

A GENERALIZED FANNES' INEQUALITY

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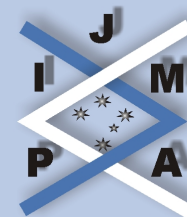
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Generalized Fannes' Inequality

S. Furuichi, K. Yanagi and
K. Kuriyama

vol. 8, iss. 1, art. 5, 2007

[Title Page](#)

[Contents](#)

[«](#) [»](#)

[◀](#) [▶](#)

Page 1 of 13

[Go Back](#)

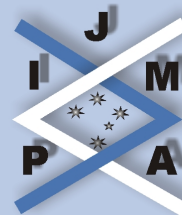
[Full Screen](#)

[Close](#)

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Abstract: We axiomatically characterize the Tsallis entropy of a finite quantum system. In addition, we derive a continuity property of Tsallis entropy. This gives a generalization of the Fannes' inequality.

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Dedicatory: Dedicated to Professor Masanori Ohya on his 60th birthday.

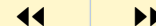
Generalized Fannes' Inequality

S. Furuichi, K. Yanagi and
K. Kuriyama

vol. 8, iss. 1, art. 5, 2007

Title Page

Contents



Page 2 of 13

Go Back

Full Screen

Close

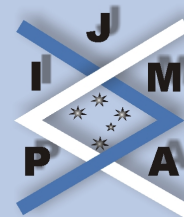
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Contents

- | | | |
|---|---|---|
| 1 | Introduction with Uniqueness Theorem of Tsallis Entropy | 4 |
| 2 | A Continuity of Tsallis Entropy | 9 |



Generalized Fannes' Inequality

S. Furuichi, K. Yanagi and
K. Kuriyama

vol. 8, iss. 1, art. 5, 2007

Title Page

Contents



Page 3 of 13

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



1. Introduction with Uniqueness Theorem of Tsallis Entropy

Three or four decades ago, a number of researchers investigated some extensions of the Shannon entropy [1]. In statistical physics, the Tsallis entropy, defined in [10] by

$$H_q(X) \equiv \frac{\sum_x (p(x)^q - p(x))}{1 - q} = \sum_x \eta_q(p(x))$$

with one parameter $q \in \mathbb{R}^+$ as an extension of Shannon entropy $H_1(X) = -\sum_x p(x) \log p(x)$, for any probability distribution $p(x) \equiv p(X = x)$ of a given random variable X , where q -entropy function is defined by $\eta_q(x) \equiv -x^q \ln_q x = \frac{x^q - x}{1 - q}$ and the q -logarithmic function $\ln_q x \equiv \frac{x^{1-q} - 1}{1 - q}$ is defined for $q \geq 0$, $q \neq 1$ and $x \geq 0$.

The Tsallis entropy $H_q(X)$ converges to the Shannon entropy $-\sum_x p(x) \log p(x)$ as $q \rightarrow 1$. See [5] for fundamental properties of the Tsallis entropy and the Tsallis relative entropy. In the previous paper [6], we gave the uniqueness theorem for the Tsallis entropy for a classical system, introducing the generalized Faddeev's axiom. We briefly review the uniqueness theorem for the Tsallis entropy below.

The function $I_q(x_1, \dots, x_n)$ is assumed to be defined on n -tuple (x_1, \dots, x_n) belonging to

$$\Delta_n \equiv \left\{ (p_1, \dots, p_n) \left| \sum_{i=1}^n p_i = 1, p_i \geq 0 \ (i = 1, 2, \dots, n) \right. \right\}$$

and to take values in $\mathbb{R}^+ \equiv [0, \infty)$. Then we adopted the following generalized Faddeev's axiom.

Generalized Fannes' Inequality

S. Furuichi, K. Yanagi and
K. Kuriyama

vol. 8, iss. 1, art. 5, 2007

Title Page

Contents



Page 4 of 13

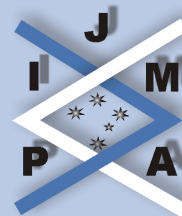
Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-575b



Title Page

Contents



Page 5 of 13

Go Back

Full Screen

Close

Axiom 1. (Generalized Faddeev's axiom)

(F1) *Continuity*: The function $f_q(x) \equiv I_q(x, 1 - x)$ with parameter $q \geq 0$ is continuous on the closed interval $[0, 1]$ and $f_q(x_0) > 0$ for some $x_0 \in [0, 1]$.

