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ON MULTIDIMENSIONAL GRÜSS TYPE INEQUALITIES

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Abstract

Contents



Home Page

Go Back

Close

Quit

Abstract

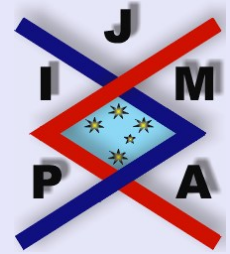
In this paper we establish some new multidimensional integral inequalities of the Grüss type by using a fairly elementary analysis.

2000 Mathematics Subject Classification: 26D15, 26D20.

Key words: Multidimensional, Grüss type inequalities, Partial Derivatives, Continuous Functions, Bounded, Identities.

Contents

1	Introduction	3
2	Statement of Results	4
3	Proof of Theorem 2.1	9
4	Proof of Theorem 2.2	11
5	Proof of Theorem 2.3	13
	References	



On Multidimensional Grüss Type Inequalities

B.G. Pachpatte

Title Page

Contents



Go Back

Close

Quit

Page 2 of 15

1. Introduction

The following inequality is well known in the literature as the Grüss inequality [2] (see also [4, p. 296]):

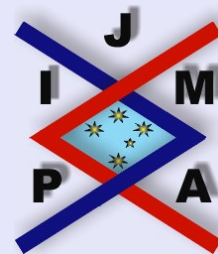
$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \left(\frac{1}{b-a} \int_a^b f(x) dx \right) \left(\frac{1}{b-a} \int_a^b g(x) dx \right) \right| \leq (M-m)(N-n),$$

provided that $f, g : [a, b] \rightarrow \mathbb{R}$ are integrable on $[a, b]$ and

$$m \leq f(x) \leq M, \quad n \leq g(x) \leq N,$$

for all $x \in [a, b]$, where m, M, n, N are given constants.

Since the appearance of the above inequality in 1935, it has evoked considerable interest and many variants, generalizations and extensions have appeared, see [1, 4] and the references cited therein. The main purpose of the present paper is to establish some new integral inequalities of the Grüss type involving functions of several independent variables. The analysis used in the proofs is elementary and our results provide new estimates on inequalities of this type.



On Multidimensional Grüss
Type Inequalities

B.G. Pachpatte

Title Page

Contents



Go Back

Close

Quit

Page 3 of 15

2. Statement of Results

In what follows, \mathbb{R} denotes the set of real numbers. Let $\Delta = [a, b] \times [c, d]$, $\Delta^0 = (a, b) \times (c, d)$, $\Omega = [a, k] \times [b, m] \times [c, n]$, $\Omega^0 = (a, k) \times (b, m) \times (c, n)$ for a, b, c, d, k, m, n in \mathbb{R} . For functions α and β defined respectively on Δ and Ω , the partial derivatives $\frac{\partial^2 \alpha(x, y)}{\partial y \partial x}$ and $\frac{\partial^3 \beta(x, y, z)}{\partial z \partial y \partial x}$ are denoted by $D_2 D_1 \alpha(x, y)$ and $D_3 D_2 D_1 \beta(x, y, z)$. Let

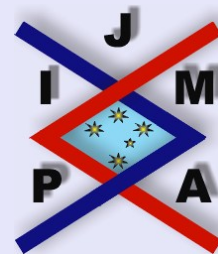
$$D = \{x = (x_1, \dots, x_n) : a_i < x_i < b_i \quad (i = 1, 2, \dots, n)\}$$

and \bar{D} be the closure of D . For a function e defined on \bar{D} the partial derivatives $\frac{\partial e(x)}{\partial x_i}$ ($i = 1, 2, \dots, n$) are denoted by $D_i e(x)$.

