A GENERALIZED FANNES' INEQUALITY

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Abstract: We axiomatically characterize the Tsallis entropy of a finite quantum system. In addition, we derive a continuity property of Tsallis entropy. This gives a generalization of the Fannes' inequality.

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1. Introduction with Uniqueness Theorem of Tsallis Entropy

Three or four decades ago, a number of researchers investigated some extensions of the Shannon entropy [1]. In statistical physics, the Tsallis entropy, defined in [10] by

$$H_q(X) \equiv \frac{\sum_x \left(p(x)^q - p(x)\right)}{1-q} = \sum_x \eta_q\left(p(x)\right)$$

with one parameter $q \in \mathbb{R}^+$ as an extension of Shannon entropy $H_1(X) = -\sum_x p(x)\log p(x)$, for any probability distribution $p(x) \equiv p(X = x)$ of a given random variable X, where q-entropy function is defined by $\eta_q(x) \equiv -x^q \ln_q x = \frac{x^q - x}{1-q}$ and the q-logarithmic function $\ln_q x \equiv \frac{x^{1-q}-1}{1-q}$ is defined for $q \geq 0$, $q \neq 1$ and $x \geq 0$.

The Tsallis entropy $H_q(X)$ converges to the Shannon entropy $-\sum_x p(x)\log p(x)$ as $q \to 1$. See [5] for fundamental properties of the Tsallis entropy and the Tsallis relative entropy. In the previous paper [6], we gave the uniqueness theorem for the Tsallis entropy for a classical system, introducing the generalized Faddeev's axiom. We briefly review the uniqueness theorem for the Tsallis entropy below.

The function $I_q(x_1, \ldots, x_n)$ is assumed to be defined on n-tuple $(x_1, \ldots, x_n)$ belonging to

$$\Delta_n \equiv \left\{ (p_1, \ldots, p_n) \middle| \sum_{i=1}^n p_i = 1\,, p_i \geq 0 \ (i = 1, 2, \ldots, n) \right\}$$

and to take values in $\mathbb{R}^+ \equiv [0, \infty)$. Then we adopted the following generalized Faddeev's axiom.

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Axiom 1. **(Generalized Faddeev's axiom)**

(F1) *Continuity*: The function $f_q(x) \equiv I_q(x, 1-x)$ with parameter $q \geq 0$ is continuous on the closed interval $[0,1]$ and $f_q(x_0) > 0$ for some $x_0 \in [0,1]$.

(F2) *Symmetry*: For arbitrary permutation $\{x'_k\} \in \Delta_n$ of $\{x_k\} \in \Delta_n$,

$$I_q(x_1, \ldots, x_n) = I_q(x'_1, \ldots, x'_n). \tag{1.1}$$

(F3) *Generalized additivity*: For $x_n = y + z$, $y \geq 0$ and $z > 0$,

$$I_q(x_1, \ldots, x_{n-1}, y, z) = I_q(x_1, \ldots, x_n) + x_n^q I_q\left(\frac{y}{x_n}, \frac{z}{x_n}\right). \tag{1.2}$$

Theorem 1.1 ([6]). *The conditions (F1), (F2) and (F3) uniquely give the form of the function $I_q : \Delta_n \to \mathbb{R}^+$ such that*

$$I_q(x_1, \ldots, x_n) = \mu_q H_q(x_1, \ldots, x_n), \tag{1.3}$$

where μ_q is a positive constant that depends on the parameter $q > 0$.

If we further impose the normalized condition on Theorem 1.1, it determines the entropy of type β (the structural a-entropy), (see [1, p. 189]).

Definition 1.1. For a density operator ρ on a finite dimensional Hilbert space $\mathbf{H}$, the Tsallis entropy is defined by

$$S_q(\rho) \equiv \frac{\mathrm{Tr}[\rho^q - \rho]}{1-q} = \mathrm{Tr}[\eta_q(\rho)],$$

with a nonnegative real number q.

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Note that the Tsallis entropy is a particular case of f-entropy [11]. See also [9] for a quasi-entropy which is a quantum version of f-divergence [3].

Let T_q be a mapping on the set $S(\mathbf{H})$ of all density operators to $\mathbb{R}^+$.