(F2) *Symmetry*: For arbitrary permutation $\{x'_k\} \in \Delta_n$ of $\{x_k\} \in \Delta_n$,

$$(1.1) \quad I_q(x_1, \dots, x_n) = I_q(x'_1, \dots, x'_n).$$

(F3) *Generalized additivity*: For $x_n = y + z$, $y \geq 0$ and $z > 0$,

$$(1.2) \quad I_q(x_1, \dots, x_{n-1}, y, z) = I_q(x_1, \dots, x_n) + x_n^q I_q\left(\frac{y}{x_n}, \frac{z}{x_n}\right).$$

Theorem 1.1 ([6]). *The conditions (F1), (F2) and (F3) uniquely give the form of the function $I_q : \Delta_n \rightarrow \mathbb{R}^+$ such that*

$$(1.3) \quad I_q(x_1, \dots, x_n) = \mu_q H_q(x_1, \dots, x_n),$$

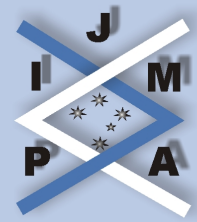
where μ_q is a positive constant that depends on the parameter $q > 0$.

If we further impose the normalized condition on Theorem 1.1, it determines the entropy of type β (the structural a -entropy), (see [1, p. 189]).

Definition 1.1. For a density operator ρ on a finite dimensional Hilbert space \mathbf{H} , the Tsallis entropy is defined by

$$S_q(\rho) \equiv \frac{\text{Tr}[\rho^q - \rho]}{1 - q} = \text{Tr}[\eta_q(\rho)],$$

with a nonnegative real number q .



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 6 of 13

Go Back

Full Screen

Close

Note that the Tsallis entropy is a particular case of f -entropy [11]. See also [9] for a quasi-entropy which is a quantum version of f -divergence [3].

Let T_q be a mapping on the set $S(\mathbf{H})$ of all density operators to \mathbb{R}^+ .

Axiom 2. We give the postulates which the Tsallis entropy should satisfy.

(T1) *Continuity:* For $\rho \in S(\mathbf{H})$, $T_q(\rho)$ is a continuous function with respect to the 1-norm $\|\cdot\|_1$.

(T2) *Invariance:* For unitary transformation U , $T_q(U^* \rho U) = T_q(\rho)$.

(T3) *Generalized mixing condition:* For $\rho = \bigoplus_{k=1}^n \lambda_k \rho_k$ on $\mathbf{H} = \bigoplus_{k=1}^n \mathbf{H}_k$, where $\lambda_k \geq 0$, $\sum_{k=1}^n \lambda_k = 1$, $\rho_k \in S(\mathbf{H}_k)$, we have the additivity:

$$T_q(\rho) = \sum_{k=1}^n \lambda_k^q T_q(\rho_k) + T_q(\lambda_1, \dots, \lambda_n),$$

where $(\lambda_1, \dots, \lambda_n)$ represents the diagonal matrix $(\lambda_k \delta_{kj})_{k,j=1,\dots,n}$.

Theorem 1.2. *If T_q satisfies Axiom 2, then T_q is uniquely given by the following form*

$$T_q(\rho) = \mu_q S_q(\rho),$$

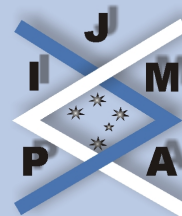
with a positive constant number μ_q depending on the parameter $q > 0$.

