_1}\right)g(a) \\ & + m_1 m_2 \left[1 - \frac{1}{\alpha_1 + 1} - \frac{1}{\alpha_2 + 1} + \frac{1}{\alpha_1 + \alpha_2 + 1} \right] f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right), \end{aligned}$$

and

$$\begin{aligned} N_2 = & \frac{f(b)g(b)}{\alpha_1 + \alpha_2 + 1} + m_2 \left[\frac{1}{\alpha_1 + 1} - \frac{1}{\alpha_1 + \alpha_2 + 1} \right] f(b)g\left(\frac{a}{m_2}\right) \\ & + m_1 \left[\frac{1}{\alpha_2 + 1} - \frac{1}{\alpha_1 + \alpha_2 + 1} \right] f\left(\frac{a}{m_1}\right)g(b) \\ & + m_1 m_2 \left[1 - \frac{1}{\alpha_1 + 1} - \frac{1}{\alpha_2 + 1} + \frac{1}{\alpha_1 + \alpha_2 + 1} \right] f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right). \end{aligned}$$



[Title Page](#)

[Contents](#)

[◀◀](#) [▶▶](#)

[◀](#) [▶](#)

Page 22 of 25

[Go Back](#)

[Full Screen](#)

[Close](#)

Proof. Since f is (α_1, m_1) -convex and g is (α_2, m_2) -convex on $[a, b]$ we have

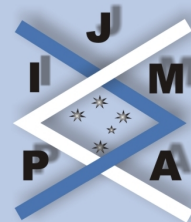
$$f\left(ta + m_1(1-t)\frac{b}{m_1}\right) \leq t^{\alpha_1}f(a) + m_1(1-t^{\alpha_1})f\left(\frac{b}{m_1}\right),$$
$$g\left(ta + m_2(1-t)\frac{b}{m_2}\right) \leq t^{\alpha_2}g(a) + m_2(1-t^{\alpha_2})g\left(\frac{b}{m_2}\right),$$

for all $t \in [0, 1]$. The functions f and g are nonnegative, hence

$$f(ta + (1-t)b)g(ta + (1-t)b) \leq t^{\alpha_1 + \alpha_2}f(a)g(a)$$
$$+ m_2t^{\alpha_1}(1-t^{\alpha_2})f(a)g\left(\frac{b}{m_2}\right) + m_1t^{\alpha_2}(1-t^{\alpha_1})f\left(\frac{b}{m_1}\right)g(a)$$
$$+ m_1m_2(1-t^{\alpha_1})(1-t^{\alpha_2})f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right).$$

Integrating both sides of the above inequality over $[0, 1]$ we obtain

$$\int_0^1 f(ta + (1-t)b)g(ta + (1-t)b) dt$$
$$= \frac{1}{b-a} \int_a^b f(x)g(x) dx$$
$$\leq \frac{f(a)g(a)}{\alpha_1 + \alpha_2 + 1} + m_2 \left[\frac{1}{\alpha_1 + 1} - \frac{1}{\alpha_1 + \alpha_2 + 1} \right] f(a)g\left(\frac{b}{m_2}\right)$$
$$+ m_1 \left[\frac{1}{\alpha_2 + 1} - \frac{1}{\alpha_1 + \alpha_2 + 1} \right] f\left(\frac{b}{m_1}\right)g(a)$$
$$+ m_1m_2 \left[1 - \frac{1}{\alpha_1 + 1} - \frac{1}{\alpha_2 + 1} + \frac{1}{\alpha_1 + \alpha_2 + 1} \right] f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right).$$



[Title Page](#)

[Contents](#)

[◀◀](#) [▶▶](#)

[◀](#) [▶](#)

Page 23 of 25

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

© 2007 Victoria University. All rights reserved.

Analogously we have

$$\begin{aligned} & \frac{1}{b-a} \int_a^b f(x)g(x) dx \\ & \leq \frac{f(b)g(b)}{\alpha_1 + \alpha_2 + 1} + m_2 \left[\frac{1}{\alpha_1 + 1} - \frac{1}{\alpha_1 + \alpha_2 + 1} \right] f(b)g\left(\frac{a}{m_2}\right) \\ & \quad + m_1 \left[\frac{1}{\alpha_2 + 1} - \frac{1}{\alpha_1 + \alpha_2 + 1} \right] f\left(\frac{a}{m_1}\right)g(b) \\ & \quad + m_1 m_2 \left[1 - \frac{1}{\alpha_1 + 1} - \frac{1}{\alpha_2 + 1} + \frac{1}{\alpha_1 + \alpha_2 + 1} \right] f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right), \end{aligned}$$

which completes the proof. \square

If in Theorem 3.2 we have $\alpha_1 = \alpha_2 = 1$, the statement of Theorem 3.2 becomes the statement of Theorem 2.4.

References

- [1] A.M. BRUCKNER AND E. OSTROW, Some function classes related to the class of convex functions, *Pacific J. Math.*, **12** (1962), 1203–1215.
- [2] S.S. DRAGOMIR AND R.P. AGARWAL, Two inequalities for differentiable mappings and applications to special means of real numbers and trapezoidal formula, *Appl. Math. Lett.*, **11**(5) (1998), 91–95.
- [3] S.S. DRAGOMIR AND G. TOADER, Some inequalities for m -convex functions, *Studia Univ. Babeş-Bolyai Math.*, **38**(1) (1993), 21–28.
- [4] M. KLARIČIĆ BAKULA, J. PEČARIĆ AND M. RIBIČIĆ, Companion inequalities to Jensen's inequality for m -convex and (α, m) -convex functions, *J. Inequal. Pure & Appl. Math.*, **7**(5) (2006), Art. 194. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=811>].
- [5] V.G. MIHEŞAN, A generalization of the convexity, *Seminar on Functional Equations, Approx. and Convex.*, Cluj-Napoca (Romania) (1993).
- [6] B.G. PACHPATTE, On some inequalities for convex functions, *RGMIA Res. Rep. Coll.*, **6**(E) (2003), [ONLINE: [http://rgmia.vu.edu.au/v6\(E\).html](http://rgmia.vu.edu.au/v6(E).html)].
- [7] C.E.M. PEARCE AND J. PEČARIĆ, Inequalities for differentiable mappings with application to special means and quadrature formulae, *Appl. Math. Lett.*, **13**(2) (2000), 51–55.
- [8] J.E. PEČARIĆ, F. PROSCHAN AND Y.L. TONG, *Convex Functions, Partial Orderings, and Statistical Applications*, Academic Press, Inc. (1992).

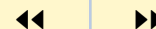


Inequalities for m -Convex and
 (α, m) -Convex Functions
M. Klaričić Bakula, M. E. Özdemir
and J. Pečarić

vol. 9, iss. 4, art. 96, 2008

Title Page

Contents



Page 24 of 25

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



- [9] G. TOADER, Some generalizations of the convexity, *Proceedings of the Colloquium on Approximation and Optimization*, Univ. Cluj-Napoca, Cluj-Napoca, 1985, 329-338.
- [10] G. TOADER, On a generalization of the convexity, *Mathematica*, **30** (53) (1988), 83-87.
- [11] S. TOADER, The order of a star-convex function, *Bull. Applied & Comp. Math.*, **85-B** (1998), BAM-1473, 347–350.

Inequalities for m -Convex and (α, m) -Convex Functions
M. Klaričić Bakula, M. E. Özdemir
and J. Pečarić
vol. 9, iss. 4, art. 96, 2008

[Title Page](#)

[Contents](#)



Page 25 of 25

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756