Axiom 2. We give the postulates which the Tsallis entropy should satisfy.

(T1) *Continuity:* For $\rho \in S(\mathbf{H})$, $T_q(\rho)$ is a continuous function with respect to the 1-norm $\|\cdot\|_1$.

(T2) *Invariance:* For unitary transformation U, $T_q(U^*\rho U) = T_q(\rho)$.

(T3) *Generalized mixing condition:* For $\rho = \bigoplus_{k=1}^n \lambda_k \rho_k$ on $\mathbf{H} = \bigoplus_{k=1}^n \mathbf{H}_k$, where $\lambda_k \geq 0$, $\sum_{k=1}^n \lambda_k = 1$, $\rho_k \in S(\mathbf{H}_k)$, we have the additivity:

$$T_q(\rho) = \sum_{k=1}^n \lambda_k^q T_q(\rho_k) + T_q(\lambda_1, \dots, \lambda_n),$$

where $(\lambda_1, \dots, \lambda_n)$ represents the diagonal matrix $(\lambda_k \delta_{kj})_{k,j=1,\dots,n}$.

Theorem 1.2. *If T_q satisfies Axiom 2, then T_q is uniquely given by the following form*

$$T_q(\rho) = \mu_q S_q(\rho),$$

with a positive constant number μ_q depending on the parameter $q > 0$.

Proof. Although the proof is quite similar to that of Theorem 2.1 in [8], we present it for readers' convenience. From (T2) and (T3), we have

$$T_q(\lambda_1, \lambda_2) = \lambda_1^q T_q(1) + \lambda_2^q T_q(1) + T_q(\lambda_1, \lambda_2),$$

which implies $T_q(1) = 0$. Moreover, by (T2) and (T3), when $p_n \neq 1$, we have

$$T_q(p_1, \dots, p_{n-1}, \lambda p_n, (1-\lambda) p_n)$$
$$= p_n^q T_q(\lambda, 1-\lambda) + (1-p_n)^q T_q\left(\frac{p_1}{1-p_n}, \dots, \frac{p_{n-1}}{1-p_n}\right) + T_q(p_n, 1-p_n)$$

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and

$$T_q(p_1,\ldots,p_{n-1},p_n)=p_n^qT_q(1)+(1-p_n)^qT_q\left(\frac{p_1}{1-p_n},\ldots,\frac{p_{n-1}}{1-p_n}\right)+T_q(p_n,1-p_n).$$

From these equations, we have

$$T_q(p_1,\ldots,p_{n-1},\lambda p_n,(1-\lambda)p_n)=T_q(p_1,\ldots,p_{n-1},p_n)+p_n^qT_q(\lambda,1-\lambda). \tag{1.4}$$

If we set $\lambda p_n=y$ and $(1-\lambda)p_n=z$ in (1.4), then for $p_n=y+z\neq 0$ we have

$$T_q(p_1,\ldots,p_{n-1},y,z)=T_q(p_1,\ldots,p_{n-1},p_n)+p_n^qT_q\left(\frac{y}{p_n},\frac{z}{p_n}\right). \tag{1.5}$$

Then for any $x,y,z\in\mathbf{R}$ such that $0\le x,y<1$, $0<z\le 1$ and $x+y+z=1$, we have

$$\begin{aligned}T_q(x,y,z)&=T_q(x,y+z)+(y+z)^qT_q\left(\frac{y}{y+z},\frac{z}{y+z}\right)\\&=T_q(y,x+z)+(x+z)^qT_q\left(\frac{x}{x+z},\frac{z}{x+z}\right).\end{aligned}$$

If we set $t_q(x)\equiv T_q(x,1-x)$, then we have

$$t_q(x)+(1-x)^qt_q\left(\frac{y}{1-x}\right)=t_q(y)+(1-y)^qt_q\left(\frac{x}{1-y}\right).$$

Taking $x=0$ and some $y>0$, we have $T_q(0,1)=t_q(0)=0$ for $q\neq 0$. Again setting $\lambda=0$ in (1.4) and using (T2), we have the reducing condition

$$T_q(p_1,\ldots,p_n,0)=T_q(p_1,\ldots,p_n).$$

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Thus T_q satisfies all conditions of the generalized Faddeev's axiom (F1), (F2) and (F3). Therefore we can apply Theorem 1.1 so that we obtain $T_q(\lambda_1, \ldots, \lambda_n) = \mu_q H_q(\lambda_1, \ldots, \lambda_n)$. Thus we have $T_q(\rho) = \mu_q S_q(\rho)$, for density operator ρ. ∎

Remark 1. For the special case $q = 0$ in the above theorem, we need the reducing condition as an additional axiom.