Proof. Although the proof is quite similar to that of Theorem 2.1 in [8], we present it for readers' convenience. From (T2) and (T3), we have

$$T_q(\lambda_1, \lambda_2) = \lambda_1^q T_q(1) + \lambda_2^q T_q(1) + T_q(\lambda_1, \lambda_2),$$

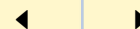
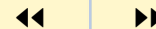
which implies $T_q(1) = 0$. Moreover, by (T2) and (T3), when $p_n \neq 1$, we have

$$\begin{aligned} & T_q(p_1, \dots, p_{n-1}, \lambda p_n, (1 - \lambda) p_n) \\ &= p_n^q T_q(\lambda, 1 - \lambda) + (1 - p_n)^q T_q\left(\frac{p_1}{1 - p_n}, \dots, \frac{p_{n-1}}{1 - p_n}\right) + T_q(p_n, 1 - p_n) \end{aligned}$$



Title Page

Contents



Page 7 of 13

Go Back

Full Screen

Close

and

$$T_q(p_1, \dots, p_{n-1}, p_n) = p_n^q T_q(1) + (1 - p_n)^q T_q\left(\frac{p_1}{1 - p_n}, \dots, \frac{p_{n-1}}{1 - p_n}\right) + T_q(p_n, 1 - p_n).$$

From these equations, we have

$$(1.4) \quad T_q(p_1, \dots, p_{n-1}, \lambda p_n, (1 - \lambda)p_n) = T_q(p_1, \dots, p_{n-1}, p_n) + p_n^q T_q(\lambda, 1 - \lambda).$$

If we set $\lambda p_n = y$ and $(1 - \lambda)p_n = z$ in (1.4), then for $p_n = y + z \neq 0$ we have

$$(1.5) \quad T_q(p_1, \dots, p_{n-1}, y, z) = T_q(p_1, \dots, p_{n-1}, p_n) + p_n^q T_q\left(\frac{y}{p_n}, \frac{z}{p_n}\right).$$

Then for any $x, y, z \in \mathbf{R}$ such that $0 \leq x, y < 1$, $0 < z \leq 1$ and $x + y + z = 1$, we have

$$\begin{aligned} T_q(x, y, z) &= T_q(x, y + z) + (y + z)^q T_q\left(\frac{y}{y + z}, \frac{z}{y + z}\right) \\ &= T_q(y, x + z) + (x + z)^q T_q\left(\frac{x}{x + z}, \frac{z}{x + z}\right). \end{aligned}$$

If we set $t_q(x) \equiv T_q(x, 1 - x)$, then we have

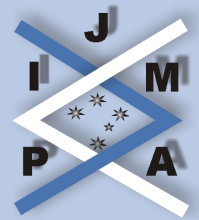
$$t_q(x) + (1 - x)^q t_q\left(\frac{y}{1 - x}\right) = t_q(y) + (1 - y)^q t_q\left(\frac{x}{1 - y}\right).$$

Taking $x = 0$ and some $y > 0$, we have $T_q(0, 1) = t_q(0) = 0$ for $q \neq 0$. Again setting $\lambda = 0$ in (1.4) and using (T2), we have the reducing condition

$$T_q(p_1, \dots, p_n, 0) = T_q(p_1, \dots, p_n).$$

Thus T_q satisfies all conditions of the generalized Faddeev's axiom (F1), (F2) and (F3). Therefore we can apply Theorem 1.1 so that we obtain $T_q(\lambda_1, \dots, \lambda_n) = \mu_q H_q(\lambda_1, \dots, \lambda_n)$. Thus we have $T_q(\rho) = \mu_q S_q(\rho)$, for density operator ρ . ■

Remark 1. For the special case $q = 0$ in the above theorem, we need the reducing condition as an additional axiom.



Generalized Fannes' Inequality

S. Furuichi, K. Yanagi and

K. Kuriyama

vol. 8, iss. 1, art. 5, 2007

Title Page

Contents



Page 8 of 13

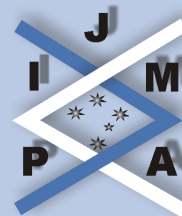
Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



Title Page

Contents



Page 9 of 13

Go Back

Full Screen

Close

2. A Continuity of Tsallis Entropy

We give a continuity property of the Tsallis entropy $S_q(\rho)$. To do so, we state a few lemmas.

Lemma 2.1. *For a density operator ρ on the finite dimensional Hilbert space \mathbf{H} , we have*

$$S_q(\rho) \leq \ln_q d,$$

where $d = \dim \mathbf{H} < \infty$.