First, we give the following notations used to simplify the details of presentation.

$$\begin{aligned} A(D_2 D_1 f(x, y)) &= A[a, c; x, y; b, d; D_2 D_1 f(s, t)] \\ &= \int_a^x \int_c^y D_2 D_1 f(s, t) dt ds - \int_a^x \int_y^d D_2 D_1 f(s, t) dt ds \\ &\quad - \int_x^b \int_c^y D_2 D_1 f(s, t) dt ds + \int_x^b \int_y^d D_2 D_1 f(s, t) dt ds, \end{aligned}$$

$$\begin{aligned} B(D_3 D_2 D_1 f(r, s, t)) &= B[a, b, c; r, s, t; k, m, n; D_3 D_2 D_1 f(u, v, w)] \\ &= \int_a^r \int_b^s \int_c^t D_3 D_2 D_1 f(u, v, w) dw dv du \\ &\quad - \int_a^r \int_b^s \int_t^n D_3 D_2 D_1 f(u, v, w) dw dv du \end{aligned}$$



On Multidimensional Grüss
Type Inequalities

B.G. Pachpatte

Title Page

Contents

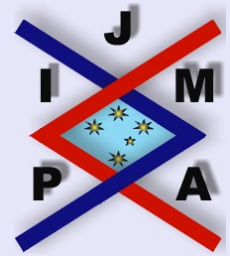


Go Back

Close

Quit

Page 4 of 15



**On Multidimensional Grüss
Type Inequalities**

B.G. Pachpatte

Title Page

Contents



Go Back

Close

Quit

Page 5 of 15

$$\begin{aligned}
 & - \int_a^r \int_s^m \int_c^t D_3 D_2 D_1 f(u, v, w) dw dv du \\
 & \quad - \int_r^k \int_b^s \int_c^t D_3 D_2 D_1 f(u, v, w) dw dv du \\
 & + \int_a^r \int_s^m \int_t^n D_3 D_2 D_1 f(u, v, w) dw dv du \\
 & \quad + \int_r^k \int_s^m \int_c^t D_3 D_2 D_1 f(u, v, w) dw dv du \\
 & + \int_r^k \int_b^s \int_t^n D_3 D_2 D_1 f(u, v, w) dw dv du \\
 & \quad - \int_r^k \int_s^m \int_t^n D_3 D_2 D_1 f(u, v, w) dw dv du,
 \end{aligned}$$

$$\begin{aligned}
 E(f(x, y)) &= E[a, c; x, y; b, d; f] \\
 &= \frac{1}{2} [f(x, c) + f(x, d) + f(a, y) + f(b, y)] \\
 & \quad - \frac{1}{4} [f(a, c) + f(a, d) + f(b, c) + f(b, d)],
 \end{aligned}$$

$$\begin{aligned}
 L(f(r, s, t)) &= L[a, b, c; r, s, t; k, m, n; f] \\
 &= \frac{1}{8} [f(a, b, c) + f(k, m, n)] \\
 & \quad - \frac{1}{4} [f(r, b, c) + f(r, m, n) + f(r, m, c) + f(r, b, n)]
 \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{4} [f(a, s, c) + f(k, s, n) + f(a, s, n) + f(k, s, c)] \\
& -\frac{1}{4} [f(a, b, t) + f(k, m, t) + f(k, b, t) + f(a, m, t)] \\
& +\frac{1}{2} [f(a, s, t) + f(k, s, t)] + \frac{1}{2} [f(r, b, t) + f(r, m, t)] \\
& +\frac{1}{2} [f(r, s, c) + f(r, s, n)].
\end{aligned}$$

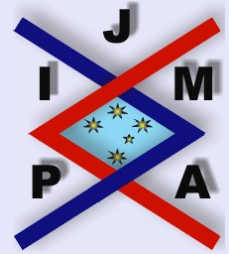
Our main results are given in the following theorems.

Theorem 2.1. Let $f, g : \Delta \rightarrow \mathbb{R}$ be continuous functions on Δ ; $D_2D_1f(x, y)$, $D_2D_1g(x, y)$ exist on Δ^0 and are bounded, i.e.

$$\begin{aligned}
\|D_2D_1f\|_\infty &= \sup_{(x,y) \in \Delta^0} |D_2D_1f(x, y)| < \infty, \\
\|D_2D_1g\|_\infty &= \sup_{(x,y) \in \Delta^0} |D_2D_1g(x, y)| < \infty.
\end{aligned}$$

Then

$$\begin{aligned}
(2.1) \quad & \left| \int_a^b \int_c^d f(x, y) g(x, y) dy dx \right. \\
& \left. - \frac{1}{2} \int_a^b \int_c^d [E(f(x, y)) g(x, y) + E(g(x, y)) f(x, y)] dy dx \right| \\
& \leq \frac{1}{8} (b-a)(d-c) \int_a^b \int_c^d (|g(x, y)| \|D_2D_1f\|_\infty \\
& \quad + |f(x, y)| \|D_2D_1g\|_\infty) dy dx.
\end{aligned}$$