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2. A Continuity of Tsallis Entropy

We give a continuity property of the Tsallis entropy $S_q(\rho)$. To do so, we state a few lemmas.

Lemma 2.1. *For a density operator* ρ *on the finite dimensional Hilbert space* $\mathbf{H}$, *we have*

$$S_q(\rho) \leq \ln_q d,$$

where $d = \dim \mathbf{H} < \infty$.

Proof. Since we have $\ln_q z \leq z - 1$ for $q \geq 0$ and $z \geq 0$, we have $\frac{x - x^q y^{1-q}}{1-q} \geq x - y$ for $x \geq 0$, $y \geq 0$, $q \geq 0$ and $q \neq 1$, Therefore the Tsallis relative entropy [5]:

$$D_q(\rho|\sigma) \equiv \frac{\mathrm{Tr}[\rho - \rho^q \sigma^{1-q}]}{1-q}$$

for two commuting density operators ρ and σ, $q \geq 0$ and $q \neq 1$, is nonnegative. Then we have $0 \leq D_q(\rho|\frac{1}{d}I) = -d^{q-1}(S_q(\rho) - \ln_q d)$. Thus we have the present lemma. ∎

Lemma 2.2. *If* f *is a concave function and* $f(0) = f(1) = 0$, *then we have*

$$|f(t+s) - f(t)| \leq \max\{f(s), f(1-s)\}$$

for any $s \in [0, 1/2]$ *and* $t \in [0,1]$ *satisfying* $0 \leq s + t \leq 1$.

Proof.

(1) Consider the function $r(t) = f(s) - f(t+s) + f(t)$. Then $r'(t) \geq 0$ since f' is a monotone decreasing function. Thus we have $r(t) \geq 0$ by $r(0) = 0$. Therefore $f(t+s) - f(t) \leq f(s)$.

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(2) Consider the function of $l(t) = f(t+s) - f(t) + f(1-s)$. Then $l'(t) \le 0$. Thus we have $l(t) \ge 0$ by $l(1-s) = 0$. Therefore $-f(1-s) \le f(t+s) - f(t)$.

Thus we have the present lemma. ∎

Lemma 2.3. *For any real number $u, v \in [0,1]$ and $q \in [0,2]$, if $|u - v| \le \frac{1}{2}$, then $|\eta_q(u) - \eta_q(v)| \le \eta_q(|u-v|)$.*

Proof. Since η_q is a concave function with $\eta_q(0) = \eta_q(1) = 0$, we have

$$|\eta_q(t+s) - \eta_q(t)| \le \max\{\eta_q(s), \eta_q(1-s)\}$$

for $s \in [0, 1/2]$ and $t \in [0,1]$ satisfying $0 \le t + s \le 1$, by Lemma 2.2. Here we set

$$h_q(s) \equiv \eta_q(s) - \eta_q(1-s), \quad s \in [0, 1/2], \;\; q \in [0,2].$$

Then we have $h_q(0) = h_q(1/2) = 0$ and $h_q''(s) \le 0$ for $s \in [0, 1/2]$. Therefore we have $h_q(s) \ge 0$, which implies

$$\max\{\eta_q(s), \eta_q(1-s)\} = \eta_q(s).$$

Thus we have the present lemma by letting $u = t + s$ and $v = t$. ∎

Theorem 2.4. *For two density operators ρ_1 and ρ_2 on the finite dimensional Hilbert space $\mathbf{H}$ with $\dim \mathbf{H} = d$ and $q \in [0,2]$, if $\|\rho_1 - \rho_2\|_1 \le q^{1/(1-q)}$, then*

$$|S_q(\rho_1) - S_q(\rho_2)| \le \|\rho_1 - \rho_2\|_1^q \ln_q d + \eta_q(\|\rho_1 - \rho_2\|_1),$$

where we denote $\|A\|_1 \equiv Tr\left[(A^*A)^{1/2}\right]$ for a bounded linear operator A.