Proof. Since we have $\ln_q z \leq z - 1$ for $q \geq 0$ and $z \geq 0$, we have $\frac{x-x^q y^{1-q}}{1-q} \geq x - y$ for $x \geq 0, y \geq 0, q \geq 0$ and $q \neq 1$, Therefore the Tsallis relative entropy [5]:

$$D_q(\rho|\sigma) \equiv \frac{\text{Tr}[\rho - \rho^q \sigma^{1-q}]}{1 - q}$$

for two commuting density operators ρ and σ , $q \geq 0$ and $q \neq 1$, is nonnegative. Then we have $0 \leq D_q(\rho|\frac{1}{d}I) = -d^{q-1}(S_q(\rho) - \ln_q d)$. Thus we have the present lemma. ■

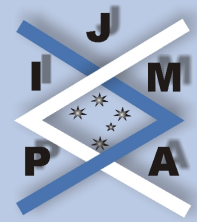
Lemma 2.2. *If f is a concave function and $f(0) = f(1) = 0$, then we have*

$$|f(t+s) - f(t)| \leq \max\{f(s), f(1-s)\}$$

for any $s \in [0, 1/2]$ and $t \in [0, 1]$ satisfying $0 \leq s+t \leq 1$.

Proof.

- (1) Consider the function $r(t) = f(s) - f(t+s) + f(t)$. Then $r'(t) \geq 0$ since f' is a monotone decreasing function. Thus we have $r(t) \geq 0$ by $r(0) = 0$. Therefore $f(t+s) - f(t) \leq f(s)$.



(2) Consider the function of $l(t) = f(t + s) - f(t) + f(1 - s)$. Then $l'(t) \leq 0$. Thus we have $l(t) \geq 0$ by $l(1 - s) = 0$. Therefore $-f(1 - s) \leq f(t + s) - f(t)$.

Thus we have the present lemma. ■

Lemma 2.3. For any real number $u, v \in [0, 1]$ and $q \in [0, 2]$, if $|u - v| \leq \frac{1}{2}$, then $|\eta_q(u) - \eta_q(v)| \leq \eta_q(|u - v|)$.

Proof. Since η_q is a concave function with $\eta_q(0) = \eta_q(1) = 0$, we have

$$|\eta_q(t + s) - \eta_q(t)| \leq \max \{ \eta_q(s), \eta_q(1 - s) \}$$

for $s \in [0, 1/2]$ and $t \in [0, 1]$ satisfying $0 \leq t + s \leq 1$, by Lemma 2.2. Here we set

$$h_q(s) \equiv \eta_q(s) - \eta_q(1 - s), \quad s \in [0, 1/2], \quad q \in [0, 2].$$

Then we have $h_q(0) = h_q(1/2) = 0$ and $h_q''(s) \leq 0$ for $s \in [0, 1/2]$. Therefore we have $h_q(s) \geq 0$, which implies

$$\max \{ \eta_q(s), \eta_q(1 - s) \} = \eta_q(s).$$

Thus we have the present lemma by letting $u = t + s$ and $v = t$. ■

Theorem 2.4. For two density operators ρ_1 and ρ_2 on the finite dimensional Hilbert space \mathbf{H} with $\dim \mathbf{H} = d$ and $q \in [0, 2]$, if $\|\rho_1 - \rho_2\|_1 \leq q^{1/(1-q)}$, then

$$|S_q(\rho_1) - S_q(\rho_2)| \leq \|\rho_1 - \rho_2\|_1^q \ln_q d + \eta_q(\|\rho_1 - \rho_2\|_1),$$

where we denote $\|A\|_1 \equiv \text{Tr} [(A^*A)^{1/2}]$ for a bounded linear operator A .

Proof. Let $\lambda_1^{(1)} \geq \lambda_2^{(1)} \geq \dots \geq \lambda_d^{(1)}$ and $\lambda_1^{(2)} \geq \lambda_2^{(2)} \geq \dots \geq \lambda_d^{(2)}$ be eigenvalues of two density operators ρ_1 and ρ_2 , respectively. (The degenerate eigenvalues are