On Multidimensional Grüss
Type Inequalities

B.G. Pachpatte

Title Page

Contents



Go Back

Close

Quit

Page 6 of 15

Theorem 2.2. Let $p, q : \Omega \rightarrow \mathbb{R}$ be continuous functions on Ω ; $D_3D_2D_1p(r, s, t)$, $D_3D_2D_1q(r, s, t)$ exist on Ω^0 and are bounded, i.e.

$$\|D_3D_2D_1p\|_\infty = \sup_{(r,s,t) \in \Omega^0} |D_3D_2D_1p(r, s, t)| < \infty,$$

$$\|D_3D_2D_1q\|_\infty = \sup_{(r,s,t) \in \Omega^0} |D_3D_2D_1q(r, s, t)| < \infty.$$

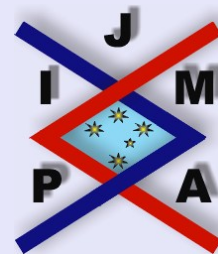
Then

$$(2.2) \quad \left| \int_a^k \int_b^m \int_c^n p(r, s, t) q(r, s, t) dt ds dr \right. \\ \left. - \frac{1}{2} \int_a^k \int_b^m \int_c^n [L(p(r, s, t)) q(r, s, t) \right. \\ \left. + L(q(r, s, t)) p(r, s, t)] dt ds dr \right| \\ \leq \frac{1}{16} (k - a) (m - b) (n - c) \\ \times \int_a^k \int_b^m \int_c^n (|q(r, s, t)| \|D_3D_2D_1p\|_\infty \\ + |p(r, s, t)| \|D_3D_2D_1q\|_\infty) dt ds dr.$$

Theorem 2.3. Let $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuous functions on \bar{D} and differentiable on D whose derivatives $D_i f(x)$, $D_i g(x)$ are bounded, i.e.,

$$\|D_i f\|_\infty = \sup_{x \in D} |D_i f(x)| < \infty,$$

$$\|D_i g\|_\infty = \sup_{x \in D} |D_i g(x)| < \infty.$$



On Multidimensional Grüss
Type Inequalities

B.G. Pachpatte

Title Page

Contents



Go Back

Close

Quit

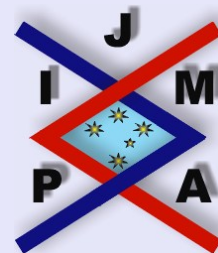
Page 7 of 15

Then

$$(2.3) \quad \left| \frac{1}{M} \int_D f(x) g(x) dx - \left(\frac{1}{M} \int_D f(x) dx \right) \left(\frac{1}{M} \int_D g(x) dx \right) \right| \\ \leq \frac{1}{2M^2} \int_D \sum_{i=1}^n (|g(x)| \|D_i f\|_\infty + |f(x)| \|D_i g\|_\infty) E_i(x) dx,$$

where

$$M = \prod_{i=1}^n (b_i - a_i), \quad E_i(x) = \int_D |x_i - y_i| dy, \\ dx = dx_1 \cdots dx_n, \quad dy = dy_1 \cdots dy_n.$$



**On Multidimensional Grüss
Type Inequalities**

B.G. Pachpatte

Title Page

Contents



Go Back

Close

Quit

Page 8 of 15

3. Proof of Theorem 2.1

From the hypotheses we have the following identities (see [6]):

$$(3.1) \quad f(x, y) = E(f(x, y)) + \frac{1}{4}A(D_2D_1f(x, y)),$$

$$(3.2) \quad g(x, y) = E(g(x, y)) + \frac{1}{4}A(D_2D_1g(x, y)),$$

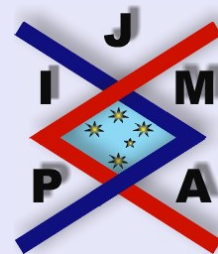
for $(x, y) \in \Delta$. Multiplying (3.1) by $g(x, y)$ and (3.2) by $f(x, y)$ and adding the resulting identities, then integrating on Δ and rewriting we have

$$(3.3) \quad \int_a^b \int_c^d f(x, y) g(x, y) dy dx \\ = \frac{1}{2} \int_a^b \int_c^d (E(f(x, y)) g(x, y) + E(g(x, y)) f(x, y)) dy dx \\ + \frac{1}{8} \int_a^b \int_c^d (A(D_2D_1f(x, y)) g(x, y) \\ + A(D_2D_1g(x, y)) f(x, y)) dy dx.$$