Proof. Let $\lambda_1^{(1)} \ge \lambda_2^{(1)} \ge \cdots \ge \lambda_d^{(1)}$ and $\lambda_1^{(2)} \ge \lambda_2^{(2)} \ge \cdots \ge \lambda_d^{(2)}$ be eigenvalues of two density operators ρ_1 and ρ_2, respectively. (The degenerate eigenvalues are

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repeated according to their multiplicity.) We set $\varepsilon \equiv \sum_{j=1}^{d} \varepsilon_j$ and $\varepsilon_j \equiv \left|\lambda_j^{(1)} - \lambda_j^{(2)}\right|$. Then we have

$$\varepsilon_j \leq \varepsilon \leq \|\rho_1 - \rho_2\|_1 \leq q^{1/(1-q)} \leq \frac{1}{2}$$

by Lemma 1.7 of [8]. Applying Lemma 2.3, we have

$$|S_q(\rho_1) - S_q(\rho_2)| \leq \sum_{j=1}^{d} \left|\eta_q\left(\lambda_j^{(1)}\right) - \eta_q\left(\lambda_j^{(2)}\right)\right| \leq \sum_{j=1}^{d} \eta_q(\varepsilon_j).$$

By the formula $\ln_q(xy) = \ln_q x + x^{1-q} \ln_q y$, we have

$$\begin{aligned}
\sum_{j=1}^{d} \eta_q(\varepsilon_j) &= -\sum_{j=1}^{d} \varepsilon_j^q \ln_q \varepsilon_j \\
&= \varepsilon \left\{ -\sum_{j=1}^{d} \frac{\varepsilon_j^q}{\varepsilon} \ln_q \left(\frac{\varepsilon_j}{\varepsilon} \varepsilon \right) \right\} \\
&= \varepsilon \left\{ -\sum_{j=1}^{d} \frac{\varepsilon_j^q}{\varepsilon} \ln_q \frac{\varepsilon_j}{\varepsilon} - \sum_{j=1}^{d} \frac{\varepsilon_j^q}{\varepsilon} \left(\frac{\varepsilon_j}{\varepsilon} \right)^{1-q} \ln_q \varepsilon \right\} \\
&= \varepsilon^q \sum_{j=1}^{d} \eta_q \left(\frac{\varepsilon_j}{\varepsilon} \right) + \eta_q(\varepsilon) \\
&\leq \varepsilon^q \ln_q d + \eta_q(\varepsilon).
\end{aligned}$$

In the above inequality, Lemma 2.1 was used for $\rho = (\varepsilon_1/\varepsilon, \ldots, \varepsilon_d/\varepsilon)$. Therefore we have

$$|S_q(\rho_1) - S_q(\rho_2)| \leq \varepsilon^q \ln_q d + \eta_q(\varepsilon).$$

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Now $\eta_q(x)$ is a monotone increasing function on $x \in [0, q^{1/(1-q)}]$. In addition, x^q is a monotone increasing function for $q \in [0, 2]$. Thus we have the present theorem. ∎

By taking the limit as $q \to 1$, we have the following Fannes' inequality (see pp.512 of [7], also [4, 2, 8]) as a corollary, since $\lim_{q\to 1} q^{1/(1-q)} = \frac{1}{e}$.

Corollary 2.5. *For two density operators ρ_1 and ρ_2 on the finite dimensional Hilbert space $\mathbf{H}$ with $\dim \mathbf{H} = d < \infty$, if $\|\rho_1 - \rho_2\|_1 \le \frac{1}{e}$, then*

$$|S_1(\rho_1) - S_1(\rho_2)| \le \|\rho_1 - \rho_2\|_1 \ln d + \eta_1(\|\rho_1 - \rho_2\|_1),$$

where S_1 represents the von Neumann entropy $S_1(\rho) = Tr[\eta_1(\rho)]$ and $\eta_1(x) = -x \ln x$.

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