Title Page

Contents

◀◀ ▶▶

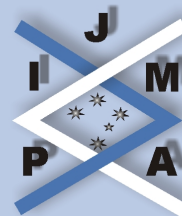
◀ ▶

Page 10 of 13

Go Back

Full Screen

Close



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 11 of 13

Go Back

Full Screen

Close

repeated according to their multiplicity.) We set $\varepsilon \equiv \sum_{j=1}^d \varepsilon_j$ and $\varepsilon_j \equiv \left| \lambda_j^{(1)} - \lambda_j^{(2)} \right|$. Then we have

$$\varepsilon_j \leq \varepsilon \leq \|\rho_1 - \rho_2\|_1 \leq q^{1/(1-q)} \leq \frac{1}{2}$$

by Lemma 1.7 of [8]. Applying Lemma 2.3, we have

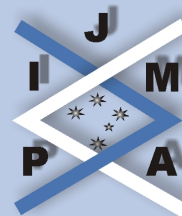
$$|S_q(\rho_1) - S_q(\rho_2)| \leq \sum_{j=1}^d \left| \eta_q \left(\lambda_j^{(1)} \right) - \eta_q \left(\lambda_j^{(2)} \right) \right| \leq \sum_{j=1}^d \eta_q(\varepsilon_j).$$

By the formula $\ln_q(xy) = \ln_q x + x^{1-q} \ln_q y$, we have

$$\begin{aligned} \sum_{j=1}^d \eta_q(\varepsilon_j) &= - \sum_{j=1}^d \varepsilon_j^q \ln_q \varepsilon_j \\ &= \varepsilon \left\{ - \sum_{j=1}^d \frac{\varepsilon_j^q}{\varepsilon} \ln_q \left(\frac{\varepsilon_j}{\varepsilon} \varepsilon \right) \right\} \\ &= \varepsilon \left\{ - \sum_{j=1}^d \frac{\varepsilon_j^q}{\varepsilon} \ln_q \frac{\varepsilon_j}{\varepsilon} - \sum_{j=1}^d \frac{\varepsilon_j^q}{\varepsilon} \left(\frac{\varepsilon_j}{\varepsilon} \right)^{1-q} \ln_q \varepsilon \right\} \\ &= \varepsilon^q \sum_{j=1}^d \eta_q \left(\frac{\varepsilon_j}{\varepsilon} \right) + \eta_q(\varepsilon) \\ &\leq \varepsilon^q \ln_q d + \eta_q(\varepsilon). \end{aligned}$$

In the above inequality, Lemma 2.1 was used for $\rho = (\varepsilon_1/\varepsilon, \dots, \varepsilon_d/\varepsilon)$. Therefore we have

$$|S_q(\rho_1) - S_q(\rho_2)| \leq \varepsilon^q \ln_q d + \eta_q(\varepsilon).$$



Now $\eta_q(x)$ is a monotone increasing function on $x \in [0, q^{1/(1-q)}]$. In addition, x^q is a monotone increasing function for $q \in [0, 2]$. Thus we have the present theorem. ■

By taking the limit as $q \rightarrow 1$, we have the following Fannes' inequality (see pp.512 of [7], also [4, 2, 8]) as a corollary, since $\lim_{q \rightarrow 1} q^{1/(1-q)} = \frac{1}{e}$.

Corollary 2.5. *For two density operators ρ_1 and ρ_2 on the finite dimensional Hilbert space \mathbf{H} with $\dim \mathbf{H} = d < \infty$, if $\|\rho_1 - \rho_2\|_1 \leq \frac{1}{e}$, then*

$$|S_1(\rho_1) - S_1(\rho_2)| \leq \|\rho_1 - \rho_2\|_1 \ln d + \eta_1(\|\rho_1 - \rho_2\|_1),$$

where S_1 represents the von Neumann entropy $S_1(\rho) = \text{Tr}[\eta_1(\rho)]$ and $\eta_1(x) = -x \ln x$.

Generalized Fannes' Inequality

S. Furuichi, K. Yanagi and
K. Kuriyama

vol. 8, iss. 1, art. 5, 2007

Title Page

Contents



Page 12 of 13

Go Back

Full Screen

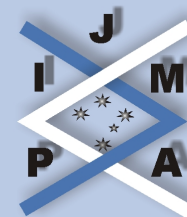
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Generalized Fannes' Inequality

S. Furuichi, K. Yanagi and
K. Kuriyama

vol. 8, iss. 1, art. 5, 2007

Title Page

Contents



Page 13 of 13

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756