From the properties of modulus and integrals, it is easy to see that

$$(3.4) \quad |A(D_2D_1f(x, y))| \leq \int_a^b \int_c^d |D_2D_1f(s, t)| dt ds,$$

$$(3.5) \quad |A(D_2D_1g(x, y))| \leq \int_a^b \int_c^d |D_2D_1g(s, t)| dt ds.$$



On Multidimensional Grüss
Type Inequalities

B.G. Pachpatte

Title Page

Contents



Go Back

Close

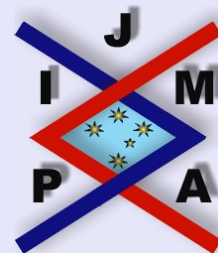
Quit

Page 9 of 15

From (3.3) – (3.5) we observe that

$$\begin{aligned}
 & \left| \int_a^b \int_c^d f(x, y) g(x, y) dy dx \right. \\
 & \quad \left. - \frac{1}{2} \int_a^b \int_c^d (E(f(x, y)) g(x, y) + E(g(x, y)) f(x, y)) dy dx \right| \\
 & \leq \frac{1}{8} \int_a^b \int_c^d (|g(x, y)| |A(D_2 D_1 f(x, y))| \\
 & \quad + |f(x, y)| |A(D_2 D_1 g(x, y))|) dy dx \\
 & \leq \frac{1}{8} \int_a^b \int_c^d \left(|g(x, y)| \int_a^b \int_c^d |D_2 D_1 f(s, t)| dt ds \right. \\
 & \quad \left. + |f(x, y)| \int_a^b \int_c^d |D_2 D_1 g(s, t)| dt ds \right) dy dx \\
 & \leq \frac{1}{8} (b-a)(d-c) \int_a^b \int_c^d (|g(x, y)| \|D_2 D_1 f\|_\infty \\
 & \quad + |f(x, y)| \|D_2 D_1 g\|_\infty) dy dx,
 \end{aligned}$$

which is the required inequality in (2.1). The proof is complete.



**On Multidimensional Grüss
Type Inequalities**

B.G. Pachpatte

Title Page

Contents



Go Back

Close

Quit

Page 10 of 15

4. Proof of Theorem 2.2

From the hypotheses we have the following identities (see [5]):

$$(4.1) \quad p(r, s, t) = L(p(r, s, t)) + \frac{1}{8}B(D_3D_2D_1p(r, s, t)),$$

$$(4.2) \quad q(r, s, t) = L(q(r, s, t)) + \frac{1}{8}B(D_3D_2D_1q(r, s, t)),$$

for $(r, s, t) \in \Omega$. Multiplying (4.1) by $q(r, s, t)$ and (4.2) by $p(r, s, t)$ and adding the resulting identities, then integrating on Ω and rewriting we have

$$(4.3) \quad \int_a^k \int_b^m \int_c^n p(r, s, t) q(r, s, t) dt ds dr$$

$$= \frac{1}{2} \int_a^k \int_b^m \int_c^n [L(p(r, s, t)) q(r, s, t)$$

$$+ L(q(r, s, t)) p(r, s, t)] dt ds dr$$

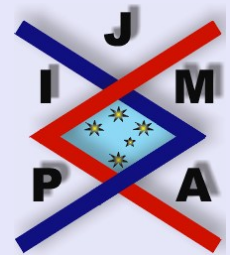
$$+ \frac{1}{16} \int_a^k \int_b^m \int_c^n (B(D_3D_2D_1p(r, s, t)) q(r, s, t)$$

$$+ B(D_3D_2D_1q(r, s, t)) p(r, s, t)) dt ds dr.$$

From the properties of modulus and integrals, we observe that

$$(4.4) \quad |B(D_3D_2D_1p(r, s, t))| \leq \int_a^k \int_b^m \int_c^n |D_3D_2D_1p(u, v, w)| dw dv du,$$

$$(4.5) \quad |B(D_3D_2D_1q(r, s, t))| \leq \int_a^k \int_b^m \int_c^n |D_3D_2D_1q(u, v, w)| dw dv du.$$



On Multidimensional Grüss
Type Inequalities

B.G. Pachpatte

Title Page

Contents



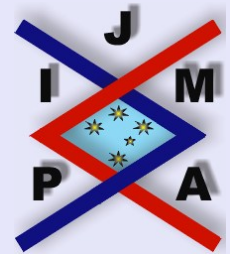
Go Back

Close

Quit

Page 11 of 15

Now, from (4.3) – (4.5) and following the same arguments as in the proof of Theorem 2.1 with suitable changes, we get the required inequality in (2.2).



**On Multidimensional Grüss
Type Inequalities**

B.G. Pachpatte

Title Page

Contents



Go Back

Close

Quit

Page 12 of 15

5. Proof of Theorem 2.3

Let $x \in \bar{D}$, $y \in D$. From the n -dimensional version of the mean value theorem, we have (see [3]):

$$(5.1) \quad f(x) - f(y) = \sum_{i=1}^n D_i f(c) (x_i - y_i),$$

where $c = (y_1 + \delta(x_1 - y_1), \dots, y_n + \delta(x_n - y_n))$, $0 < \delta < 1$.

Integrating (5.1) with respect to y , we obtain

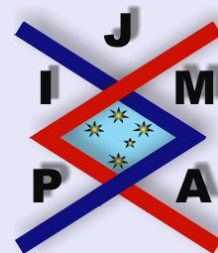
$$(5.2) \quad f(x) \operatorname{mes} D = \int_D f(y) dy + \sum_{i=1}^n \int_D D_i f(c) (x_i - y_i) dy,$$

where $\operatorname{mes} D = \prod_{i=1}^n (b_i - a_i) = M$. Similarly, we obtain

$$(5.3) \quad g(x) \operatorname{mes} D = \int_D g(y) dy + \sum_{i=1}^n \int_D D_i g(c) (x_i - y_i) dy.$$

Multiplying (5.2) by $g(x)$ and (5.3) by $f(x)$ and adding the resulting identities, then integrating on D and noting that $\operatorname{mes} > 0$, we have

$$(5.4) \quad \int_D f(x) g(x) dx = \frac{1}{M} \left(\int_D f(x) dx \right) \left(\int_D g(x) dx \right) + \frac{1}{2M} \left[\int_D g(x) \left(\sum_{i=1}^n \int_D D_i f(c) (x_i - y_i) dy \right) dx \right]$$



On Multidimensional Grüss
Type Inequalities

B.G. Pachpatte

Title Page

Contents



Go Back

Close

Quit

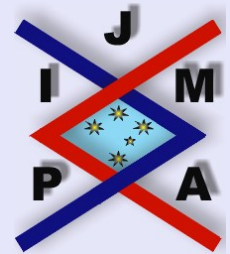
Page 13 of 15

$$+ \int_D f(x) \left(\sum_{i=1}^n \int_D D_i g(c) (x_i - y_i) dy \right) dx \Big].$$

From (5.4) we observe that

$$\begin{aligned} & \left| \frac{1}{M} \int_D f(x) g(x) dx - \left(\frac{1}{M} \int_D f(x) dx \right) \left(\frac{1}{M} \int_D g(x) dx \right) \right| \\ & \leq \frac{1}{2M^2} \int_D \sum_{i=1}^n (|g(x)| \|D_i f\|_\infty + |f(x)| \|D_i g\|_\infty) E_i(x) dx. \end{aligned}$$

This is the required inequality in (2.3). The proof is complete.



**On Multidimensional Grüss
Type Inequalities**

B.G. Pachpatte

Title Page

Contents



Go Back

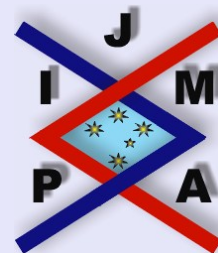
Close

Quit

Page 14 of 15

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On Multidimensional Grüss Type Inequalities

B.G. Pachpatte

Title Page

Contents



Go Back

Close

Quit

Page 15 of 15